Lecture 9 Strong Markov Property

Def $(X_t, \mathcal{F}_t)_{t\geq 0}$ homogeneous Markov with state space $(\mathcal{X}, \mathcal{G})$ has strong M.P. if

$\forall\ h: [0,\infty) \times \mathcal{X}^{[0,\infty)} \to \mathbb{R}$

$(s, x) \longmapsto h(s,x)$

measurable on $\mathcal{B}_{[0,\infty)} \times \mathcal{G}^{[0,\infty)}$ bounded

$\forall\ \mathcal{F}_t$ Markov time τ

$$(*)\quad \mathbb{E}\{h(\tau, \theta_\tau \circ X) \mid \mathcal{F}_{\tau+}\} I_{\tau<\infty} = g(\tau, X_\tau) I_{\tau<\infty}$$

$$g(s,x) := \mathbb{E}_x h(s, X)$$

$\mathbb{P}_x$ law of MP (X_t) started at $X_0 = x$ $\square$

Recall : $x \mapsto \mathbb{E}_x f(X)$ measurable on $\mathcal{G}$ $\forall\ f \in b\mathcal{G}^{[0,\infty)}$.

(*) Analogously $(s,x) \mapsto g(s,x) = \mathbb{E}_x h(s, X)$ measurable on $\mathcal{B}_{[0,\infty)} \times \mathcal{G}$, $\forall\ h \in b(\mathcal{B}_{[0,\infty)} \times \mathcal{G}^{[0,\infty)})$

(**) X_t progr meas on $\mathcal{F}_t$, τ a stopping time, $\tau < \infty$ a.s. $\Rightarrow$ $X_\tau \in m\mathcal{F}_\tau$

X_∞ exists

τ $\mathcal{F}_t$ Markov ($\mathcal{F}_{t+}$ stopping)

τ finite a.s $\Rightarrow$ $X_\tau \in m\mathcal{F}_{\tau+}$

In general $X_\tau I_{\tau<\infty} \in m\mathcal{F}_{\tau+}$

$g(\tau, X_\tau) I_{\tau<\infty} \in m\mathcal{F}_{\tau+}$

$Z_t = g(t, X_t)$

$$(*) \quad \underbrace{\mathbb{E}[h(\tau, \theta_\tau \circ X) | \mathcal{F}_{\tau+}]}_{m\mathcal{F}_{\tau+}} \underbrace{I_{\tau<\infty}}_{m\mathcal{F}_{\tau+}} = \underbrace{g_h(\tau, X_\tau) I_{\tau<\infty}}_{m\mathcal{F}_{\tau+}.}$$

<u>Proposition</u> $(X_t, \mathcal{F}_t)_{t\geq 0}$ strong Markov, τ <u>stopping</u>

Then $\forall\, h \in b(\mathcal{B}_{[0,\infty)} \times \mathcal{G}^{[0,\infty)})$:

① $\mathbb{E}[h(\tau, \theta_\tau \circ X) | \mathcal{F}_\tau] I_{\tau<\infty} = g_h(\tau, X_\tau) I_{\tau<\infty}$.

② $(X_t, \mathcal{F}_{t+})_{t\geq 0}$ is Markov

③ $\forall\, h \in b\mathcal{G}^{[0,\infty)}$ $s \in [0,\infty)$ determ.

$$\mathbb{E}[h(X) | \mathcal{F}_s] = \mathbb{E}[h(X) | \mathcal{F}_{s+}]$$

<u>Proof</u> ① $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau+}$. take condition. exp of

$(*)$ given $\mathcal{F}_\tau$ τ, $I_{\tau<\infty}$, $g(\tau, X_\tau) \in m\mathcal{F}_\tau$

$$\mathbb{E}\left[\mathbb{E}(h(\tau,\theta_\tau\circ X)\mid\mathcal{F}_{\tau+})I_{\tau<\infty}\mid\mathcal{F}_\tau\right]=$$

$$=\mathbb{E}\left[g(\tau,X_\tau)I_{\tau<\infty}\mid\mathcal{F}_\tau\right]$$

$$\mathbb{E}\left[\mathbb{E}[h(\tau,\theta_\tau\circ X)\mid\mathcal{F}_{\tau+}]\mid\mathcal{F}_\tau\right]I_{\tau<\infty}$$

$$=g(\tau,X_\tau)I_{\tau<\infty} \quad \square$$

(2) X_t is $\mathcal{F}_{t+}$ MP. Take $\tau=t_0$ determ

$$h(s,X)=I_{\{X(t)\in B\}}$$

$$\mathbb{E}\left[I_{X(t_0+t)\in B}\mid\mathcal{F}_{t_0}\right]=\mathbb{E}_{X_{t_0}}I_{X(t)\in B}$$

$\mathcal{F}_{t_0+}$

$$p_t(x,B):=\mathbb{P}_x(X(t)\in B)$$

$$\mathbb{P}(X_{t_0+t}\in B\mid\mathcal{F}_{t_0})=p_t(x,B)$$

$\Rightarrow$ p_t is the Markov semigroup.

$$\mathbb{P}(X_{t_0+t}\in B\mid\mathcal{F}_{t_0+})=p_t(x,B)$$

$\Rightarrow$ $\mathcal{F}_{t+}$ Markov.

(3) $\mathbb{E}[h(X) \mid \mathcal{F}_{s+}] = \mathbb{E}[h(X) \mid \mathcal{F}_s]$.

by monotone class argument. consider

$$h(X) = \prod_{i=1}^{n} I_{X_{t_i} \in B_i} \qquad B_i \in \mathcal{G}$$

$$t_1 < \dots < t_k \leq s < t_{k+1} < t_n$$

$$\mathbb{E}[h(X) \mid \mathcal{F}_s] = \mathbb{E}\Big[\prod_{i=k+1}^{n} I_{X_{t_i} \in B_i} \mid \mathcal{F}_s\Big] \prod_{i=1}^{k} I_{X_{t_i} \in B_i}$$

$$\mathbb{E}[h(X) \mid \mathcal{F}_{s+}] = \mathbb{E}[\;\text{''}\; \mid \mathcal{F}_{s+}] \qquad \text{''}$$

sufficient consider $\quad h(X) = \prod_{i=1}^{m} I_{X_{t_i} \in B_i}$

$$s < t_1 < t_2 < \dots < t_m \qquad t_i = s + u_i \quad u_i > 0$$

$$h(X) = h_0(\theta_s \circ X) \qquad h_0(X) = \prod_{i=1}^{m} I_{X_{u_i} \in B_i}$$

$$\mathbb{E}[h(X) \mid \mathcal{F}_{s+}] = \mathbb{E}[h_0(\theta_s \circ X) \mid \mathcal{F}_{s+}]$$

$$= g_h(X_s) \qquad g_h(x) = \mathbb{E}_x h_0(X)$$

$$\mathbb{E}[h(X) \mid \mathcal{F}_s] = g_h(X_s)$$

$$\Rightarrow \mathbb{E}[h(X) \mid \mathcal{F}_s] = \mathbb{E}[h(X) \mid \mathcal{F}_{s+}]$$

$\square$

To check $(X_t, \mathcal{F}_t)$ strong Markov sufficient to consider

- τ bdd Markov times.
- $h(s, X) = I_{X_u \in B}$ $\forall u \geq 0$, $B \in \mathcal{G}$.

$$\mathbb{E}[h(\tau, \theta_\tau \circ X) | \mathcal{F}_{\tau+}] = \mathbb{P}(X_{\tau+u} \in B | \mathcal{F}_{\tau+})$$

$$g(s,x) = \mathbb{E}_x I_{X_u \in B} = p_u(x, B)$$

<u>Theorem</u> $(X_t, \mathcal{F}_t)_{\geq 0}$ homogeneous Markov progr meas on $(\mathcal{X}, \mathcal{G})$, with semigroup $(p_t)_{t \geq 0}$.

If $\forall B \in \mathcal{G}, u \in \mathbb{R}_{\geq 0}, \tau$ bdd Markov

$$\mathbb{P}(X_{\tau+u} \in B | \mathcal{F}_{\tau+}) = p_u(X_\tau, B)$$

then $(X_t, \mathcal{F}_t)_{t \geq 0}$ is strong Markov. ∎

<u>Proof sketch</u>: $\boxed{\text{Step 1}}$: bdd Markov τ $\Rightarrow$ gen. Markov τ

Start τ general. $\tau_n = \tau \wedge n$.

$$\mathbb{P}(X_{\tau_n+u} \in B | \mathcal{F}_{\tau_n+}) I_{\tau \leq n} = p_u(X_{\tau_n}, B) I_{\tau \leq n}$$

$$I_{\tau \leq n} \in m\mathcal{F}_{\tau_n+}$$

$$\mathbb{E}\{I_{X_{\tau_n+u} \in B} \cdot I_{\tau \leq n} | \mathcal{F}_{\tau_n+}\} = p_u(X_{\tau_n}, B) I_{\tau \leq n}$$

$$Z := \left[I_{X_{\tau+u} \in B} - p_u(X_\tau, B) \right] I_{\tau<\infty}$$

$$\mathbb{E}\left\{ \left[I_{\tau+u \in B} - p_u(X_\tau, B) \right] \underset{I_{\tau<\infty}}{\uparrow} I_{\tau \le n} \,\middle|\, \mathcal{F}_{\tau_n +} \right\} = 0$$

$$\mathbb{E}(Z \mid \mathcal{F}_{\tau_n +}) \, I_{\tau \le n} = 0$$

(Rmk: $\mathbb{E}[Z \mid \mathcal{G}_{\tau \wedge n}] I_{\tau \le n} = \mathbb{E}[Z \mid \mathcal{G}_\tau] I_{\tau \le n}$)

Apply to $\mathcal{G}_t = \mathcal{F}_{t+}$, $\tau_n = \tau \wedge n$

$$\mathbb{E}[Z \mid \mathcal{F}_{\tau +}] I_{\tau \le n} = 0$$

$n \uparrow \infty$ $\quad \mathbb{E}[Z \mid \mathcal{F}_{\tau+}] I_{\tau<\infty} = 0 \quad$ a.s.

(*) $\Rightarrow$ $\mathbb{P}(X_{\tau+u} \in B \mid \mathcal{F}_{\tau+}) I_{\tau<\infty} = p_u(X_\tau, B) I_{\tau<\infty}$.

$\boxed{\text{Step 2}}$ $\quad h(s, x) = I_{x(u) \in B} \Rightarrow h \in b(\mathcal{B}_{[0,\infty)} \times \mathcal{G}^{[0,\infty)})$

(*) $\mathbb{E}[I_{X_{\tau+u} \in B} I_{\tau<\infty} \mid \mathcal{F}_{\tau+}] = (p_u I_B)(X_\tau) I_{\tau<\infty}$

$$(p_u f)(x) = \int_{\mathcal{X}} f(y) \, p(x, dy)$$

(*) $\quad \varphi \in SF \qquad \varphi = \sum_{i=1}^{k} a_i I_{B_i}$

$$\mathbb{E}\{ \varphi(X_{\tau+u}) I_{\tau<\infty} \mid \mathcal{F}_{\tau+} \} = (p_u \varphi)(X_\tau) I_{\tau<\infty}$$

$\Rightarrow \quad \forall f \in b\mathcal{G} \qquad$ by bdd conv.

$$\mathbb{E}[I_{\tau<\infty} f(X_{\tau+u}) \mid \mathcal{F}_{\tau+}] = I_{\tau<\infty}(P_u f)(X_\tau) \qquad (*)$$

Next consider

$$h(s,x) = f_0(s) \prod_{\ell=1}^{n} f_\ell(x(t_\ell))$$

$f_0 \in b\mathcal{B}_{\mathbb{R}}. \qquad f_\ell \in b\mathcal{G}.$

$t_1 < t_2 < \dots < t_n$

Follows by $(*)$ and recursion formula

Arbitrary h by Monotone class. $\blacksquare$