b) If $A \gg 1$, it means $V_0$ is large compared to the zero-point energy, so WKB can be used. The classical action is

$$S_c = \int_{-L}^{L} dx \sqrt{2mV(x)} = \sqrt{2mV_0} L \int_{-1}^{1} dx \sqrt{(1-x^2)^2}$$

$$= \sqrt{2mV_0} L \left( 2 \int_{0}^{1} dx [1-x^2] \right)$$

$$= \frac{4}{3} \sqrt{2mV_0} L$$

So tunnel splitting is

$$\Delta \sim \mathcal{O} \left[ \exp \left( -\frac{S_c}{\hbar} \right) \right]$$

The prefactor is tough; it needs units of energy, hence $V_0$, but it can depend on $[S_c/\hbar]$. 
To get the power right requires doing the full WKB calculation.

The ground state energy is

the tunnel splitting comes from

making the harmonic appear about

one of the minima, say \( x = L \)

\[
U(x) = U_0 \left( \frac{2L^2}{x-L} \right)^2 + \ldots
\]

Thus, the ground state energy is that of a harmonic oscillator

\[
E = \frac{\hbar^2}{2L^2} \sqrt{8V_0 \xi} + \ldots = \sqrt{2} V_0 \sqrt{\frac{\hbar^2}{V_0 L^2 \xi}} + \ldots
\]

\[
= \sqrt{2} V_0 \xi \frac{\hbar}{2} + \ldots
\]
If \( A \ll 1 \), then the ground state is much more spread out than the double well.

\( \implies \) the localization is over length \( A \gg L \), so

\[ V(x) \approx \frac{V_0}{L^4} x^4 + \ldots \]

Make a variational guess (of any form) with width \( A \)

\[ E = \frac{\hbar^2}{2m} \frac{C_1}{A^2} + \frac{V_0}{L^4} C_2 A^4 \]

where \( C_1 \) \& \( C_2 \) depend on shape of wave function but not on \( A \).

Minimize w.r.t. \( A \)
\[- \frac{c_1 \hbar^2}{m} \frac{1}{\lambda^3} + \frac{4 c_2 V_0}{L^4} \lambda^3 = 0, \]

\[
\implies \lambda^{8/3} = \left( \frac{L^4 \hbar^2}{m V_0} \right) \left( \frac{c_1}{4 c_2} \right),
\]

\[
E = V_0 \frac{1}{m^{2/3}} \frac{\hbar}{L^{4/3}} \left[ \frac{1}{2} \left( \frac{c_1}{4 c_2} \right)^{-\frac{1}{3}} + \left( \frac{c_1}{4 c_2} \right)^{2/3} \right]
\]

\[
= V_0 \lambda^{-2/3} \left[ \cdots \right]
\]

\[\cdots 9.3, \quad \cdots 1^{st} \text{ exc. level} \]

\[\cdots \cdots\]