5/25/2021 Stochastic integral wrt BM

[Mörters-Peres Chapter 7]

$(B_t, \mathcal{F}_t)_{t\geq 0}$ Wiener process. Want to define

(*) $\int_0^\infty H_t \, dB_t$ H_t ~~adapted~~ progressive

Rmk 1) (*) is a random var

2) Lebesgue-Stieltjes integral does not work

$$\|B\|_{TV[0,t]} = \sup_n \sup_{0=t_0<t_1<\dots<t_n=t} \sum_{\ell=0}^{n-1} |B_{t_{\ell+1}} - B_{t_\ell}|$$

$$= \infty .$$

$(H_t)_{t\geq 0}$ a step process if $\exists\ 0=t_0\leq t_1\leq\dots\leq t_{k+1}$ and rv $A_i \in m\mathcal{F}_{t_i}$ st

$$H_t(\omega) = \sum_{i=0}^{k} A_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

For such process. define

$$\tilde{\int_0^\infty} H_t \, dB_t := \sum_{i=0}^{k} A_i (B_{t_{i+1}} - B_{t_i})$$

Define

$\int H(t,\omega)^2 \, \lambda(dt) \, \mathbb{P}(d\omega)$

$$\|H\|_2 := \left(\int_0^\infty \mathbb{E}[H_t^2] \, dt \right)^{1/2}$$

Lebesgue on $\mathbb{R}_{\geq 0}$

norm in $L^2(\Omega \times \mathbb{R}_{\geq 0}, \mathbb{P}\times\lambda)$

Lemma H progressive, $\|H\|_2 < \infty$. Then there exists $\{H_n\}$ step processes st $\|H - H_n\|_2 \xrightarrow{n\to\infty} 0$ $\square$

Proof $H \geq 0$ wlog.

1) Assume $0 \leq H(t,\omega) \leq C$. $H(t,\omega) = 0$ $\forall t \geq T$

$t \mapsto H(t,\omega)$ a.e ω.

Define $H_n(t,\omega) = H([t]_n, \omega)$

$[t]_n = \sup\left\{\frac{k}{n} : \frac{k}{n} \leq t\right\}$. $\qquad \| \; L \sum_{i\geq 0} \overbrace{H(\underbrace{\tfrac{i}{n}}_{\in \mathcal{F}_{i/n}},\omega)}^{A_i} 1_{t\in[\frac{i}{n},\frac{i+1}{n})}$

$\lim_{n\to\infty} H_n(t,\omega) = H(t,\omega)$ a.e.

By dom. $\|H_n - H\|_2 \to 0$ $\square$

2) Assume $0 \leq H \leq C$. $H(t,\omega) = 0$ $\forall t \geq T$.

$$H_n(t,\omega) = n \int_{t-1/n}^{t} H(s,\omega)\, ds. \in m\mathcal{F}_t$$

$$\left[\; g_n(t) = n\int_{t-1/n}^{t} g(s)\, ds \qquad g_n(t) \to g(t) \text{ a.e.t.} \; \right]$$

$H_n(t,\omega) \to H(t,\omega)$ a.e.

$\Rightarrow$ $\|H_n - H\|_2 \to 0$ by DOM.

3) Assume $0 \le H(t,\omega) \le C$.

define $H_n(t,\omega) = H(t,\omega)\, 1_{t \le n}$

$$\|H_n - H\|_2^2 = \int_0^\infty \int_\Omega \underbrace{H(t,\omega)^2 1_{t>n}}_{\le H^2(t,\omega)} \, \mathbb{P}(d\omega)\, \lambda(dt)$$

$\to 0$ by DOM $\quad (\|H\|_2 < \infty.)$

4) Do not assume anything. $(\|H\| < \infty)$

$$H_n(t,\omega) = H(t,\omega) \wedge n. \qquad \blacksquare$$

<u>Thm</u> For H progr, $\|H\|_2 < \infty$, let $\{H_n\}$ be step proc. st $\|H_n - H\|_2 \to 0$.

$$H_n = \sum_{i=0}^{k(n)} A_i(\omega)\, 1_{(t_i(n), t_{i+1}(n)]}(t)$$

Then the following limit exists in L^2:

$$\int_0^\infty H_t \, dB_t := \lim_{n\to\infty} \sum_{i=0}^{k(n)} A_i \left(B_{t_{i+1}(n)} - B_{t_i(n)}\right)$$

The limit is indep of $\{H_n\}$. Further

$$\left\| \int_0^\infty H_t \, dB_t \right\|_{L^2(\mathbb{P})} = \|H\|_2 \leftarrow L^2(\mathbb{P}\times\lambda) \qquad \square$$

<u>Rmk</u> 1 $\quad H \longmapsto \int_0^\infty H_t \, dB_t \;;\; L^2(\mathbb{P}\times\lambda) \to L^2(\mathbb{P})$

Hilbert space isometry.

Wts A_0 is dense in A

"Proof". A_0 dense in A_1

A_1 dense in A_2

A_2 dense in A.

<u>Corollary</u> If H is a step process $H_t = \sum_{i=0}^{k} A_i 1_{(t_i, t_{i+1}]}$

then

$$\int_0^\infty H_t \, dB_t = \sum_{i=0}^{k} A_i (B_{t_{i+1}} - B_{t_i})$$

$\square$

<u>Rmk 2</u>

$$\Big\| \underbrace{\int_0^\infty H_t \, dB_t}_{X} - \underbrace{\int_0^\infty H_{n,t} \, dB_t}_{X_n} \Big\|_{L^2} = \| H - H_n \|_{L^2(\mathbb{P} \times \lambda)}$$

Thm states $X_n \xrightarrow{L^2} X$; by taking a subseq. can ensure

$$\sum_{n=1}^{\infty} \| H - H_n \|_2 < \infty .$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \| X - X_n \|_{L^2} < \infty .$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \mathbb{P}(|X - X_n| \geq \epsilon) < \infty \quad \forall \epsilon > 0$$

$\Rightarrow$ by BC1. $X_n \xrightarrow{a.s.} X$.

$$\int H_n \, dB \xrightarrow{a.s.} \int H \, dB$$

<u>Proof</u> For $G = \sum_{i=0}^{k} A_i \mathbb{1}_{[t_i, t_{i+1}]}$ a step process

$$\left\| \int_0^\infty G_t \, dB_t \right\|_{L^2(\mathbb{P})} = \| G \|_{L^2(\mathbb{P}\times\lambda)}. \qquad \text{indeed}$$

$$\left\| \int_0^\infty G_t \, dB_t \right\|_{L^2}^2 = \mathbb{E}\left\{ \left(\sum_{i=0}^{k} A_i \left(B_{t_{i+1}} - B_{t_i}\right) \right)^2 \right\} = *$$

(note that $A_i \left(B_{t_{i+1}} - B_{t_i}\right) \in L^2(\mathbb{P})$

$\mathbb{E}\left(A_i^2 \left(B_{t_{i+1}} - B_{t_i}\right)^2 \right\} = \mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)^2 \right] \mathbb{E}(A_i^2) < \infty$)

$$* = \sum_{i=0}^{k} \mathbb{E}\Big(\overbrace{A_i^2}^{\in m\mathcal{F}_{t_i}} \left(B_{t_{i+1}} - B_{t_i}\right)^2 \Big)$$

$$+ 2 \sum_{i<j\le k} \mathbb{E}\Big\{ \underbrace{A_i A_j \left(B_{t_{i+1}} - B_{t_i}\right)}_{\in m\mathcal{F}_{t_j}} \left(B_{t_{j+1}} - B_{t_j}\right) \Big\}$$

indep of $B_{t_{j+1}} - B_{t_j}$

$$= \sum_{i=1}^{k} \mathbb{E}(A_i^2) \left(t_{i+1} - t_i\right)$$

$$= \int_0^\infty \mathbb{E}(G_t^2) \, dt = \| G \|_{L^2(\mathbb{P}\times\lambda)}^2 .$$

Since $\| H_n - H \|_2 \to 0$, hence $\{H_n\}$ Cauchy seq in $L^2(\mathbb{P}\times\lambda)$.

$\Rightarrow \int_0^\infty H_{n,t} \, dB_t$ is Cauchy in $L^2(\mathbb{P})$.

$\Rightarrow \int_0^\infty H_{n,t} \, dB_t \xrightarrow{L^2(\mathbb{P})} X =: \int_0^\infty H_t \, dB_t$.

Limit is indep of sequence

Take a different $\{\tilde{H}_n\}$ $\|\tilde{H}_n - H\|_2 \to 0$

$$\left\| \int \tilde{H}_n \, dB - \int H_m \, dB \right\|_{L^2} = \| \tilde{H}_n - H_m \|_2$$

Take limit $m \to \infty$

$$\left\| \int \tilde{H}_n \, dB - \int H \, dB \right\|_{L^2} = \underbrace{\| \tilde{H}_n - H \|_2}_{\to 0 \text{ by assumption}}.$$

$$\lim_{n \to \infty} \left\| \int \tilde{H}_n \, dB - \int H \, dB \right\|_{L^2} = 0.$$

Finally $\|H_n - H\|_2 \to 0$ $\quad \left\| \int H_n \, dB - \int H \, dB \right\|_{L^2} \to 0$

$$\underbrace{\|H_n\|_2 = \left\| \int H_n \, dB \right\|_{L^2}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\|H\|_2 \qquad\qquad \left\| \int H \, dB \right\|_{L^2}$$

$$\Rightarrow \quad \|H\|_2 = \left\| \int H \, dB \right\|_{L^2}.$$

$\forall t \in [0,\infty]$ define $H^t(s,\omega) = H(s,\omega) 1_{s\le t}$

and

$$\int_0^t H_s \, dB_s := \int_0^\infty (H^t)_s \, dB_s .$$

<u>Rmk</u> If $H_t = \sum_{i=0}^k A_i 1_{[t_i, t_{i+1}]}(t)$

$t \in (t_\ell, t_{\ell+1}]$

$$\int_0^t H_s \, dB_s = \sum_{i=1}^{\ell-1} A_i (B_{t_{i+1}} - B_{t_i}) + A_\ell (B_t - B_{t_\ell})$$ ☒

(H^t step process...)

$\left(\int_0^t H_s \, dB_s .\right)_{t\ge 0}$ is a stochastic process.

<u>Thm</u> Assume H adapted progr $\int_0^t \mathbb{E} H_s^2 \, ds < \infty$. $\forall t$

$M_t := \int_0^t H_s \, dB_s .$

M_t has a continuous modif which is a MG

with $\mathbb{E} M_t = \mathbb{E} M_0 = 0$ $\forall t$. ☒

$H_t = 1.$ $M_t = B_t .$

$\mathcal{H} = \ell_2$

$T_k x = (\underbrace{0 \ldots 0}_{k}, x_1, x_2, \ldots)$

$\|T_k x\| = \|x\|$

$\mathcal{H}_k = \{x : (x_1 \ldots x_k) = 0\}$

subspace of $\mathcal{H}$ of codimension k

$M_t \qquad d\langle M \rangle_t \text{ "} = \mathbb{E}(dM_t^2 \mid \mathcal{F}_t)\text{"}$

$dt.$

$$\int f(x) \cos\left(\frac{x^2}{2}\right) dx$$

$$\int f(x) e^{-x^2/2} dx \qquad \checkmark$$