Solutions

QUAL – DAY 1
Thursday, January 5th

General Physics I
Quantum Mechanics I
Special Relativity
Classical Mechanics
Answers to General Physics I

a) The Equipartition Theorem states that the average value for the kinetic energy in either of the transverse directions \((x,y)\) is \(\frac{1}{2} kT\), where \(k\) is Boltzmann's constant. Thus, \(E_\perp = \frac{1}{2} kT = \frac{1}{2} m_0 V_\perp^2 = \frac{1}{2} p_\perp^2/m_0\)

\[ p_{\perp\text{rms}} = (mkT)^{1/2} \]

b) \(\theta_{\text{rms}} = p_{\perp\text{rms}}/p_z = p_{\perp\text{rms}}/\gamma\beta m_0 c = (kT/m_0 c^2)^{1/2}/\gamma\beta \quad [\gamma = 1+eV/m_0 c^2; (\gamma\beta)^2 = \gamma^2 - 1] \)

If \(T = 1200\, \text{K}, kT/m_0 c^2 = 1200\, \text{K} \times 1.4 \times 10^{-23}\, \text{J/K} /9.1 \times 10^{-31}\, \text{kg} /9 \times 10^{16}\, \text{m}^2/\text{s}^2 = 2 \times 10^{-7}\)

\[(\gamma\beta)^2 = \gamma^2 - 1 = (1 + eV/m_0 c^2)^2 - 1\]

\[ (\gamma\beta(1\, \text{kV}))^2 = (1 + 1/511)^2 - 1 \sim 2/511 = 3.91 \times 10^{-3} \]

\[ (\gamma\beta(1\, \text{MV}))^2 = (1 + 1000/511)^2 - 1 \sim 8 \]

\[ \beta\gamma(1\, \text{kV}) = 0.06 \]

\[ \beta\gamma(1\, \text{MV}) = 2.8 \]

\[ \theta(1\, \text{kV})_{\text{rms}} = 4.5 \times 10^{-4} / 0.06 = 7.5 \times 10^{-3} \text{ rad} = 7.5 \text{ mr} \]

\[ \theta(1\, \text{MV})_{\text{rms}} = 4.5 \times 10^{-4} / 2.8 = 1.6 \times 10^{-4} \text{ rad} = 0.16 \text{ mr} \]

c) The phase space area of one of the transverse planes at the exit of the gun is given as \(\pi\) a \(p_{\perp\text{rms}}\). Liouville’s Theorem assures that the phase space area (two-dimensional volume) will be conserved as the beam travels along the perfect transport system. Therefore, the smallest spot at the end will occur with the biggest transverse momentum at the end, which also means the largest angle.

The maximum angle available to the beam as it goes from the lens to the final plane is given by

\[ \theta_{\text{lens}} = (r_{\text{lens}} + r_{\text{min}})/L \sim r_{\text{lens}}/L, \] so the maximum \(p_\perp\) is \(p_{\perp\text{lens}} = \gamma\beta m_0 c r_{\text{lens}}/L\)

Since \(p_\perp\) doesn’t change in a drift space, \(p_\perp\) at the minimum spot size is also \(p_{\perp\text{lens}}\). Thus \(r_{\text{min}} p_{\perp\text{lens}} = a p_{\perp\text{rms}}\)

\[ r_{\text{min}} = a (p_{\perp\text{rms}} / p_{\perp\text{lens}}) = (a p_{\perp\text{rms}} L) / (\gamma\beta m_0 c r_{\text{lens}}) = (a / r_{\text{lens}}) \theta_{\text{rms}} L \]

\[ r_{\text{min}} = (a / r_{\text{lens}}) \theta_{\text{rms}} L \]
Quantum I Solution  January 2006

A quantum-mechanical particle of mass $m$ and energy $E$ moves in a one-dimensional attractive potential $V(x) = -\lambda \delta(x)$ localized at the origin (assume that $\lambda$ is positive throughout this problem).

(a) (3 points) What are the dimensions of the parameter $\lambda$? Sketch the potential. For a bound state with negative energy $E = -E_b$, sketch the qualitative form of the wave function. Define $\kappa = \sqrt{2mE_b}/\hbar$ and use the Schrödinger equation to find the wave function and binding energy $E_b$ of the bound state. How is $E_b$ related to $\lambda$? How is the spatial size of the bound-state wave function related to the binding energy $E_b$? Are there other bound states?

By inspection, the parameter $\lambda$ is an energy times a length, since a delta function has the dimension of its inverse argument.

The ground state is symmetric and decays exponentially away from the origin. For negative energy $E = -E_b$, rewrite the Schrödinger equation as

$$\psi'' + \frac{2m\lambda}{\hbar^2} \delta(x) \psi = \kappa^2 \psi$$

(1)

where $\kappa = \sqrt{2mE_b}/\hbar$. For $x \neq 0$, the solutions are $e^{\pm \kappa x}$, and choose $\psi(x) = e^{-\kappa|x|}$. This solution is continuous at origin. Integrate (1) from $-\epsilon$ to $\epsilon$ to get

$$\psi'(\epsilon) - \psi'(-\epsilon) + \frac{2m\lambda}{\hbar^2} \psi(0) = 0$$

(2)

For $\epsilon \to 0$, these conditions give

$$\kappa = \frac{m\lambda}{\hbar^2} = \frac{\sqrt{2mE_b}}{\hbar}$$

(3)

Note that $1/\kappa$ characterizes the spatial size of the bound state and thus depends on both $E_b$ and on $\lambda$. Equivalently, we have $E_b = m\lambda^2/2\hbar^2$. There is only one solution, so there is only one bound state.

(b) (3 points) For positive $E = \hbar^2 k^2/2m > 0$, use the Schrödinger equation for a particle incident from the left to find the transmission and reflection amplitudes $t$ and $r$ (express your answers in terms of the dimensionless quantity $\alpha = m\lambda/\hbar^2 k$). Show that the transmission probability $T$ is given by $T = (1 + \alpha^2)^{-1}$, and find the corresponding reflection probability $R$. Sketch $R$ and $T$ as functions of $E$.

For scattering states, $E$ is positive, defining the wave number $k = \sqrt{2mE}/\hbar$. Assume an incident wave from the left, and write $\psi_l = e^{ikx} + re^{-ikx}$, where $r$ is the reflection amplitude. The right side has only a transmitted wave with $\psi_r = te^{ikx}$, where $t$ is the transmission amplitude. Continuity at the origin gives $t = 1 + r$. The discontinuity condition (2) gives

$$ik[t - (1 - r)] + \frac{2m\lambda}{\hbar^2} t = 0$$

(4)
and simple algebra yields

\[ t = \frac{1}{1 - i\alpha} \]  

(5)

where \( \alpha = m\lambda/h^2k \) is a dimensionless parameter. A similar analysis gives \( r = i\alpha/(1 - i\alpha) \).

The particle flux is proportional to \( |\psi|^2 \), which is 1 for the incident wave. For the transmitted wave, it is \( |t|^2 \), which gives the transmission probability \( T = |t|^2 = (1 + \alpha^2)^{-1} \).

Since \( \alpha^2 = m\lambda^2/2Eh^2 = E_b/E \), we have \( T = E/(E_b + E) \), which vanishes as \( E \to 0 \) and approaches 1 for large \( E \). Similarly, the reflection probability is

\[ R = \alpha^2/(1 + \alpha^2) = E_b/(E_b + E) \], which is 1 for small \( E \) and vanishes as \( E \to \infty \).

(c) (1 point) If instead the potential is repulsive with \( V(x) = \lambda\delta(x) \), discuss briefly the effect on your conclusions in (a) and (b).

If the potential is positive with \( V(x) = \lambda\delta(x) \), there is no bound state since the potential is everywhere non-negative. A detailed analysis confirms this conclusion (try it). For the scattering solution, the final probabilities depend only on \( \lambda^2 \) and are the same for attractive and repulsive potentials.

(d) (1 point) Now generalize the potential to have two attractive centers

\[ V(x) = -\lambda[\delta(x + a) + \delta(x - a)] \]

symmetrically placed about the origin. How can you classify the bound states? What is the symmetry of the ground state? If \( a \) is large, sketch various possible bound-state wave functions. How many bound states do you expect? What happens as \( a \) decreases?

The potential is symmetric, so the solutions can be classified as even or odd. The ground state will be even with no nodes, since nodes increase the kinetic energy of a state. If \( a \) is large, then there is essentially one localized bound state at \( a \) and another at \( -a \), and the overlap is very small. Thus the symmetric and antisymmetric combinations have almost the same energy, with the ground state symmetric. This suggests that there will be two bound states, one even (lower) and one odd (higher). As \( a \) decreases, the ground state remains symmetric, but the odd state may disappear, since the gradient (kinetic) energy would become large near the origin midway between the two attractive wells.

(e) (2 points) For the ground state of this symmetric potential, show that the eigenvalue condition is

\[ \tanh(\kappa a) = \frac{2\gamma}{\kappa a} - 1 \]

where \( \gamma = m\lambda a/h^2 \) and \( \kappa = \sqrt{2mE_b/h} \). Use graphical methods to conclude that there is one such bound state for any positive \( \gamma \). Discuss briefly what happens for the odd bound state.

Assume an even solution with energy \( E = -E_b = -\hbar^2\kappa^2/2m \). The wave function has the form \( \psi_1(x) = A \cosh \kappa x \) for \( 0 \leq x < a \), and \( \psi_2(x) = Be^{-\kappa x} \) for \( a < x \). Continuity at \( x = a \)
gives $A \cosh \kappa a = Be^{-\kappa a}$; discontinuity of slope similar to (2) gives

$$-\kappa Be^{-\kappa a} - \kappa A \sinh \kappa a + \frac{2m\lambda}{\hbar^2} Be^{-\kappa a} = 0$$

(6)

Together, these conditions give the eigenvalue equation

$$\tanh \kappa a = \frac{2\gamma}{\kappa a} - 1$$

(7)

where $\gamma = m\alpha \lambda / \hbar^2$. As a function of $\kappa a$, the left side rises linearly from zero and grows toward 1. The right side diverges to $\infty$ for small $\kappa a$ and vanishes at $\kappa a = 2\gamma$. There is one root in the interval $0 < \kappa a < 2\gamma$, and this holds for all positive $\gamma$.

For the odd state, the corresponding eigenvalue equation differs from (7) in that the left side is $\coth \kappa a$. Defining $u = \kappa a$, the eigenvalue is the root of $u \coth u = 2\gamma - u$. The minimum of the left side is 1 at $u = 0$, and the left side increases monotonically with $u$. The right side has its maximum value $2\gamma$ at $u = 0$ and decreased linearly. Evidently, there is a solution only if $2\gamma > 1$. Thus the product $\lambda a$ must exceed $\hbar^2 / 2m$ for a second bound state; otherwise, only the symmetric ground state exists.
Solution

The trajectory of an object with uniform acceleration is a hyperbola in space-time

$$X^2 - c^2 T^2 = R^2$$

By differentiating twice with respect to time and setting $T=0$ one finds that the acceleration is given by

$$a = \frac{c^2}{R}$$

The trajectory can also be described parametrically by

$$X = R \cosh w = \left(\frac{c^2}{a}\right) \cosh w$$

$$T = \left(\frac{R}{c}\right) \sinh w = \left(\frac{c}{a}\right) \sinh w$$

The proper time along the trajectory is easily computed to be $\tau = \left(\frac{c}{a}\right) w$

Thus

$$T = \left(\frac{c}{a}\right) \sinh \frac{a \tau}{c}$$
If we now substitute $a = g = 10 \text{ m/s/s}$, $c = 3 \times 10^8 \text{ m/s}$, and $\tau = 10 \text{ years (approx } 3 \times 10^8 \text{ sec)}$, we find that the first half of the outward journey lasts 20,000 years in the Earth's rest frame.

Multiplying by 4, we find the total trip lasts 80,000 years in Joe's frame. That is how much Joe ages.
4. Hope springs eternal

(a) First of all, we need to calculate the energy of a massive spring. The potential energy is easy, since it obviously doesn’t change with mass; $V = \frac{1}{2} k x^2$. (Potential energy is distributed uniformly throughout the spring)

To get the kinetic energy, we need to think about how the spring moves. Imagine that at some instant the spring has length $x$, and a brief instant $dt$ later, it has length $x + v dt$. Let’s coordinatize the points along the spring by a variable $\xi$ which runs from zero to $x$ initially. The bit of spring at position $\xi$ has density $M/x$.

Now, because the spring moves linearly, since the point at $\xi$ was initially a fraction $\xi/x$ of the way down the spring, its new position must be the same fraction of the way down the stretched spring:

$$\frac{\xi + d\xi}{x + v dt} = \frac{\xi}{x},$$

or

$$\frac{d\xi}{dt} = \frac{v}{x}.$$

The total kinetic energy of the spring is therefore

$$T = \int_{0}^{x} \frac{1}{2} \left( \frac{M}{x} \right) \dot{\xi}^2 d\xi$$

$$= \int_{0}^{x} \frac{1}{2} \frac{Mv^2}{x^3} \xi^2 d\xi$$

$$= \frac{1}{6} Mv^2.$$

(b) If the spring is oriented vertically and initially compressed by two meters, and then all of this potential energy is converted first into kinetic and then into gravitational energy, $V = \frac{1}{2} k(x_0 - l_0)^2 = mg h$, so

$$h = \frac{kAx^2}{2mg} = 204 m.$$

He leaps quite far over the building.

(c) The kinetic energy of Lox plus the spring is now

$$T = \frac{1}{6} M \dot{z}^2 + \frac{1}{2} m \dot{z}^2 = \frac{1}{2} (m + M/3) \dot{z}^2,$$

and so the Lagrangian is

$$\mathcal{L} = \frac{1}{2} (m + M/3) \dot{z}^2 - \frac{1}{2} k \dot{z}^2 - mgx - \frac{1}{2} Mgx.$$

The last term is the gravitational potential energy of the spring; the factor of $1/2$ is for the position of its center of mass, or equivalently from letting

$$V = \int \rho g x dx = \frac{M}{x} g \frac{z}{2} = \frac{1}{2} Mgx.$$
We can eliminate the linear term by completing the square;
\[
\mathcal{L} = \frac{1}{2}(m + M/3)\dot{x}^2 - \frac{1}{2}k \left( x + \frac{(m + M/2)g}{2k} \right)^2 + \text{const.}
\] (44)

This is therefore a simple harmonic oscillator in the variable \( \ddot{x} = x + (m + M/2)g/2k \) with frequency
\[
\omega = \sqrt{\frac{k}{m + M/3}} = 8.34 \text{rad/sec}.
\] (45)

The spring is initially compressed to \( x = -2m \), or \( \ddot{x} = -1.90m \), and initially has no velocity. The motion of the spring is therefore given by
\[
\ddot{x} = -1.90m \cos \omega t.
\] (46)

However, the motion of Lex is not so simple, since he gets launched. In particular, if at any point the spring is decelerating, Lex will leave the platform; it can only push, not pull. He will therefore be launched vertically when the spring ceases to accelerate, i.e. when \( \ddot{x} = 0 \), i.e. at \( \omega t = \pi/2 \). At this instant, his velocity is at its peak,
\[
v = \omega \ddot{x}_0 = 15.82 m/s
\] (47)

which gives him a maximum height of
\[
mgh = \frac{1}{2}mv^2 \implies h = \frac{v^2}{2g} = 12.8 m
\] (48)

not enough to leap over even a moderately tall building.

Since his acceleration is \( \ddot{x} = -\omega^2 \ddot{x}_0 \cos \omega t \), his peak acceleration occurs at \( t = 0 \), when it is
\[
\ddot{x}(0) = \omega^2 \ddot{x}_0 = 132 m/s = 13.48 g.
\] (49)

(For comparison, had he had a massless spring, we would have \( \ddot{x} = x + 5mm \), an initial compression of \( \ddot{x}_0 = -2m \), and \( \omega = \sqrt{k/m} = 31.6 \text{rad/sec} \). This gives \( v_{\text{max}} = \omega \ddot{x}_0 = 63 m/s \), \( h = 204 m \) (as above), and \( a_{\text{max}} = \omega^2 \ddot{x}_0 = 2000 m/s = 204g \), which while it would be highly propulsive would also be rather fatal.)

(d) (Bonus) Well, obviously Lex would leap higher if the spring were compressed further - more potential energy in the spring means more launch kinetic energy. In particular,
\[
h = \frac{v^2}{2g} = \frac{\omega^2 \ddot{x}_0^2}{2g}
\] (50)

so jump height is proportional to the square of compression. Acceleration is \( \omega^2 \ddot{x}_0 \), so it goes linearly with compression. A more compressed spring would increase his jump distance but also increase acceleration, and given the results above this would kill him. Cutting the spring while retaining the material would change the spring constant \( k \) (half the length = twice the spring constant), and thus change \( \omega \), which would scale both the jump height and the peak acceleration equally; again, it wouldn't help him much.

In fact, the only other thing left to change is the spring mass, and changing that similarly changes \( \omega \); but in general, anything which scales \( \omega \) doesn't help much, since it increases both peak acceleration and jump height. We may thus conclude that using springs to leap over tall buildings in a single bound is a pretty stupid idea.