Qualifying Exam 2000

Statistical Mechanics

In a highly simplified model, liquid $^3$He may be treated as a Fermi gas with binding energy $\varepsilon$/atom.

(a) assuming the liquid has interparticle spacing $r$ of $O(3\text{Å})$ show that the liquid may be described as a highly degenerate Fermi gas for $T \leq 0.1^\circ\text{K}$ and show that the ground state kinetic energy per atom may be expressed as $K = \frac{3}{2}\varepsilon_F$, where $\varepsilon_F$ is the Fermi energy.

(b) By representing $\varepsilon$ approximately in terms of the energy of a simple cubic lattice of $^3$He atoms with pairwise interaction energy $U(r)$, where $U$ may be expanded about its minimum as

$$U(r) \approx -\varepsilon_0 + \frac{1}{2}\varepsilon_1 (r - r_0)^2,$$  \hspace{1cm} (1)

use the fact that $P \to -\frac{\partial U}{\partial V}$ as $T \to 0$ to compute the equilibrium density at $T = 0$ and $P = 0$ as a function of $\varepsilon_1$.

(c) Now consider the $^3$He vapor. Assume it may be treated using Maxwell Boltzmann statistics for $T \leq 0.1^\circ\text{K}$, determine the vapor pressure of the liquid, in zero applied magnetic field, as a function of the temperature $T$, in the vicinity of $T \to 0$. (Hint: the the pressure of a classical gas is given in terms of the chemical potential, $\mu = -\frac{\partial F}{\partial N}$, by $P \propto e^{\beta \mu}$ where $\beta = 1/K_B T$.)

(d) in an applied magnetic field, $H$, the spin up particles of wavevector $k$ have energy $\varepsilon_k - g\mu_B H/2$ and the spin down particles have energy $\varepsilon_k + g\mu_B H/2$, where $\varepsilon_k = \hbar^2 k^2 / 2m$. Estimate the dependence on $H$ of the equilibrium density at $T = 0$ and $P = 0$. 

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STATISTICAL MECHANICS – solutions

(a)

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2M_{He}} \approx 10^{-27} \times 4\pi^2/(9 \times 10^{-16} \times 8000)$$

$$\approx 5.5 \times 10^{-15} \text{erg} \approx 4^o K$$

(1)

$$\langle E \rangle = \frac{\int_0^{k_F} k^2 dk k^2 k^2 / 2M}{\int k^2 dk} = (3/5)\varepsilon_F$$

(2)

(b)

$$U/N = -3(\varepsilon_0 + \frac{1}{2}\varepsilon_1 (r - r_0)^2) + \frac{3}{5}\varepsilon_F$$

(3)

and $k_F$ can be written in terms of the number density, $n$, via

$$N/vol = n = \frac{4\pi}{(2\pi)^3} \int_0^{k_F} dk k^2 = \frac{k_F^3}{6\pi^2}$$

(4)

hence

$$\frac{\partial E}{\partial r} = 6\varepsilon_1 (r - r_0) - 2\frac{\alpha}{r^3} = 0$$

(5)

where $\alpha = (\hbar^2 / 2M_{He})(6\pi^2)^{2/3}$. Expanding $r/r_0 \approx 1 + \eta$ one finds

$$6\varepsilon_1 \eta \approx \frac{2\alpha}{r_0^3}$$

(6)

hence

$$\eta \approx \left( \frac{\alpha / r_0^3}{3\varepsilon_1 r_0} \right)$$

(7)

and $n_{eq} \approx r_0^3(1 + 3\eta)$. 

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(c) chemical potential is given by

\[ \mu = \varepsilon_{\text{min}} + \varepsilon_F \]  

where \( \varepsilon_{\text{min}} \) is the value of \( U/N \) at its minimum \((< 0) \). Hence

\[ P \propto \exp(-(|\varepsilon_{\text{min}}| - \varepsilon_F)/k_BT) \]  

(d)

\begin{align*}
    k^2_{F_1} &= k^2_F + \frac{2M}{\hbar^2} g\mu_B H/2 \\
    k^2_{F_1} &= k^2_F - \frac{2M}{\hbar^2} g\mu_B H/2
\end{align*}

hence from Eqn. (5)

\[ \delta \rho \propto (1 + \eta)\nu^{-3} + (1 + \eta)\nu^{-3} - 2 \]

\[ \approx -(3\alpha k^4_F + 3\alpha k^4_{F_1} - 6\alpha k^4_F) = \mathcal{O}(-H^2) \]
General Physics I

Note that for all parts of this question careful physical or order of magnitude arguments, not involving detailed calculations or exact numerical factors, will get full credit. Dimensional analysis errors are unacceptable.

1) What is the gravitational field \( \vec{g} \) of a point mass in \( 3 + n \) spatial dimensions? (Hint: You can use Gauss' Law)

2) What are the units of the analog of Newton's gravitational constant, \( G_{N(3+n)} \), in \( 3 + n \) dimensions? (Hint: You may use any system of units (CGS, MKS, etc.) you wish, including the particle physicist's units where \( h = c = 1 \).

3) Imagine now that gravity becomes 4 (space) dimensional at distances less than \( L \), while all ordinary particles continue to live in 3 spatial dimensions. Compute the gravitational self-energy of a 3-dimensional sphere of radius \( r < L \) and mass \( M \).

4) Continuing from part 3, consider now a star of radius \( R \gg L \) and write an expression for the fractional shift in its gravitational self-energy as a result of having 4-dimensional gravity at distances less than \( L \).
General Physics I: Solutions

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1) \[ \bar{g} = \frac{G_{N(3+n)}m}{r^{2+n}} \]

2) Using \( h = c = 1 \)
\[ [G_{N(3+n)}] \sim \frac{1}{m^{2+n}} \]

3) By physical argument the energy \( E \) must be proportional to \( \sim G_{N4}M^2 \) therefore by dimensional analysis
\[ E \sim G_{N4} \frac{M^2}{r^2} \]

By matching the 3-D and 4-D behavior at \( r = L \) we see that
\[ G_{N4} \sim G_N L \]

so that
\[ E \sim G_N \frac{LM^2}{r^2} \]

4) The normal (3-D) Newtonian self-energy is
\[ E_{\text{normal}} \sim G_N \frac{M^2}{R} \]

To compute the shift, separate the star into \( N \sim \left( \frac{R}{L} \right)^3 \) regions, of size \( L \) and mass \( M_L \sim \rho L^3 \). Each region has self-energy from part 3
\[ E \sim G_{N4} \frac{M_L^2}{L^2} \]

thus the total shift will be
\[ \Delta E_{\text{shift}} \sim G_{N4} M^2 L^2 R^3 = G_N \frac{M^2 L^2}{R^3} \]

where we used \( \rho = M/R^3 \) in the second step. The fractional shift is
\[ \frac{\Delta E_{\text{shift}}}{E_{\text{normal}}} \sim \left( \frac{L}{R} \right)^2 \]
Qualifying Exam Problem
Quantum Mechanics

1. Obtain normalized eigenvectors and corresponding eigenvalues for the Hamiltonian

\[ \mathcal{H} = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + k_3 xy \quad , \tag{1} \]

for which the spring constants satisfy \( k_1 k_2 > k_3^2 \).

2. Same question, but for the case \( k_1 = k_2 = k_3 \).

Solution

The trick to this problem is to rotate the coordinate system so as to decouple the oscillators. The potential energy may be written in terms of a dynamical matrix, per

\[ \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + k_3 xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} k_1 & k_3 \\ k_3 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad . \tag{2} \]

The eigenvalues of this matrix are determined by

\[ \text{Det} \begin{bmatrix} k_1 - k & k_3 \\ k_3 & k_2 - k \end{bmatrix} = (k_1 - k)(k_2 - k) - k_3^2 = 0 \quad , \tag{3} \]

which gives

\[ k = \frac{k_1 + k_2}{2} \pm \sqrt{(\frac{k_1 - k_2}{2})^2 + k_3^2} \ . \tag{4} \]

The corresponding eigenvectors are then either column of the cofactor matrix.
\[
\text{cof} \begin{bmatrix} k_1 - k & k_3 \\ k_3 & k_2 - k \end{bmatrix} = (k_1 - k)(k_2 - k) - k_3^2 = \begin{bmatrix} k_2 - k & -k_3 \\ -k_3 & k_1 - k \end{bmatrix}.
\] (5)

Thus denoting the two eigenvalues by \(k_a\) and \(k_b\) and letting

\[
\cot(\theta) = \frac{k_1 - k_2}{2k_3} + \sqrt{(\frac{k_1 - k_2}{2k_3})^2 + 1},
\] (6)

in terms of which we have

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},
\] (7)

and

\[
\mathcal{H} = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} \right) + \frac{1}{2} k_a a^2 + \frac{1}{2} k_b b^2.
\] (8)

The normalized eigenstates are then

\[
\psi_{mn}(a, b) = \frac{1}{\ell_a \ell_b} \phi_m(x/\ell_a) \phi_n(y/\ell_b),
\] (9)

where

\[
\phi_n(z) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{z^2/2} (\frac{\partial}{\partial z})^n e^{-z^2}
\] (10)

\[
\omega_a = \sqrt{\frac{k_a}{M}}, \quad \omega_b = \sqrt{\frac{k_b}{M}}
\] (11)

\[
\ell_a = \sqrt{\frac{\hbar}{M \omega_a}}, \quad \ell_b = \sqrt{\frac{\hbar}{M \omega_b}}
\] (12)

and the corresponding eigenvalues are

\[
E_{mn} = (m + \frac{1}{2}) \hbar \omega_a + (n + \frac{1}{2}) \hbar \omega_b.
\] (13)

This formal solution does not work for the special case, however, because the \(b\) eigenvalue of the dynamical matrix is zero. We then have the freely-propagating bound states

\[
\psi_{mk}(a, b) = \frac{1}{\sqrt{\ell_a L^2}} \phi_m(a/\ell_a) e^{ikb},
\] (14)

where \(L\) is the length of the “universe”, and corresponding eigenvalues

\[
E_{mk} = (m + \frac{1}{2}) \hbar \omega_a + \frac{\hbar^2 k^2}{2M}.
\] (15)
Electricity and Magnetism

1a) A charged particle with mass $m$ and charge $q$ oscillates in a harmonic potential with spring constant $k$. The initial amplitude of the oscillation is $x$. Estimate how long it takes for $\frac{1}{2}$ of the energy to be radiated away in electromagnetic radiation.

1b) A distribution of charge is described by the time-dependent charge density

$$\rho(r) = \frac{q_0}{a} \sin(\omega t) \frac{1}{r^2} e^{-\frac{r}{a}}$$

where $q_0, a$ and $\omega$ are constants and $r$ is the distance from the origin. Find the spatial current density $\vec{j}$. Estimate the rate of energy loss by radiation.
Electricity and Magnetism

Solution

1a)

\[ E_{\text{init}} = kx^2 \]

Dipole radiation formula for power emitted:

\[ P \approx \frac{d_0^2 \omega^4}{c^3} \quad \text{(Exact:} \frac{d_0^2 \omega^4}{3c^3} \text{)} \]

where

\[ d_0 = \text{Maximum Dipole} = qx \]

\[ P \approx \frac{q^2 x^2 \omega^4}{c^3} \]

\[ Pt \sim kx^2 \]

\[ \left( \frac{q^2 x^2 \omega^4}{c^3} \right)^{-1} kx^2 \sim t \]

so we have

\[ t \sim \frac{c^3 k}{q^2 \omega^4} \]

1b) The charge in a ball of radius \( r \) is

\[ Q = 4\pi \int_0^r x^2 \rho(x)dx \]

\[ Q = 4\pi \frac{q_0}{a} \sin(\omega t) \int_0^r \left(1 - \frac{x}{a}\right) e^{-\frac{x}{a}}dx \]

\[ = 4\pi q_0 \sin(\omega t) \int_0^\frac{r}{a} (1 - y)e^{-y}dy \]

\[ = 4\pi q_0 \sin(\omega t) \frac{r}{a} e^{-\frac{r}{a}} \]

From continuity we have:

\[ \frac{dQ}{dt} = -4\pi r^2 J_r \]

where \( J_r \) is the current. Thus:

\[ J_r = \frac{-q_0 \omega \cos \omega t}{ra} e^{-\frac{r}{a}} \]

Energy loss=0 because the charge is spherically symmetric.
Special Relativity

Suppose that a relativistic particle with charge $q$, initial energy $E_0$ and momentum $\mathbf{p} = p_0 \hat{x}$ enters a region with a uniform electric field $F_E$ in the $\hat{y}$ direction. The particle accelerates in the $\hat{y}$ direction.

a) One consequence is that the particle's $\hat{x}$ velocity decreases. Compute its $\hat{y}$ velocity, if $v_x$ has decreased by a factor of two.

b) How long does the particle have to spend in the electric field to obtain the velocity you computed in part a)? Give your answer in terms of the parameters above.

c) How far has the particle traveled in the $\hat{x}$ direction when it obtains this velocity? [Hint: $\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}(x/a)$, $\sinh x = (e^x - e^{-x})/2$.]
Solution – Special Relativity

a) The electrostatic force is in the \( \hat{y} \) direction \( q F_E \hat{y} \), so we have \( p_x = p_0 \) and \( p_y = q F_E t \). Thus the energy of the particle is

\[
E^2 = m^2 c^4 + p_0^2 c^2 + (q F_E t)^2 c^2 = E_0^2 + (q F_E t)^2 c^2
\]

Now \( p_x = m c \beta \gamma = E v_x / c^2 \), which is constant, so when \( v_x = 1/2 v_{x,i} \) we have \( E = 2E_0 \). So we need \( (q F_E t) c = 3^{1/2} E_0 \). Finally, we get

\[
v_y = p_y(t) c^2 / E = q F_E t c^2 / (2E_0) = 3^{1/2} E_0 c / (2E_0) = (3^{1/2} / 2) c
\]

b) We can find the time required to achieve this acceleration from \( t = p_y / (q F_E) \). Substituting in for \( p_y = E v_y / c^2 \) we get

\[
t_f = (E / c^2 3^{1/2} / 2c) / (q F_E) = 3^{1/2} E_0 / (c q F_E)
\]

c) We now need to figure out how far the particle travels in the \( \hat{x} \) direction in this time. Well, we know that \( dx / dt = v_x = p_0 c^2 / (E_0^2 + (q F_E c t)^2) ^{1/2} \) so we can integrate over the time \( t_f \) above:

\[
L = \int_0^L dx = \left( \frac{p_0 c^2}{q F_E c} \right) \int_0^{t_f} [(E_0 / q F_E c)^2 + t^2]^{-1/2} dt
\]

Using the provided integral, this gives

\[
L = \left( \frac{p_0 c}{q F_E} \right) \sinh^{-1} (q F_E c t_f / E_0)
\]

Which we can simplify, using the value of \( t_f \) above to

\[
L = \left( \frac{p_0 c}{q F_E} \right) \sinh^{-1} (3^{1/2})
\]
General Physics II

Note: Substantial partial credit will be given for answers based on proper dimensional analysis, even if the results are off by factors of order unity.

An electric charge $Q$ is spread over the surface of a hollow, extremely thin, spherical conductor. The interior of the conductor is evacuated. The conductor has mass $M$ and is made of a deformable material, so that its radius can freely vary within the range of interest for this problem. The sphere is placed in a gaseous medium of pressure $p$ and density $\rho$.

(a) The sphere is allowed to slowly adjust its radius until it comes into pressure equilibrium with the gas. What is its final radius $R_0$? Express your result in terms of one or more of the following quantities: $M$, $Q$, $p$, and $\rho$. Assume that only gas pressure and electrostatic forces are significant. In particular, neglect the elastic self-force of the conducting material.

(b) Suppose that the sphere is isotropically expanded or contracted a small amount from the equilibrium radius $R_0$ found in part (a). Derive an expression for the angular frequency, $\omega$, of small oscillations about equilibrium. Assume that the amplitude $\Delta R$ of the oscillations is small, $\Delta R \ll R_0$. Again, express your result in terms of one or more of the quantities $M$, $Q$, $p$, and $\rho$.

(c) The small oscillations in part (b) will produce acoustic waves that propagate outward in the gas. These waves will result in the damping of the oscillations. Derive an expression for the “damping time” $t_d$, the characteristic time for the oscillation energy to diminish by a factor of $\sim 2$. Assume that the waves carry off only a very small fraction of the total energy in a single oscillation period. Again, express your results in terms of one or more of $M$, $Q$, $p$, and $\rho$. For this part of the problem, full credit will be given for answers correct to within a factor of a few (provided the right physics is used to arrive at them).
General Physics II: Solutions

(a) The electrostatic pressure of the sphere is

\[ p_{el} = -\frac{dU_{el}}{dV} = -\frac{1}{4\pi r^2} \frac{dU_{el}}{dr}, \]

where \( U_{el}(r) \) is its electrostatic self-energy. \( U_{el} \) is most simply calculated in terms of the electric field it produces:

\[ U_{el}(R) = \int_{R}^{\infty} 4\pi r^2 \frac{E^2}{8\pi} dr \]

where \( R \) is the radius of the sphere and \( E \) is the electric field it produces. The field outside is, by Gauss’ law, \( E(r) = Q/r^2 \). Thus

\[ U_{el}(R) = \int_{R}^{\infty} 4\pi r^2 \frac{Q^2}{8\pi r^4} dr = \frac{Q^2}{2R}. \]

The electrostatic pressure is thus

\[ p_{el}(R) = -\frac{1}{4\pi R^2} \frac{d}{dR} \left( \frac{Q^2}{2R} \right) = \frac{Q^2}{8\pi R^4}. \]

The equilibrium radius \( R_0 \) is achieved when the electrostatic pressure equals the gas pressure, \( p_{el}(R_0) = p \). Using the above equation and solving for \( R_0 \) then yields

\[ R_0 = \left( \frac{Q^2}{8\pi p} \right)^{\frac{1}{4}}. \]

An alternative approach to this problem would have been to note that the electrostatic force per unit area on the sphere is \( f_{el} = \sigma E/2 \), where \( \sigma \) is the surface charge density. By symmetry the charge is uniformly distributed, so \( \sigma = Q/4\pi R^2 \), so \( f_{el} = Q^2/8\pi R^4 \). Equating \( f_{el} \) with \( p \) yields the same result for \( R_0 \) as found above.

(b) There are two terms in the effective potential energy for the sphere: one due to gas pressure and the other, already found above, for electrostatic repulsion. The pressure term is

\[ U_{pr}(R) = \int_{R}^{R} pA(r)dr = \frac{4}{3}\pi R^3 p \]

Hence the total potential energy is

\[ U(R) = U_{pr}(R) + U_{el}(R) = \frac{4}{3}\pi R^3 p + \frac{Q^2}{2R} \]

The first derivative is \( U'(R) = 4\pi R^2 p - Q^2/2R^2 \), which vanishes, as expected, at \( R = R_0 \). The second derivative is

\[ U''(R) = 8\pi p R + \frac{Q^2}{R^3} \]
evaluating at \( R = R_0 \) yields

\[
U''(R_0) = \frac{2Q^2}{R^3}
\]

Thus, the total energy for small oscillations about equilibrium may be written

\[
E_{osc} = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} \left( \frac{2Q^2}{R_0^3} \right) r^2
\]

where \( r = R - R_0 \) is the displacement of the sphere’s surface. The above equation describes a simple harmonic oscillator of angular frequency

\[
\omega = \sqrt{\frac{2Q^2}{MR_0^3}}
\]

Inserting the result from part (a) for \( R_0 \), we find

\[
\omega = Q^{1/2} M^{-1/2} 2^{1/2} (8\pi)^{3/8} p^{3/8}
\]

(c) Let the oscillation amplitude be \( \Delta r \). This will also be the amplitude of the acoustic wave motion. The maximum velocity of the wave motion (and that of the oscillation) is thus \( \omega \Delta r \). The energy in the oscillation is thus \( E_{osc} = M \omega^2 \Delta r^2 / 2 \), while the energy density in the acoustic waves is \( \epsilon_{ac} = \rho \omega^2 \Delta r^2 / 2 \). The energy flux associated with the waves – energy per unit area per unit time – is \( f = c_s \epsilon_{ac} \), where \( c_s \approx \sqrt{p/\rho} \) is the speed of sound. The rate at which the waves carry away energy is thus

\[
\frac{dE}{dt} = 4\pi R_0^2 f = 2\pi R_0^2 \sqrt{\rho} \omega^2 \Delta r^2
\]

The decay time for the oscillation is therefore

\[
t_d = \frac{E_{osc}}{dE/dt} = \frac{M}{4\pi R_0^2 \sqrt{\rho}} \frac{1}{\omega^2 \Delta r^2}
\]

Finally, plugging in our original expression for \( R_0 \) as a function of \( Q \) and \( p \), we obtain

\[
t_d = \frac{1}{\sqrt{2\pi \rho}} \frac{M}{Q}
\]
Qualifying Exam 1999, Quantum Mechanics

1) Consider the problem of a single quantum mechanical particle moving in one space dimension, subjected to the potential

\[ V(x) = \begin{cases} \frac{1}{2}kx^2, & \text{for } x > 0 \\ \infty, & \text{for } x < 0. \end{cases} \quad (1) \]

The Hamiltonian is given by

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x). \]

Determine the eigenvalues and the eigenstates of this problem.

2) Consider the problem of two identical spin 1/2 fermions moving in one space dimension.

The Hamiltonian is given by

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_2^2} + V(x_1 - x_2) \]

where the interaction potential between particles \( x_1 \) and \( x_2 \) is given by

\[ V(x_1 - x_2) = \frac{1}{2}k(x_1 - x_2)^2 \]

a) What is the symmetry property of the orbital wave function \( \Psi(x_1, x_2) \) when the total spin \( S \) of the system is \( S = 1 \) and \( S = 0 \) respectively.

b) For each of the two cases listed above, determine the eigenvalues and eigenstates of this problem.
Solution

1) Since the potential is infinite when $x < 0$, the wave function must vanish in this region. In particular, $\Psi(x = 0) = 0$. For $x > 0$, the problem is identical to the usual harmonic oscillator problem. However, only the odd solutions of the usual harmonic oscillator problem satisfy the condition $\Psi(x = 0) = 0$. Therefore, the eigenvalue of this problem is given by

$$E_n = (n + \frac{1}{2})\hbar\omega, \quad \omega^2 = \frac{k}{m}$$

where $n$ is an odd integer.

The eigen-wave-functions are the usual harmonic oscillator wave functions $\Psi_n(x)$ when $x > 0$ and vanishes when $x < 0$.

2a) For $S = 1$, the spin wave function is symmetric under exchange, therefore the Pauli principle implies that the orbital wave function must be antisymmetric, i.e.

$$\Psi_a(x_1, x_2) = -\Psi_a(x_2, x_1).$$

For $S = 0$, the spin wave function is antisymmetric, and the orbital wave function must be symmetric, i.e.

$$\Psi_s(x_1, x_2) = \Psi_s(x_2, x_1).$$

2b) In reduced coordinates, the center-of-mass degree of freedom is free, and propagates as simple plane waves. The relative degree of freedom is a simple harmonic oscillator problem subjected to the above symmetry constraint. Therefore, we obtain the following eigenvalues for the relative degree of freedom:

$$E_n = (n + \frac{1}{2})\hbar\omega \quad \{ \begin{array}{l} n = \text{even}, \quad \text{for } S = 0 \\ n = \text{odd}, \quad \text{for } S = 1 \end{array} \}$$

(2)

The eigenstates are the usual harmonic oscillator wave functions with even or odd $n$, for $S = 0$ or $S = 1$ respectively.
1. Find the Lagrangian of a pendulum made with a massless stick of length $l$ connecting the masses $m_1$ and $m_2$. The pendulum is free to swing in a plane around $m_1$ and the hinge in $m_1$ can slide with no friction over a horizontal segment (see Figure). Write the Lagrange equations and find what curve describes the point at $m_2$ during (arbitrarily large) oscillations.

2. This problem can be used as first step towards the design of parts of the fuselage of an airplane. While several approximations have to be made to obtain an analytical result, you will find that your solution resembles quite closely to real cases.

A solid of revolution is exposed to a uniform stream of gas with velocity parallel to the revolution axis of the solid. Assuming that the gas density $\rho$ is low (no molecule-molecule collisions) and that molecules hitting the body with an angle $\theta$ are mirror reflected by the surface of the solid, what shape of the solid minimizes the force of drag? Show the exact form of the functional but then continue the calculations keeping only the lowest order terms in $\theta$. In such approximation $\sin \theta \approx y'$ and $\sqrt{1 + y'^2} \approx 1$. Discuss the limits of this case.
Solutions:

1. The system has two degrees of freedom: we use as coordinates the position of $m_1$ along $AB$ (called $x$) and the angle of the stick (of length $l$) relative to the vertical (we call this $\phi$). We can write the rectangular coordinates of the two masses $x_1, y_1$ and $x_2, y_2$ as function of the generalized coordinates $x, \phi$:

\[
\begin{align*}
  x_1 &= x \\
  y_1 &= 0 \\
  x_2 &= x + l \sin \phi \\
  y_2 &= l \cos \phi
\end{align*}
\]

and

\[
\begin{align*}
  \dot{x}_1 &= \dot{x} \\
  \dot{x}_2 &= \dot{x} + l \dot{\phi} \cos \phi \\
  \dot{y}_2 &= -l \dot{\phi} \sin \phi
\end{align*}
\]

The kinetic and potential energies can then be written as functions of the generalized coordinates:

\[
T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\phi}^2 \cos^2 \phi + 2l \dot{x} \dot{\phi} \cos \phi + l^2 \dot{\phi}^2 \sin^2 \phi)
\]

or

\[
T = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{m_2}{2} (l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi)
\]

and

\[
U = m_2 g y_2 = -m_2 l \dot{\phi} \cos \phi
\]

We can then easily write the Lagrangian

\[
\mathcal{L} = T - U = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{m_2}{2} (l^2 \dot{\phi}^2 + 2l \dot{x} \dot{\phi} \cos \phi) + m_2 g l \cos \phi
\]

Since $x$ is absent from the Lagrangian we have

\[
\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{and} \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt} ((m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos \phi) = 0
\]

so that

\[
P_x \equiv (m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos \phi = \text{const.}
\]
is a conserved quantity of the motion. This quantity is in fact the linear momentum along $x$, as we can promptly recognize by writing

$$P_x = m_1 \dot{x}_1 + m_2 \dot{x}_2$$

(9)

and transforming to generalized coordinates. We can then assume $P_x = 0$ without loss of generality and integrate in $t$ the Lagrange equation

$$(m_1 + m_2) \dot{x} + m_2 l \dot{\phi} \cos \phi = 0$$

(10)

obtaining

$$(m_1 + m_2) x + m_2 l \sin \phi = c$$

(11)

where $c$ is a constant. This formula, incidentally, expresses the fact the the center-of-mass of the system sits still at the horizontal position $c$. For simplicity we set $c = 0$, meaning that we center the coordinate frame so that its origin coincides with the position of the fixed center-of-mass. This means

$$(m_1 + m_2) x + m_2 l \sin \phi = 0$$

(12)

that, re-written in terms of rectangular coordinates using (1), can be used to find the curve described by $m_2$.

$$(m_1 + m_2)^2 x_2^2 + m_1^2 y_2^2 - m_1^2 l^2 = 0$$

(13)

and dividing by $m_1^2 l^2$

$$\left( \frac{m_1 + m_2}{m_1} \right)^2 x_2^2 + \frac{y_2^2}{l^2} = 1.$$  

(14)

This is the equation of an ellipse of semi-axes

$$\begin{cases} a_y = l \\ a_x = \frac{m_1 l}{m_1 + m_2} \end{cases}$$

(15)

Since $\frac{a_y}{a_x} = \frac{m_1 + m_2}{m_1} \geq 1$ the $y$ semi-axis is always longer than the $x$. This is shown in the Figure.
It is interesting to analyze some limit-cases:

- $m_1 \gg m_2$: then $a_x = a_y = l$ that is the case of the regular plane pendulum.
- $m_1 \ll m_2$: then $a_x = \frac{m_1}{m_2}l \to 0$ that is to say $m_1$ is very light and slides back and forth in $x$ while $m_2$ moves essentially on a line up and down. Note that in this case the center-of-mass is very close to $m_2$ that, as we said, does not move along $x$ at all.
2. We want to use a variational method to find the curve that minimizes the force of drag. As shown in the figure the recoil force for a mass \( m \) hitting the surface with an angle \( \theta \) is \( F_\perp = \frac{ \Delta p_\perp }{ \Delta t } \) where \( \Delta t \) is the collision time and \( \Delta p_\perp \) is the component of the momentum change locally perpendicular to the surface.

![Diagram of force](image)

If \( p \) is the momentum of a gas molecule \( \Delta p_\perp = 2p \sin \theta \) and

\[
F_\perp = \frac{2mv \sin \theta}{\Delta t}
\]

(16)

We can easily find that \( \frac{\Delta m}{\Delta t} = \rho v \sin \theta \) where \( A \sin \theta \) is the surface seen by the incoming stream of gas. Hence

\[
F_\perp = 2 \sin \theta \rho v^2 \sin \theta A = 2 \rho v^2 \sin^2 \theta A
\]

(17)

and the pressure is

\[
P = \frac{F_\perp}{A} = 2 \rho v^2 \sin^2 \theta
\]

(18)

The drag acts in the direction \( y \) parallel to the motion and on a ring of width \( ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx \) and radius \( y(x) \) will be

\[
df = P2\pi y \sqrt{1 + y'^2} \sin \theta dx
\]

(19)

that can be integrated as

\[
f = \int_0^l 4\pi \rho v^2 \sin^3 \theta y \sqrt{1 + y'^2} dx
\]

(20)

where \( l \) is the length of the solid. The problem can be solved now by finding the function \( y(x) \) that minimizes \( f \). In order to find \( y \) in analytic form we now make the approximation suggested that is \( \sin \theta \simeq \tan \theta = \frac{dy}{dx} = y' \) and \( \sqrt{1 + y'^2} \simeq 1 \). This assumes that \( \theta \) is small, like it would be around the hump of a Boeing 747 (if it would fly in the low density conditions described). Near the nose of the aircraft this approximation of course does not hold.... so this is the limit of our solution.

We get
\[ f = 4\pi \rho v^2 \int_0^l yy'dx \quad (21) \]

and the functional
\[ g(y, y') = yy'^3 \quad (22) \]

We can then calculate the Euler's equation
\[ \frac{\partial g}{\partial y} - \frac{d}{dx} \frac{\partial g}{\partial y'} = 0 \quad (23) \]

that results in
\[ y'^3 - 3y''y'y = 0 \quad (24) \]

While a trivial solution is \( y = \text{const} \), we can find other solution by multiplying both sides by \( y' \), obtaining
\[ y'^4 - 3y''y'^2y = 0 \quad (25) \]

that is
\[ (y'^3y)' = 0 \quad (26) \]

or
\[ y'^3y = c^3 \quad (27) \]

where \( c^3 \) is a constant. The
\[ y' = \frac{c}{y^{1/3}} \quad (28) \]

can then be integrated yielding
\[ y = (cx + a)^{3/4} \quad (29) \]

where \( a \) is another constant. We can now easily identify the integration constants in terms of boundary conditions for the problem:
\[ c = \frac{R^{4/3}}{l} \quad (30) \]

and
\[ a = 0 \]  \hspace{1cm} (31)

So in our approximations the minimum drag contour is given by

\[ y = R \left( \frac{x}{l} \right)^{3/4} \]  \hspace{1cm} (32)