Physics Qualifying Exam, January 2007

Classical Mechanics

Possibly relevant constants:

\[ G \approx 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \quad k \approx 1.4 \times 10^{-23} \text{ J K}^{-1}. \]

For the earth,

\[ R \approx 6.4 \times 10^6 \text{ m} \quad M \approx 6.0 \times 10^{24} \text{ kg} \quad g \approx 9.8 \text{ m s}^{-2}. \]

All answers need only be accurate to one significant figure.

A. Pencil Balancing

At room temperature in a vacuum, how long can a pencil remain balanced on its point? Assume a perfectly sharp point that does not slide. Specifically, what is the maximum time for it to hit the table? (A pencil hanging by its point has an oscillation frequency of about 1.5 Hz.)

B. Skyhook

In a science fiction story of this name, Arthur C. Clarke envisions an orbiting ‘elevator’ of length \( L \) extending radially from a fixed location just above the ground at the equator. It is not attached to the earth (of radius \( R \)), so the only external force is gravitational. Thus the angular velocity of all locations on the elevator is the same as that of the earth.

a) Idealize the orbiting elevator as a (thin) rod with uniform mass density \( \rho \) and cross sectional area \( A_c \). Estimate the required value of \( L/R \).

b) Estimate the maximum stress (force per unit area) induced within the rod. Could it be made from ordinary materials, which fail at a stress \( \sim 10^8 \text{ N m}^{-2} \)?
Skyhook directly above the equator
**Electromagnetism**

A pure magnetic dipole moment \( \mathbf{m} = m\hat{\mathbf{z}} \) is located at a distance \( x \) from a circular loop of wire oriented in the \( yz \) plane. The dipole is located directly on the axis of the loop as shown below. The loop has a radius \( a \), where \( a \ll x \). The dimensions of the dipole are very much less than \( x \) or \( a \).

a) The loop initially carries a current \( I \) in the anticlockwise direction when the loop viewed from the dipole. Calculate the change in potential energy of the dipole if the direction of the dipole is reversed. Express your answer in terms of \( I, a, m \) and \( x \).

b) The current in the loop is now turned off. Assume now that \( \mathbf{m} \) points in the \( \hat{\mathbf{x}} \) direction. Calculate the force required to move \( \mathbf{m} \) at a constant velocity \( v\hat{\mathbf{x}} \) towards the loop where \( v \ll c \), the speed of light. Neglect the self-inductance of the loop. Express your answer in terms of \( m, a, v, x \) and \( R \), where \( R \) is the resistance of the loop.

c) Explain qualitatively how your answer to part b would change if the self-inductance is not, in fact, negligible and discuss the limits under which the self-inductance, \( L \), can be assumed to be negligible.
A helicopter requires power $P$ to hover near the surface of the earth. Assume that all parts of the helicopter operate in the subsonic regime and neglect the viscosity of air. Neglect all friction in the helicopter machinery.

(a) What power is required for a scale model of the same helicopter, built with the same materials but with all dimensions reduced by a factor of 2?

(b) What power would be required for the original, full scale helicopter to hover on a planet where the atmosphere is twice as dense as that on Earth, and where objects weigh 3 times their terrestrial weight?

(c) Defining the “efficiency” of an engine as its power-to-mass ratio, discuss whether it is easier to build small or large hovering machines. (Assume that the mass of the machine is dominated by the mass of the engine, as is the case in reality.) Can you relate this to flying animals?

(d) Although, as suggested above, you have neglected the viscosity of air, explain how in real life viscosity plays a role in the conservation of energy for this problem.
In the laser cooling of atoms, six laser beams are used to “cool” a sample of atoms located in the region where the six beams cross, as illustrated in the figure below. The six laser beams consist of three orthogonal pairs. Atoms absorb photons from the laser beams and emit them in a random direction. If an atom repeatedly absorbs photons from a particular laser beam, emitting a photon in a random direction after each absorption, the atom will experience a net change in momentum in the direction of the laser beam. Since the atoms will “see” a doppler-shifted frequency for the laser photon, the frequency of light from the lasers can be tuned so that one laser in a pair slows down atoms moving in one direction while the other laser slows atoms moving in the opposite direction.

In the following questions, accurate physical arguments, not involving detailed calculations or exact numerical factors, will get full credit. Aim for results that are correct to within a factor of a few.
You may need some or all of the following information:
The atomic mass of sodium is 23.
A sodium atom will absorb light with a wavelength near 589 nm if the light is within \( \approx 10 \text{ MHz} \) of the resonant frequency.
The mass of a proton is \( 1.67 \times 10^{-27} \text{ kg} = 938 \text{ MeV}/c^2 \); \( h = 6.6 \times 10^{-34} \text{ J-s} = 4.1 \times 10^{-21} \text{ MeV-s} \); \( k_B = 1.4 \times 10^{-23} \text{ J/K} = 8.6 \times 10^{-8} \text{ eV/K} \).

(a) Calculate the number of “yellow” photons of wavelength \( \lambda \approx 589 \text{ nm} \) that must be absorbed to stop a sodium atom initially at room temperature \( (v \approx 600 \text{ m/s}) \).

(b) What is the minimum time needed to cool a sodium atom?

(c) Suppose you use the same yellow photons discussed in (a) to slow down sodium atoms from 600 m/s. As the sodium atom slows down and “stops”, by how much does the frequency of the yellow photons appear to change due to the Doppler shift? To compensate for this change in the frequency seen by the atom, the frequency of the yellow light must by changed as the atom slows down so that the light will remain in resonance with the atom. What is the minimum number of times that the frequency must be changed, if it were changed in steps?

(d) One of the possible limiting factors in how much one can slow an atom is due to the recoil velocity that an atom gains when it emits a single photon. What is this limiting “recoil temperature” for sodium atoms cooled with yellow photons?

[Note that temperatures below the recoil temperature have been achieved with laser cooling with a method called “velocity-selective coherent population trapping”.]
Quantum Mechanics 2, January 2007

In the following problem ignore everything but the spins of the electrons.

Consider two electrons (spin $\frac{1}{2}$) prepared in an initial state $|S\rangle$. The spin of electron-one is called $\vec{\sigma}$ and the spin of electron-two is $\vec{\tau}$. (Each has 3 components, $\sigma_1$, $\sigma_2$, $\sigma_3$, and similarly for $\tau$.) The state $|S\rangle$ satisfies

$$(\vec{\sigma} + \vec{\tau})|S\rangle = 0 \quad (1)$$

a) Working in the $\sigma_3$, $\tau_3$ basis, construct $|S\rangle$.

b) Suppose that the spin of electron-one is measured along the 3$^{rd}$ axis and the spin of electron-two is measured along an axis $\overrightarrow{n}$ at 45 degrees to the 3$^{rd}$ axis. What is the probability that both spins will be along their respective measurement axes? In other words, what is the probability that spin-one is pointing along the 3$^{rd}$ axis and spin-two is pointing along the $\overrightarrow{n}$ axis?
Quantum Mechanics I, January 2007

The general nonrelativistic Schrödinger equation for the wave function \( \psi(r, t) \) describing a particle of mass \( m \) moving in a potential \( V(r) \) is

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi.
\]

(a) Prove that the probability density \( \rho(r, t) \) obeys the equation \( \partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0 \), where

\[
\mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*).
\]

What is the explicit form of \( \rho \)? Explain why this equation represents a conservation law.

(b) Consider a one-dimensional problem valid for \( 0 \leq x < \infty \), with boundary condition \( \psi(x = 0) = 0 \). Consider a free particle with energy \( E \) in this bounded space. What is the allowed range of \( E \)? Find the corresponding wave function (use \( k = \sqrt{2mE/\hbar} \) for the wave number) and interpret it in terms of incident and reflected waves. How does the incident probability current compare with the reflected probability current? Is probability density conserved?

(c) Now add a potential

\[
V(x) = \begin{cases} 
V_0, & \text{for } 0 \leq x < a \\
0, & \text{for } a < x < \infty 
\end{cases}
\]

For \( x > a \), explain why the solution for a particle with energy \( E > 0 \) now has the form \( \psi \propto \sin(kx + \delta) \), where \( \delta \) is called the phase shift. In the limit \( V_0 \to \infty \), find the phase shift and interpret this result. Rewrite this wave function so that (apart from an overall irrelevant factor) it looks like the incident wave from (b) plus a reflected wave with an additional phase. How do the incident and reflected probability currents compare with those in (b)?

(d) Explain carefully how the sign of \( \delta \) is related to the sign of \( V_0 \). What happens to \( \delta \) as \( V_0 \) varies from large and positive (discuss what "large" means here) through zero to a negative value? Why do you need to assume that \( E \) is positive? Solve the time-independent Schrödinger equation in the separate regions \( 0 \leq x < a \) and \( a < x \) to find the phase shift \( \delta \) for \( E > V_0 \) in terms of \( k \) and \( k' = [2m \left(E - V_0 \right)]^{1/2}/\hbar \). Repeat for \( E < V_0 \) in terms of \( \kappa = [2m \left(V_0 - E \right)]^{1/2}/\hbar \). Take the limit \( V_0 \to \infty \) and rederive the result from (c).

(e) Explain briefly why this problem is directly relevant to the \( s \)-wave scattering by the spherically symmetric potential (\( r \) is now the radial coordinate)

\[
V(r) = \begin{cases} 
V_0, & \text{for } 0 \leq r < a \\
0, & \text{for } a < r < \infty.
\end{cases}
\]

[The physics of this problem is essential in understanding the effect of a "Feshbach" resonance on a gas of ultracold fermions.]
The Photon Gas

Consider a photon gas in a cube of linear dimension $L$ in 3 spatial dimensions at temperature $T$.

In this problem, ignore constants of order one. Dimensional analysis and simple physical arguments will take you a long way!

1. When distance, mass, time and temperature are measured in Planck units ($\ell_P, M_P, t_P, T_P$), then $c = k_B = \hbar = G = 1$. Using well-known relations in physics, argue that when Planck units are used, energy, temperature and inverse length all have the same dimensions.

2. For each of the following parts, express your answer using Planck units.

(a) What is the Helmholtz free energy $F(T)$ for this system?

(b) What is the internal energy $E(T)$ of the system (namely the average energy $\langle E \rangle$ as a function of $T$)?

(c) What is the entropy $S(T)$?

(d) What is the heat capacity of this system?

(e) What is the average energy of a photon?

(f) What is the average number of photons $\langle N \rangle$? Compare this to your expression for the entropy.

3. On general grounds, what do you expect the root mean square fluctuation $\Delta N$ in the number of photons to be?

4. Compute $\langle N \rangle$ for an oven one meter cubed at 200°C, up to constants of order one.

You may wish to use some of the following constants: $c = 3.0 \times 10^8$ m/s, $\hbar = 1.1 \times 10^{-34}$ J·s = $6.6 \times 10^{-16}$ eV·s, $k_B = 8.6 \times 10^{-5}$ eV/K.
Solution:

1. When distance, mass, time and temperature are measured in Planck units \((\ell_P, M_P, t_P, T_P)\), then \(c = k_B = \hbar = G = 1\). Using well-known relations in physics, argue that when Planck units are used, energy, temperature and inverse length all have the same dimensions.

   From thermodynamics, \(E \propto k_B T\), where the constant of proportionality is dimensionless; therefore, when Planck units are used \((k_B = 1)\), energy \(E\) and temperature \(T\) have the same dimensions.

   From quantum mechanics, \(E = \hbar \omega = \hbar c k\), where \(k = 2\pi/\lambda\); therefore, when Planck units are used \((\hbar = c = 1)\), energy \(E\) and inverse-wavelength \(\lambda^{-1}\) have the same dimensions.

   Therefore, when Planck units are used, energy, temperature and inverse-length all have the same dimensions.

2. For each of the following parts, express your answer using Planck units.

   Use the fact that the only energy scales in the problem are \(T\) (or \(E\)) and that the extensive quantities must be proportional to \(L^3\). (Extensive quantities are those that depend on the size or extent of the system.)

   (a) What is the Helmholtz free energy \(F(T)\) for this system?
       \(F(T)\) is extensive and has dimensions of \(T\) so \(F(T) \approx L^3 T^4\).

   (b) What is the internal energy \(E(T)\) (namely the average energy \(\langle E \rangle\) as a function of \(T\))? 
       Same as \(F\).

   (c) What is the entropy \(S(T)\)?
       Entropy is dimensionless and extensive, so \(S \approx L^3 T^3\).

   (d) What is the heat capacity of this system?
       The heat capacity is dimensionless and extensive, so it is same as \(S\).

   (e) What is the average energy of a photon?
       The only energy scale is \(T\).
(f) What is the average number of photons $\langle N \rangle$? Compare this to your expression for the entropy.
$\langle N \rangle$ is dimensionless and extensive (or divide $E$ by $T$). Same as $S$.

3. On general grounds, what do you expect the root mean square fluctuation $\Delta N$ in the number of photons to be?
Like all simple statistical processes, $N^{1/2}$.

4. Compute $\langle N \rangle$ for an oven one meter cubed at 200°C, up to constants of order one.
Calculate $\langle N \rangle \approx L^3 T^3 = \left( k_B / (\hbar c) \right)^3 L^3 T^3$. Use $\hbar c \approx 200$ MeV·fm and $k_B \approx 10^{-10}$ MeV/K to find $\langle N \rangle \approx 10^{16}$. 

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Special Relativity, January 2007

Bob and John escape from prison in two different spacecraft, each flying with speed 0.6c in different directions. Five days later police discover that Bob and John have escaped; they send their only spaceboat to catch John. Not realizing he is being followed, John turns off his engine to avoid detection and continues to fly with a constant speed of 0.6c. What a tragic mistake...

(a) If the police spaceboat flies at a speed of 0.8c to catch John and return to the prison, how many days pass in the reference frame of the prison between the moment John and Bob escape, and when the spaceboat returns to the prison with John?

(b) How many days of freedom does John enjoy before the police catch him, from John’s point of view?

Meanwhile, after flying for three days (in the reference frame of the prison) with constant speed 0.6c, Bob lands on the planet Io, spends two days there playing in a casino, wins a lot of money, refuels his spacecraft, and then suddenly realizes that he is in danger. He decides he must fly away as fast as possible. Therefore, he starts flying again, with constant acceleration a.

Since the police have only one spaceboat, they cannot chase both John and Bob when they realize they are gone. Therefore, five days after Bob’s escape (in the reference frame of the prison), just as Bob is leaving the planet Io, the police send a radio signal from the prison, which is supposed to trigger an explosion on Bob’s spacecraft.

Bob knows about this danger but he has a hope that the radiowave will never reach his spacecraft if he flies away with sufficiently large acceleration. He remembers that a spacecraft moving with constant acceleration follows a hyperbolic trajectory in space-time: $x^2 - c^2t^2 = R^2$, where $R$ is some constant, but he forgets what $R$ is equal to, and he must figure out a few other things as well...

(c) How is $R$ related to the acceleration $a$? [Hint: consider the case near $t = 0$, where $t$ is measured in the reference frame of the prison.]

(d) Explain why acceleration is constant along the trajectory $x^2 - c^2t^2 = R^2$. 

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(e) Draw the space-time diagram of Bob's spacecraft leaving the planet Io, and find the minimum value of acceleration \( a \) that Bob will need if he wants to avoid the intersection of his trajectory with the light cone corresponding to the radio signal.
1 The Classical Mechanics Problem

1.1 Balancing the Pencil

The angle $\theta$ the pencil makes with the vertical is given by

$$\theta(t) = \theta_0 e^{\omega t}$$  \hfill (1)

The velocity at time $t = 0$ is

$$\dot{\theta} = \theta_0 \omega$$  \hfill (2)

The kinetic energy is

$$K.E = \frac{1}{2} I (\dot{\theta})^2$$  \hfill (3)

Equating this kinetic energy to $k_B T$, we have

$$\theta_0 = \frac{1}{\omega} \sqrt{\frac{2k_B T}{I}}$$  \hfill (4)

The pencil falls when $\theta(t) \approx 1$; i.e.

$$\frac{1}{\omega} \sqrt{\frac{2k_B T}{I}} e^{\omega t} \approx 1$$  \hfill (5)

which gives,

$$e^{\omega t} \approx \omega \sqrt{\frac{I}{2k_B T}}$$  \hfill (6)

Let us estimate the RHS. With Bob’s numbers for the mass and length of the pencil, we have

$$\omega \sqrt{\frac{I}{2k_B T}} \approx 10^9$$  \hfill (7)

and, $\omega \approx 10$ sec$^{-1}$.

So we have to solve for $t$ where

$$e^{10t} \approx 10^9$$  \hfill (8)

which yields $t \approx 2$ seconds. This is Bob’s answer. Given that the RHS is a big number (like $10^9$) and the exponential also involves a big number
(ω), it isn't kosher to approximate the exponential by a linear approximation (which was Sandy's worry I think).

1.2 b. The Sky Hook

Let the linear mass density of the rod be ρ and the Earth's angular velocity be ω.

The only external force acting on the rod is gravity. The total gravitational force is:

\[ F_g = \int_{x=R}^{x=R+L} dx \frac{GM\rho}{x^2} = \frac{GM\rho L}{R(R + L)} \]  

(9)

The acceleration of the center of mass is given by:

\[ a_{CM} = \frac{1}{\rho L} \int_{x=R}^{x=R+L} dx \rho x \omega^2 = (R + \frac{L}{2})\omega^2 \]  

(10)

Using Newton's law,

\[ F_{net} = Ma_{CM} \]  

(11)

we have,

\[ \frac{GM}{R^2(1 + L/R)} = R\omega^2(1 + \frac{L}{2R}) \]  

(12)

Solving for \( \frac{L}{R} \), we get,

\[ \frac{L}{R} \approx \frac{1}{2} \sqrt{\frac{8GM}{\omega^2 R^3}} \approx 20 \]  

(13)

Let us estimate the stress in the rod. To do so, we calculate the tension \( T(y) \) at a point \( y \) above the surface of the Earth.

\( T(y) \) satisfies:

\[ T(y) + \int_{x=R}^{x=R+y} dx \frac{GM\rho}{x^2} = \int_{x=R}^{x=R+y} dx \rho \omega^2 x \]  

(14)

which gives
\[ T(y) = \rho \omega^2 R^2 y \left( \frac{GM}{R} \left( 1 + \frac{y}{R} \right) - \left( 1 + \frac{y}{2R} \right) \right) \]  

(15)

Plugging in numbers, we have:

\[ T(y) \approx \rho \times 10^8 \text{Nm/(Kg)} \]  

(16)

For a rod of cross-sectional area A, the linear mass density \( \rho \) is \( \rho_V A \) where \( \rho_V \) is the volume density. For a typical material, \( \rho_V \) is \( \approx 10^3 \text{Kg/m}^3 \).

Plugging in, we have:

\[ \frac{T(y)}{A} \approx 10^{11} \text{N/m}^2 \]  

(17)

And hence normal materials will not be able to support the skyhook.
Solutions

a) The change in energy of the dipole is obtained by considering the internal energy of a magnetic dipole in a magnetic field, given by:

\[ U = -\mathbf{m} \cdot \mathbf{B} \]  

We are told that initially \( \mathbf{m} \) is pointing in the \( \hat{x} \) direction, and that the current flows in the anticlockwise direction as viewed from the dipole. The B-field from the loop thus points in the \( -\hat{x} \) direction. The initial energy is thus \( mB \), where \( B \) is the magnetic field from the loop, and the final energy is \( -mB \) and the change in energy is \( -2mB \). The field \( B \) can be determined from the Biot-Savart law as:

\[ \mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \]  

From which:

\[ B_x = \frac{\mu_0 I \sin \theta}{4\pi r^2} \oint_C d\mathbf{l} \]  

\[ B = \frac{\mu_0 I a^2}{2(x^2 + a^2)^{3/2}} \approx \frac{\mu_0 I a^2}{2x^3} \]  

since \( x \gg a \). Consequently the change in energy when the dipole is reversed is:

\[ \Delta U = -\frac{\mu_0 I a^2 m}{x^3} \]  

b) The motion of the dipole towards the loop creates a changing magnetic flux through the loop. We are told that the dipole is a pure dipole and so the magnetic vector potential is given by:

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^2} \]  

at large distances from the dipole. The field is given by:

\[ \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3} \left( 2\cos \theta \ \hat{\rho} + \sin \theta \ \hat{\theta} \right) \]  

At the loop, \( a \ll x \), therefore \( \hat{\rho} \approx \hat{z} \), \( \theta \sim 0 \) from which the field from the dipole at the loop is given by:

\[ B = \frac{\mu_0 m}{2\pi x^3} \hat{z} \]  

The flux through the current loop is then:
\[ \Phi = BA = \frac{\mu_0 a^2 m}{2x^3} \]  

(9)

The change in flux as \( x \) decreases induces an emf in the loop and thus a current. Ignoring the self-inductance, the current flowing in the loop is then:

\[ I = \frac{1}{R} \frac{\partial \Phi}{\partial t} = \frac{3\mu_0 ma^2 v}{2x^4 R} \]  

(10)

Using eq 4, this in turn causes a magnetic field at the dipole of:

\[ B = \frac{3\mu_0^2 ma^4 v}{4x^7 R} \]  

(7)

Let’s consider the direction. The flux increases as the dipole moves towards the loop, so the current generates a field in the \(-x\) direction. The force on the dipole is then calculated from the potential energy:

\[ U = mB = \frac{3\mu_0^2 m^2 a^4 v}{4x^7 R} \]  

(8)

The force is then:

\[ F = \frac{\partial U}{\partial x} = \frac{21\mu_0^2 m^2 a^4 v}{4x^8 R} \]  

(9)

c) The current induced in the loop is time dependent, increasing with time. The effect of the self-inductance is to oppose that change in current and thus to reduce the induced current in the loop from that calculated in equation 6. If we include the self-inductance, then equation 6 should actually be:

\[ IR + L \frac{dI}{dt} = \frac{d\Phi}{dt} = \frac{3\mu_0 ma^2 v}{2x^4} \]  

(10)

The self-inductance term can be ignored if:

\[ L \frac{dI}{dt} \ll IR \]  

(11)

In this regime, then:

\[ \frac{dI}{dt} = \frac{dI}{dx} \frac{dx}{dt} = \frac{3\mu_0 ma^2 v}{x} \frac{4v}{2x^4 R} = \frac{4v I}{x} \]  

\[ \frac{dI}{dt} = \frac{dx}{dt} = \frac{dx}{dt} = \frac{3\mu_0 ma^2 v}{x} \frac{4v}{2x^4 R} = \frac{4v I}{x} \]  

(12)

from eq 11. So in order to neglect the self-inductance term:

\[ \frac{4v I}{x} \ll IR \]

or:

\[ L \ll \frac{Rx}{4v} \]
So the assumption holds for small velocities and large distances from the coil. For a constant velocity, the assumption becomes less valid as we get nearer the coil (as will our other assumption that $x \ll a$).
Solutions:

(a) The parameters involved in the problem are the power required to keep the chopper hovering, $P$, the acceleration due to gravity, $g$, the linear scale of the chopper, $l$, the average density of the chopper, $\rho_H$ and the density of air, $\rho_{air}$. The shape of the helicopter will be described by a set of dimensionless parameters that refer each dimension to the scale. These dimensionless parameters do not change by shrinking the craft and hence can be ignored. If we dwell on the details of a real helicopter we may be tempted to introduce the viscosity of air, speed of sound and speed of the blades of the rotor (the chopper as a whole has no velocity, since it is simply hovering). However these must be non-essential details because I could ignore how the blades work and simply describe the chopper as a machine that "blows down" a mass of air and hovers this way.

So a generic relationship for $P$ is

$$P = kg^{a}l^{\beta} \rho_{H}^{\gamma} \rho_{air}^{\delta}$$  \hspace{1cm} (1)

where $k$ is a dimensionless constant and $\alpha$, $\beta$, $\gamma$, $\delta$ are the exponents we are set to find.

The physical dimensions of the quantities introduced above are:

$$[P] = L^{2} M / T^{3}$$  \hspace{1cm} (2)

$$[g] = L / T^{2}$$  \hspace{1cm} (3)

$$[l] = L$$  \hspace{1cm} (4)

$$[\rho_H] = [\rho_{air}] = M / L^{3}$$  \hspace{1cm} (5)

Hence we can write 3 simultaneous equations for the exponents:

$$2 = \alpha - 3(\beta + \delta) + \gamma$$  \hspace{1cm} (6)

$$-3 = -2\alpha$$  \hspace{1cm} (7)

$$1 = \beta + \delta$$  \hspace{1cm} (8)

Note that, since there are 4 unknowns we cannot entirely solve the system. We get

$$\alpha = 3/2$$  \hspace{1cm} (9)

$$\beta = 1 - \delta$$  \hspace{1cm} (10)

$$\gamma = 7/2$$  \hspace{1cm} (11)

which can be used to write
\[ P = kg^{3/2} l^{7/2} \rho_H^{1-\delta} \rho_{air} \delta \]  \hspace{1cm} (12)

So the power \( P \) scales with the \( 7/2 \) power of the linear dimensions and a helicopter of half size will require \( \frac{1}{2}^{7/2} = 0.088 \) the power of the full-size helicopter.

(b) The expression for the power \( P \) still has unknown exponents for the air density and the helicopter density. We can pin down the value of \( \delta \) with the following argument: \( P \) must depend on the weight of the helicopter and so the term \( \rho_H \) can only appear in the product \( g \rho_H \). (Conversely, \( \rho_H \) by itself makes reference to the inertia of the craft, which clearly has no reason to enter the problem since the helicopter is stationary.) That means that the exponent of \( \rho_H \) is \( 3/2 \) and, therefore, \( \delta = -1/2 \). So the full expression for the power is

\[ P = kg^{3/2} l^{7/2} \frac{3/2}{\rho_H \rho_{air}}. \]  \hspace{1cm} (13)

Hence on the planet with atmospheric density twice that on Earth and with thrice the earth’s gravity it will take \( 2^{-1/2} \times 3^{3/2} = 3.7 \) times the power required to fly the same helicopter on Earth.

(c) The efficiency of the engine is defined as \( \eta = P/M \). Since we said that in scaling from small to large machines we keep the same materials and hence the same density, \( \eta \propto P/l^3 \propto \sqrt{l} \). Therefore it takes engines with higher efficiency (larger \( \eta \)) to run larger hovering machines. This is the reason why many small insects can hover but most birds cannot. In fact the only hovering bird is tiny. The development of large helicopters, initially dreamed of by Leonardo, was dogged for a long time by the unavailability of engines with high enough power-to-mass ratio. Only with the advent of turbines did helicopters become feasible.

(d) The fact that we require a steady power to hover means that we are throwing away energy in some sort of continuous fashion. Even with the simple assumption that the chopper stays up by blowing air down in some sort of generic way we should wonder where this constant stream of air goes. Here enters the viscosity that makes the stream stop at some point due to turbulence. In the case of a real helicopter (or a hummingbird) there is also lots of turbulence around the blades (or wings). All this turbulence converts the mechanical energy that we keep supplying into heat.
Solutions:

(a) Calculate the number of “yellow” photons of wavelength $\lambda \approx 589$ nm that must be absorbed to stop a sodium atom initially at room temperature ($v \approx 600$ m/s).

Use conservation of momentum. Because $v/c$ is much less than 1, we can use nonrelativistic kinematics.

The initial momentum of the atom is

$$p_a = m_a v = 23 m_p v = 23(1.67 \times 10^{-27} \text{ kg})(600 \text{ m/s}) = 2.3 \times 10^{-23} \frac{\text{kg} \cdot \text{m}}{\text{s}}.$$ 

The momentum carried by an individual photon of wavelength $\lambda = 589$ nm is

$$p_\gamma = \frac{E}{c} = \frac{h \nu}{c} = \frac{h}{\lambda} = \frac{6.6 \times 10^{-34} \text{ J} \cdot \text{s}}{589 \times 10^{-9} \text{ m}} = 1.12 \times 10^{-27} \frac{\text{kg} \cdot \text{m}}{\text{s}}.$$ 

Therefore, the number of photons needed to stop the atom is

$$N_\gamma = \frac{p_a}{p_\gamma} = \frac{2.3 \times 10^{-23} \frac{\text{kg} \cdot \text{m}}{\text{s}}}{1.12 \times 10^{-27} \frac{\text{kg} \cdot \text{m}}{\text{s}}} \approx 20,000.$$ 

(b) What is the minimum time needed to cool a sodium atom?

We are given the information that a sodium atom will absorb light with a wavelength near 589 nm if the light is within $\approx 10$ MHz of the resonant frequency. In other words, the width of the resonance ($\Gamma$) is 10 MHz; but this is roughly the inverse of the lifetime of the state: $\Gamma^{-1} \approx 10^{-7}$ s. In part (a), we found that 20,000 photons must be absorbed (and emitted) to cool a sodium atom. Therefore, the minimum time needed to cool the atom is approximately

$$20,000 \times 10^{-7} \text{s} = 2 \times 10^{-3} \text{s} = 2 \text{ ms}.$$ 

[More precisely, the lifetime of sodium is $(2\pi\Gamma)^{-1} = 1.6 \times 10^{-8}$ s = 16 ns. Therefore, the minimum cooling time is closer to $20,000 \times 16$ ns = 0.3 ms. Full credit will be given for the response without the factor of 2\pi.]
(c) Suppose you use the same yellow photons discussed in (a) to slow down sodium atoms from 600 m/s. As the sodium atom slows down and "stops", by how much does the frequency of the yellow photons appear to change due to the Doppler shift? To compensate for this change in the frequency seen by the atom, the frequency of the yellow light must be changed as the atom slows down so that the light remains in resonance with the atom. What is the minimum number of times that the frequency must be changed, if it were changed in steps?

Since the atoms are non-relativistic, we can use the non-relativistic approximation for the Doppler shift:

\[ \nu = \nu_0 \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \approx \nu_0 (1 + \frac{v}{c}). \]

In other words, \( \Delta \nu \equiv \nu - \nu_0 = \frac{v}{c} \nu_0 \). Therefore, at \( v = 600 \text{ m/s} \), the Doppler shift in the frequency is

\[ \Delta \nu = -\frac{v}{c} \nu_0 = \frac{600 \text{ m/s}}{589 \times 10^{-9} \text{ m}} = 10^9 \text{ Hz} = 1000 \text{ MHz}. \]

Since the Doppler shift of 1000 MHz is 100 times larger than the 10 MHz range in frequency over which the sodium atoms will absorb the 589-nm photons, and since the Doppler shift is linear in \( v \), the frequency will shifted out of resonance every time the speed of the atom drops by 1%, or by 6 m/s. Therefore, if the laser frequency is shifted in steps, it must be shifted at least 100 times.

Note that we also implicitly used the fact that \( \frac{\Delta \nu}{\nu_0} \) is much less than 1, since the frequency of visible light is \( \mathcal{O}(10^{14}) \) Hz, and therefore \( \nu \approx \nu_0 \).

(d) One of the possible limiting factors in how much one can slow an atom is due to the recoil velocity that an atom gains when it emits a single photon. What is this limiting "recoil temperature" for sodium atoms cooled with yellow photons?

From part (a), 20,000 photons are required to slow the atom from 600 m/s. For this nonrelativistic regime, the change in velocity is linear with the number of photons; therefore, the velocity gained by an atom at rest when it emits a single photon is \( (600 \text{ m/s})/20,000 = 0.03 \text{ m/s} \). The kinetic energy of the atom is

\[ K = \frac{1}{2} mv^2 = \frac{1}{2} (23)(1.67 \times 10^{-27} \text{ kg})(0.03 \frac{\text{m}}{\text{s}})^2 = 1.7 \times 10^{-29} \text{ J}. \]
Set the kinetic energy equal to $\frac{1}{2}k_B T$ to find the temperature $T$:

\[
T = \frac{2K}{k_B} = \frac{2(1.7 \times 10^{-29} \text{ J})}{1.4 \times 10^{-23} \text{ J/K}} = 2.4 \times 10^{-7} \text{ K} = 2.4 \mu\text{K}.
\]
Solution

a) $|S\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$ (where u and d (up and down) refer to spin along 3 axis)

b) Define projection operators $\frac{1}{2}(1 + \sigma_3)$ and $\frac{1}{2}(1 + \pi)$ where $\pi = \frac{1}{\sqrt{2}}(\tau_3 + \tau_1)$. The projection operator $P$ onto the desired result is the product $\frac{1}{2}(1 + \sigma_3) \times \frac{1}{2}(1 + \pi)$. The probability is $\langle S|P|S\rangle$. The result is $\frac{(2 - \sqrt{2})}{8}$. 
Solutions

a) $0.6(t + 5) = 0.8t$, so $t = 15$ days. He returns to prison in $(2t + 5) = 35$ days

b) He will enjoy freedom for $(t + 5)\sqrt{1 - 0.6^2} = 16$ days

c) $R = \frac{c^2}{a}$

d) $x^2 - c^2t^2$ is an invariant, so one can always change the coordinates to make $t = 0$, which returns us to the previous problem.

e) If he will be at a distance greater than $R$ from the prison, the light cone will never touch the hyperbola $x^2 - c^2t^2 = R^2$.

Numerically, $a = \frac{c^2}{R} = \frac{3 \times 10^8 m/sec}{0.6 \times 3 \times 24 \times 3600 sec} \approx 1929 m/sec^2 \approx 200g$

Tough life for the tough man...
The Photon Gas

Work in units where $k_B = c = \hbar = 1$. Note that temperature $T$ now has units of energy, or inverse length.

Consider a photon gas in a cube of linear dimension $L$ in $D$ spatial dimensions at temperature $T$.

In all the following ignore constants of order one. Dimensional analysis and simple physical arguments will take you a long way!

The only energy scales in the problem are $T$ (or $E$) and extensive quantities must be proportional to $L^3$.

1. Give the density of states $D(E)$ for this system

   $D(E)$ has dimensions of $1/E$ and must be extensive, so $D(E) \sim L^3 E^2$ is the only allowed answer.

2. What is the Helmholtz Free Energy $F(T)$ for this system?
   $F(T)$ is extensive and has dimensions of $T$ so $F(T) \sim L^3 T^4$.

3. What is the average energy $\langle E \rangle$ as a function of $T$?
   Like $F$

4. What is the entropy $S(T)$?
   Dimensionless and extensive, $S \sim L^3 T^3$

5. What is the heat capacity of this system?
   Dimensionless and extensive, like $S$

6. What is the average energy of a photon?
   Only scale is $T$
7. What is the average number of photons $< N >$? Compare this to your expression for the entropy.
   \textbf{Dimensionless, extensive (or divide E by T), answer like S.}

8. On general grounds, what do you expect the root mean square fluctuation $\Delta N$ in the number of photons to be?
   Like all simple statistical processes, $1/N^{1/2}$

9. Compute $< N >$ for an oven one meter cubed at 400 degrees Fahrenheit, up to constants of order one.

   \textbf{Calculate} $N \sim L^3 T^3 = (k_B/\hbar c)^3 L^3 T^3$. Use $\hbar c \sim 200 \text{MeV fermi}$
   and $k_B \sim 10^{-10} \text{MeV/K}$. \textbf{Find} $N \sim 10^{16}$. 
Quantum Mechanics I, January 2007—Solution

(a) Rewrite the Schrödinger equation in the form

$$ \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \nabla^2 \psi + \frac{1}{\hbar^2} V \psi. $$

Form the combination $\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$. The term involving the potential $V$ cancels from the right side. The remaining terms (involving the Laplacian) are easily rewritten as $-\nabla \cdot j$ with $j$ as given in the problem. Thus the probability density has the form $\rho = |\psi|^2$. The equation $\partial \rho/\partial t = -\nabla \cdot j$ is recognizable as directly analogous to the conservation equation for charge in electricity or for mass in hydrodynamics. If you integrate over a small volume element, it says that change in the probability inside the volume arises only from the influx of probability current through the surface.

(b) In one dimension, assume a stationary state with $\psi(x, t) = \psi(x) \exp(-iEt/\hbar)$. Rewrite the one-dimensional Schrödinger equation in the form

$$ \psi''(x) = -\frac{2m [E - V(x)]}{\hbar^2} \psi(x). $$

where the prime denotes a derivative. For this part of the question, $V$ vanishes, and the solutions are sines and cosines if $E$ is positive (otherwise, the solution involves growing exponentials and is not normalizable). The only solution that vanishes at $x = 0$ is $\psi(x) = A \sin(kx)$, where $A$ is a constant and $k = \sqrt{2mE}/\hbar$. This free-particle solution can be written as $\psi = \frac{i}{2} iA (e^{-ikx} - e^{ikx})$. The first term is an incoming wave moving to the left (because of the time dependence $e^{-i\omega t}$, with $\omega = E/\hbar$). The probability current for this part is $j_i = -\hbar k/m$. The second term is the reflected wave moving to the right with probability current $j_r = \hbar k/m$. Since these two terms add to give zero, the probability density does not change.

(c) With the square-well potential, the solution for $x < a$ involves an altered wave number that can be either real or imaginary. For $x > a$, however, the solution is still a linear combination of sines and cosines (as long as $E > 0$). Any such linear combination can be written as $\psi(x) = C \sin(kx + \delta)$, with $C$ a constant. This can be said to define the phase shift $\delta$. In the limit $V_0 \to \infty$, the wave function must vanish in the region $x < a$, so that $\psi(x = a) = 0$ acts like a boundary condition for the outer region. Evidently, the resulting phase shift must be $\delta = -ka$. Since $\delta$ is negative, it says that the nodes in $\psi$ are moved out, which is to be expected for a hard boundary at $x = a$. This wave function can be rewritten as $\psi(x) = \frac{i}{2} iC e^{-i\delta} (e^{-ikx} - e^{ikx})$. Apart from the overall phase factor, this wave function has the same form as in (b), but with the additional factor $e^{i\delta}$ for the reflected wave. The incident and reflected probability currents for $x > a$ are the same as for (b).

(d) If $V_0 > 0$, the potential is everywhere repulsive on zero. Hence the wave length in the region $x < a$ is longer (or imaginary), and the nodes of the wave function are always shifted outward. Thus the phase shift $\delta$ is negative. Conversely, if $V_0 < 0$, the wave length in the
well is shorter than in the outer region, and the nodes are shifted inward. In this case, the phase shift is positive. It is clear that if $V_0 \gg E$, then $\psi \approx \sin k(x - a)$, which means that $\delta \approx -ka$. As $V_0$ decreases, the nodes move inward and reach those for a free particle when $V_0 = 0$. Thus the phase shift is negative for any effectively repulsive potential. For $V_0 < 0$, the nodes are shifted inward relative to those for a free particle, and the phase shift is positive.

For $E > V_0$, the solution of the Schrödinger equation in the inner region must be $A \sin k'x$, where $k' = [2m(E - V_0)]^{1/2}/\hbar$. The continuity of value and slope at $x = a$ leads to the condition

$$\frac{\tan k'a}{k'} = \frac{\tan (ka + \delta)}{k}.$$  

This equation can be solved to yield

$$\delta = -ka + \arctan \left( \frac{k}{k'} \tan k'a \right).$$

For $E < V_0$, the inner solution is $A \sinh \kappa x$, where $\kappa = \sqrt{2m(V_0 - E)/\hbar}$. A similar calculation yields

$$\delta = -ka + \arctan \left( \frac{k}{\kappa} \tanh \kappa a \right).$$

If $V_0 \to \infty$, then $\kappa \gg 1$ and the second term here vanishes, reproducing the previous result $\delta \to -ka$. In principle, if $V_0$ is sufficiently large and negative, there can be bound states with $E < 0$, but they are not of the form studied here (they must vanish exponentially for large $x$).

(e) For a spherically symmetric potential $V(r)$, the time-independent Schrödinger equation has the form given in the introduction with the left side replaced by $E\psi$. The $s$-wave solution is spherically symmetric and can be written as $\psi(r) = u(r)/r$, where $u$ obeys the equation

$$u''(r) = -\frac{2m [E - V(r)]}{\hbar^2} u(r).$$

This is exactly the same as the one-dimensional equation, because $u(r = 0)$ must vanish to cancel the extra factor of $1/r$ in this $s$-wave $\psi(r)$. The new feature is the possibility of scattering away from the direction of the incident beam, which leads to the differential cross section.