Lecture 3 Starting from $(\tilde{X}_t)_{t\in T}$ satisfying KC

We constructed a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ $(X_t)_{t\in \mathbb{T}}$ with cont. sample paths

$$E^X_A := \{\omega\in\Omega : X_\cdot(\omega)\in A\} \in \mathcal{F} \Leftarrow A\in \mathcal{B}^{\mathbb{T}}$$

– $A = \pi^{-1}_{t_1,..t_k}(B)$, $B\in \mathcal{B}_{\mathbb{R}^k}$ true

$\left(\pi_{t_1..t_k} : \mathbb{R}^{\mathbb{T}} \to \mathbb{R}^k \quad \omega \mapsto (\omega(t_1)..\omega(t_k))\right)$

Because $\{X_\cdot(\omega)\in A\} = \{(X_{t_1}(\omega)... X_{t_k}(\omega))\in B\}$

$$= \{(\tilde{X}_{t_1}(\omega),.. \tilde{X}_{t_k}(\omega))\in B\} \,\Delta\, N$$

$\{(\tilde{X}_{t_1}(\omega),.. \tilde{X}_{t_k}(\omega))\in B\} \in \mathcal{F}$

N negligible. $\mathbb{P}(N)=0$.

– $\{A : E^X_A\in\mathcal{F}\}$ is a σ algebra.

since $\{\pi^{-1}_{t_1..t_k}(B) : B\in\mathcal{B}_{\mathbb{R}^k}\}$ generate $\mathcal{B}^{\mathbb{T}}$

$$\Rightarrow \{A : E^X_A\in\mathcal{F}\} \supseteq \mathcal{B}^{\mathbb{T}}.$$

$$E^X_{C,A} = \{\omega : X_\cdot(\omega)\in A\} \cap \{\omega : X_\cdot(\omega)\in C(\mathbb{T})\}$$
$$\in \mathcal{F} \quad \forall A\in\mathcal{B}^{\mathbb{T}}.$$

Question: $\mathcal{F}_{C(\mathbb{T})} = \{A\cap C(\mathbb{T}) : A\in\mathcal{B}^{\mathbb{T}}\}$.

(For any $\tilde{A}\in\mathcal{F}_{C(\mathbb{T})}$ $\{X\in\tilde{A}\}\in\mathcal{F}$)

$\mathbb{T} = [a,b]^d$

Lemma $C(\Pi)$, $d(x,y) = \sup_{t\in\Pi} |x(t)-y(t)|$, $\mathcal{B}_{C(\Pi)}$

Then $\mathcal{F}_{C(\Pi)} = \mathcal{B}_{C(\Pi)}$ $\square$

Proof $\boxed{\mathcal{B}_{C(\Pi)} \subseteq \mathcal{F}_{C(\Pi)}}$ $\mathcal{B}^\Pi = \sigma(\{\pi_{t_1 \ldots t_k}^{-1}(A_1 \times \ldots A_k) : A_1 \ldots A_k \in \mathcal{B}\})$

$B_{C(\Pi)}(x,r) = \{ z \in C(\Pi) : \sup_{t \in \underbrace{\Pi \cap \mathbb{Q}^d}} |x(t)-z(t)| < r \}$ $= A \cap C(\Pi)$

[countably represented $\in \mathcal{B}^\Pi$ $\subseteq C(\Pi)$

$B_{C(\Pi)}(x,r) \in \mathcal{F}_{C(\Pi)}$

generate $\mathcal{B}_{C(\Pi)}$ $\{ B_{C(\Pi)}(r,x) : r = 1/n, x$ countable dense set$\}$

$O \subseteq C(\Pi)$ $O =$ countable union

$\boxed{\mathcal{F}_{C(\Pi)} \subseteq \mathcal{B}_{C(\Pi)}}$

$\mathcal{F}_{C(\Pi)} = \sigma(\{O\})$: $O = \{x \in C(\Pi) : x(t_1) \in O_1 \ldots x(t_n) \in O_n\}$

$O_1, \ldots O_n \in \mathbb{R}$ open.

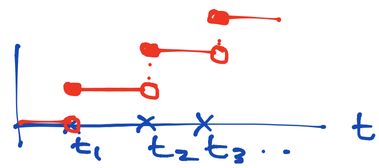
Such O is in $\mathcal{B}_{C(\Pi)}$ because it is an open set in unif. topology.

$x \in O$: $\exists \epsilon$: $B_{C(\Pi)}(x,\epsilon) \subseteq O$. $\square$

– KC also generalizes to inf. intervals.

What about processes that are not cont and do not have cont. modif?

eg Poisson process N_t

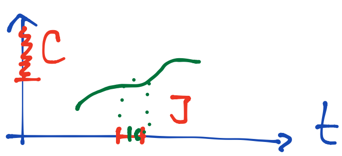


<u>Def</u> $(X_t)_{t\in\mathbb{T}}$ $\mathbb{T}=[a,b)$ is <u>separable</u> if $\exists$ countable dense set $\mathbb{D}\subseteq\mathbb{T}$ and a negligible set $N\in\mathcal{F}$ $\mathbb{P}(\mathcal{F})=0$. st

$\forall C$ closed in $\mathbb{R}$ $\forall J\subseteq\mathbb{T}$ open interval

$$\{\omega : X_t(\omega)\in C\ \forall t\in J\cap\mathbb{D}\} =$$
$$\{\omega : X_t(\omega)\in C\ \forall t\in J\}\cup N_{C,J}$$

$N_{C,J}\subseteq N$



<u>Lemma</u> $(X_t)_{t\in\mathbb{T}}$ is $\mathbb{D}$-separable iff $\forall\omega\notin N$ $\forall t\in\mathbb{T}$ $\exists(t_k)_{k\geq1}\subseteq\mathbb{D}$ st $t_k\to t$ (dependent on ω)

and $\underline{X_{t_k}(\omega)\to X_t(\omega)}$.

<u>Main examples</u>

- X_t right continuous. $\forall t$

$\mathbb{D}=[a,b)\cap\mathbb{Q}^{(2)}$, (t_k) fixed nonrandom

$t_k\in\mathbb{D}$ $t_k\downarrow t$.

RCLL (left limits).

$t \in [0,1) = \mathbb{T}$

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = t_* \\ 0 & \text{ow} \end{cases} \qquad \forall \omega$$

$\forall t \quad \exists (t_k) \subseteq \mathbb{D}$

$$X_{t_k}(\omega) \to X_t(\omega)$$

If $\mathbb{D} \not\ni t_*$ impossible

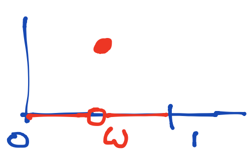
Can take $\mathbb{D} = t_* \cup [\mathbb{Q} \cap [0,1)]$

$(q_k)_k$ prob distr on $\mathbb{N}$

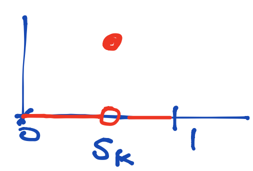
$(s_k)_{k \in \mathbb{N}} \subseteq [0,1)$ not in $\mathbb{Q}$

$(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{N}, 2^{\mathbb{N}}, q)$

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = s_k \\ 0 & \text{ow} \end{cases} \iff \omega = k.$$



$\mathbb{D} = \mathbb{Q}$



w prob q_k

$\mathbb{D} = \mathbb{Q} \cup \{(s_k)_{k \geq 1}\}$

Proposition Any cont time SP with values in $\mathbb{R}$ admits a separable modif with values in $\overline{\mathbb{R}}$. $\square$

We want to be able to compute $\int_0^1 X_t(\omega)\, dt = I(\omega)$

Def $\mathbb{T}$ an interval. $(X_t)_{t\in\mathbb{T}}$ measurable

if $X : (t,\omega) \mapsto X_t(\omega)$ is a meas fct

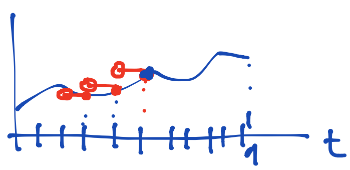
wrt $\overline{\mathcal{B}_{\mathbb{T}}} \times \mathcal{F}$. $\square$

Proposition (X_t) right cont $\Rightarrow$ measurable. $\square$

Proof Consider $\mathbb{T} = [0,1]$, $\mathbb{D}_\ell = \{t_0 = 0, t_1, \dots, t_{k_\ell} = 1\}$

$t_i < t_{i+1}$ $\qquad |t_{i+1} - t_i| \le 2^{-\ell}$.

$$X_t^{(\ell)}(\omega) = X_0(\omega)\,\mathbb{I}_{\{0\}}(t) + \sum_{j=1}^{k_\ell} X_{t_j}(\omega)\,\mathbb{I}_{(t_{j-1},t_j]}(t)$$



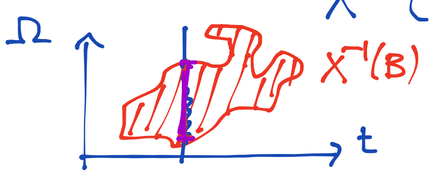
[$\forall t \in [0,1]$ we know $X_t^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}$

also $\forall t_1 \dots t_k$

$(X_{t_1} \dots X_{t_k})^{-1}(B) \in \mathcal{F}$ $\forall B \in \mathcal{B}_{\mathbb{R}^k}$

We want to look at $X_t(\omega) = X(t,\omega)$

$X^{-1}(B) \in \overline{\mathcal{B}} \times \mathcal{F}$ $\forall B$



$B = [a,b)$]

$B \in \mathcal{B}$

$$(X^{(\ell)})^{-1}(B) = \{(t,\omega) \in [0,1]\times\Omega : X_t^{(\ell)}(\omega) \in B\}$$

$$= \{0\} \times X_0^{-1}(B) \quad \bigcup_{j=1}^{k_\ell} \big[\underbrace{(t_{j-1}, t_j]}_{\in \mathcal{B}} \times \underbrace{X_{t_j}^{-1}(B)}_{\in \mathcal{F}} \big] \in \mathcal{B} \times \mathcal{F}$$

$$X^{(\ell)} \in m(\mathcal{B} \times \mathcal{F})$$

$\mathbb{D}_\ell \subseteq \mathbb{D}$

$|\mathbb{D}_\ell| = k_\ell$

$|t_\ell - t_{\ell-1}| \leq 2^{\ell}$

$$X^{(\ell)}(t,\omega) = X(\pi_\ell(t), \omega)$$

$$\pi_\ell(t) = \min\{\tilde{t} \in \mathbb{D}_\ell, \ \tilde{t} \geq t\}.$$

$(X(\pi_\ell(t), \omega) \to X(t,\omega))$

$$|\pi_\ell(t) - t| \leq 2^{\ell} \qquad \pi_\ell(t) \downarrow t \quad \text{as } \ell \downarrow \infty$$

by RC $\quad \underline{X^{(\ell)}(t,\omega) \to X(t,\omega)}$ as $\ell \to \infty$ $\forall t, \omega$

$$\Rightarrow \quad X \in m(\mathcal{B} \times \mathcal{F}).$$

$\square$

This implies that $\forall \omega$ $\quad t \mapsto X_t(\omega)$ is in $m\mathcal{B}$.
Hence if $h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is Borel
then

$$\int_a^b h(t, X_t(\omega))\, dt$$ is a well defined r.v.

and

$$\mathbb{E}\Big[\int_a^b h(t, X_t(\omega))\, dt\Big] = \int_a^b \mathbb{E}[h(t, X_t)]\, dt$$

By Fubini

Same if $\displaystyle\int_a^b \mathbb{E}[|h(t,X_t)|]\, dt < \infty$.

$\square$

<u>Def</u> $(X_t)_{t \in [0,\infty)}$ has indep increments

$\forall t \geq 0, \ h \geq 0 \quad (X_{t+h} - X_t)$ is indep of $\mathcal{F}_t^X$

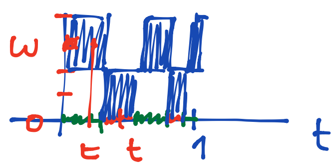
$(\mathcal{F}_t^X = \sigma(\{X_s : s \le t\}).\ )$ $\square$

Poisson process $N_t = \{ i : t_i \le t \}$

<u>Rmk</u> Indep increments iff $\forall n,\ 0 \le t_1 < t_2 < \dots \le t_n < \infty$

$$(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}) \in \mathbb{R}^n$$

are mutually indep.



Not measurable

separable ?

$[0,1] = A \cup B$

Countable dense sets $C_A \subseteq A$ $C_B \subseteq B$.

$\mathbb{D} = C_A \cup C_B$

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}})$$

$$X : (\Omega, \mathcal{F}) \to (C(\mathbb{T}), \mathcal{B}_{C(\mathbb{T})})$$

$$\{\omega : X(\omega) \notin C(\mathbb{T})\} \in \mathcal{F}$$