Stats 310A Session 4

November 8, 2026

1 Uniform integrability

The dominated convergence theorem states that $X_n \overset{a.s.}{\to} X_\infty$ and $|X_n| \le Y$ for some integrable Y implies that $X_n \overset{L_1}{\to} X_\infty$ and $\mathbb{E}[X_n] \to \mathbb{E}[X_\infty]$. In this section, we explore *uniform integrability*, a useful concept that allows us to relax both conditions assumed in the dominated convergence theorem and still get L_1 convergence. We will follow Section 1.3.4 in Dembo's Notes.

Definition 1 (Uniform integrability)**.** *A collection of R.V.-s $\{X_\alpha, \alpha \in \mathcal{I}\}$ is called* uniformly integrable *(U.I.) if*

$$\lim_{n\to\infty} \sup_{\alpha\in\mathcal{I}} \mathbb{E}[|X_\alpha| \mathbf{1}\{|X_\alpha| > M\}] = 0.$$

Let us show that U.I. is indeed a relaxation of that $|X_\alpha| \le Y$ for some integrable Y.

Lemma 1.1. *Let Y be integrable and suppose that $|X_\alpha| \le Y$ for all α, then $\{X_\alpha\}$ is U.I.. In particular, any finite collection of integrable R.V.-s is U.I.. Further, if X_α is U.I. then $\sup_\alpha \mathbb{E}[|X_\alpha|] < \infty$.*

Proof That $\{X_\alpha\}$ is U.I. follows from that

$$\sup_\alpha \mathbb{E}[|X_\alpha| \mathbf{1}\{|X_\alpha| \ge M\}] \le \mathbb{E}[|Y| \mathbf{1}\{|Y| \ge M\}] \to 0 \ \text{ as } M \to \infty.$$

If we have a finite collection $\{X_k\}_{k=1}^n$ that are integrable, then $Y = \sum_{k=1}^n |X_k|$ is integrable and dominates X_k. Suppose $\{X_\alpha\}$ is U.I., then for all M we have

$$\sup_\alpha \mathbb{E}[|X_\alpha|] \le M + \sup_\alpha \mathbb{E}[|X_\alpha| \mathbf{1}\{|X_\alpha| > M\}].$$

The second term goes to zero as $M \to \infty$, so has to be finite for some M. This M yields a finite value on the RHS, so gives a finite upper bound on $\sup_\alpha \mathbb{E}[|X_\alpha|]$. □

To further understand U.I., consider the following example, which shows $\sup_\alpha \mathbb{E}[|X_\alpha|] < \infty$ does not necessarily give U.I..

Example 1: Let X_n be binary R.V.-s with $\mathbb{P}(X_n = 0) = 1 - 1/n$ and $\mathbb{P}(X_n = n) = 1/n$. Then $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] = 1$ for all n but X_n is not U.I.: for any M, take $n = \lceil M \rceil$, we have $\mathbb{E}[|X_n| \mathbf{1}\{|X_n| \ge M\}] = \mathbb{E}[|X_n|] = 1$. ◇

We are now ready to state the main convergence theorem.

Theorem 1 (Vitali's convergence theorem)**.** *Suppose $X_n \overset{p}{\to} X_\infty$, then the following are equivalent:*

(a) $\{X_n\}$ is U.I..

(b) $X_n \xrightarrow{L_1} X_\infty$.

(c) X_n *is integrable for all* $n \le \infty$ *and* $\mathbb{E}[|X_n|] \to \mathbb{E}[|X_\infty|]$.

Proof "$(a) \implies (b)$". We first deal with the case that $|X_n| \le M$ for some finite M. For all n and $\varepsilon > 0$, define

$$B_{n,\varepsilon} = \{\omega : |X_n(\omega) - X_\infty(\omega)| > \varepsilon\}.$$

As $X_n \xrightarrow{p} X_\infty$, we have $\mathbb{P}(B_{n,\varepsilon}) \to 0$ as $n \to \infty$ for all ε. In particular, we have $\mathbb{P}(|X_\infty| \ge M + \varepsilon) \le \mathbb{P}(B_{n,\varepsilon})$. Letting $n \to \infty$ gives $|X_\infty| \le M + \varepsilon$ almost surely, which after taking $\varepsilon \to 0$ gives that $|X_\infty| \le M$ almost surely. Hence, $|X_n - X_\infty| \le 2M$, which allows us to bound

$$\mathbb{E}[|X_n - X_\infty|] = \mathbb{E}[|X_n - X_\infty|\mathbf{1}\{B_{n,\varepsilon}\}] + \mathbb{E}[|X_n - X_\infty|\mathbf{1}\{B_{n,\varepsilon}^c\}] \le 2M\mathbb{P}(B_{n,\varepsilon}) + \varepsilon.$$

Taking $n \to \infty$, we conclude that $\limsup_{n\to\infty} \mathbb{E}[|X_n - X_\infty|] \le \varepsilon$, so as ε is arbitrary we get $\mathbb{E}[|X_n - X_\infty|] \to 0$.

We now show the general case where $\{X_n\}$ is U.I. by applying a truncation argument. Define the truncation function

$$\varphi_M(x) = x\mathbf{1}\{|x| \le M\}.$$

As $X_n \xrightarrow{p} X_\infty$ and φ_M is continuous, we have $\varphi_M(X_n) \xrightarrow{p} \varphi_M(X_\infty)$. The R.V.-s $\varphi_M(X_n)$ are bounded in $[-M, M]$, so by the bounded case, we get $\mathbb{E}[|\varphi_M(X_n) - \varphi_M(X_\infty)|] \to 0$.

As $\{X_n\}$ is U.I., we have $\sup_n \mathbb{E}[|X_n|] = c < \infty$ by Lemma 1.1, which gives that

$$c = \sup_n \mathbb{E}[|X_n|] \ge \sup_n \mathbb{E}[|\varphi_M(X_n)|] \ge \lim_{n\to\infty} \mathbb{E}[|\varphi_M(X_n)|] = \mathbb{E}[\varphi_M(X_\infty)].$$

The R.V.-s $|\varphi_M(X_\infty)|$ is an increasing sequence as $M \uparrow \infty$ and converges to $|X_\infty|$. By the monotone convergence theorem, we have

$$\mathbb{E}[|X_\infty|] = \lim_{M\to\infty} \mathbb{E}[|\varphi_M(X_\infty)|] \le c,$$

so X_∞ is integrable.

As $\{X_n\}$ is U.I. and X_∞ is integrable, for any $\varepsilon > 0$, there exists some M such that $\sup_n \mathbb{E}[|X_n|\mathbf{1}\{|X_n| \ge M\}] \le \varepsilon$ and $\mathbb{E}[|X_\infty|\mathbf{1}\{|X_\infty| \ge M\}] \le \varepsilon$. By the triangle inequality, we have

$$\begin{aligned}\mathbb{E}[|X_n - X_\infty|] &\le \mathbb{E}[|X_n - \varphi_M(X_n)|] + \mathbb{E}[|\varphi_M(X_n) - \varphi_m(X_\infty)|] + \mathbb{E}[|\varphi_M(X_\infty) - X_\infty|] \\ &\le 2\varepsilon + \mathbb{E}[|\varphi_M(X_n) - \varphi_M(X_\infty)|].\end{aligned}$$

Letting $n \to \infty$, we get $\limsup_{n\to\infty} \mathbb{E}[|X_n - X_\infty|] \le 2\varepsilon$. Taking $\varepsilon \downarrow 0$ gives that $\mathbb{E}[|X_n - X_\infty|] \to 0$, the desired result.

That $(b) \implies (c)$ is immediate, and we will skip the proof of $(c) \implies (a)$. □

2 Constructing measures from densities

We have seen in the last HW that measures can be defined via densities – indeed, this is one standard way of defining continuous random variables in elementary probability. This section gives a measure-theoretic treatment of such construction and in particular show that it satisfies the composition rule.

Proposition 1. *Fix a measure space $(\mathbb{S}, \mathcal{F}, \mu)$. Every $f \in m\mathcal{F}_+$ induces a measure $f\mu$ on $(\mathbb{S}, \mathcal{F})$ via $f\mu(A) = \mu(f\mathbf{1}_A)$ for all $A \in \mathcal{F}$. These measures satisfy the composition relation $h(f\mu) = (hf)\mu$ for all $f, h \in m\mathcal{F}_+$. Further, $h \in L^1(\mathbb{S}, \mathcal{F}, f\mu)$ if and only if $fh \in L^1(\mathbb{S}, \mathcal{F}, \mu)$ and then $(f\mu)h = \mu(fh)$.*

Proof We give a proof sketch for the composition rule $h(f\mu) = (hf)\mu$. By definition of the measure operation, this requires showing the identity

$$(f\mu)(h\mathbf{1}_A) = \mu(fh\mathbf{1}_A) \quad \text{for all } A \in \mathcal{F}.$$

We will use the standard machine to show this.

(1) $h = \mathbf{1}_B$ is an indicator ($B \in \mathcal{F}$), then $h\mathbf{1}_A = \mathbf{1}_{A \cap B}$, giving that

$$(f\mu)(h\mathbf{1}_A) = (f\mu)(A \cap B) = \mu(f\mathbf{1}_{A\cap B}) = \mu(fh\mathbf{1}_A).$$

(2) The above identity is linear in the component $\mathbf{1}_B$, so extends to h being simple functions.

(3) For general $h \in m\mathcal{F}_+$, there exists simple functions $h_n \uparrow h$. Part (2) gives that $(f\mu)(h_n\mathbf{1}_A) = \mu(fh_n\mathbf{1}_A)$. Further, $h_n\mathbf{1}_A \uparrow h\mathbf{1}_A$ and $fh_n\mathbf{1}_A \uparrow fh\mathbf{1}_A$ for all $A \in \mathcal{F}$ and $f \in m\mathcal{F}_+$, so by monotone convergence we have that

$$(f\mu)(h\mathbf{1}_A) = \lim_{n\to\infty} (f\mu)(h_n\mathbf{1}_A) = \lim_{n\to\infty} \mu(fh_n\mathbf{1}_A) = \mu(fh\mathbf{1}_A).$$

□

Remark The measure μ is called the *base measure*, the function f the *density*, and $f\mu$ the *induced measure*. Elementary probability theory often uses the Lebesgue measure on $\mathbb{R}$ (or $\mathbb{R}^n$) as the base measure. Given two measures f, g, there exists μ such that $g = f\mu$ iff g is *absolutely continuous* w.r.t. f, in which case μ is called their *Radon-Nikodym derivative*.