In this problem, you will derive some interesting properties of black holes using simple dimensional analysis and general physical arguments. Remember that the properties of a black hole depend only on their mass $M$ and three fundamental constants:

$$G = 6.67 \times 10^{-11} \frac{m^3}{kg \cdot s^2}, \quad c = 3 \times 10^8 \frac{m}{s}, \quad \hbar = 1.05 \times 10^{-34} \frac{m^2 \cdot kg}{s}.$$

1) Estimate the size of a black hole of mass $M$ (the Schwarzschild radius $R_g$).

2) Find an approximate expression for the Hawking temperature $T_H$ (more exactly, the Hawking energy $k_B T_H$), which gives the typical energy of photons and other particles emitted by a black hole.

Hint: Hawking radiation is a quantum effect, which disappears for $\hbar \to 0$, so look for an expression proportional to $\hbar$.

3) Estimate the average wavelength of photons emitted in Hawking radiation and compare this wavelength to the Schwarzschild radius $R_g$.

4) Estimate the amount of energy per unit time (i.e., the power) emitted by a black hole of mass $M$. Here dimensional considerations should be complemented by the Boltzmann equation, which says that the power of a source is proportional to the area of its surface multiplied by $(k_B T)^4$.

5) About how long does it take for a black hole to evaporate?

6) Estimate the mass and radius of the least massive black hole that would not have evaporated during the time since the Big Bang (approximately 14 billion years).
General Physics I: Solutions

1) This is a classical effect, so \( \hbar \) should not be involved. The resulting combination must increase when \( M \) increases, and must have dimension of length. Here it is:

\[
R_g \sim MG/c^2
\]

2)

\[
kT_H \sim \frac{hc^3}{MG}
\]

3) Compton wavelength is \( \lambda = \frac{hc}{k_BT_H} \).

\[
\lambda \sim hc(k_BT_H)^{-1} \sim R_G
\]

This is an interesting and instructive conclusion: The typical wavelength of the photons in Hawking radiation is the same as the size of the black hole.

4)

\[
\frac{dE}{dt} \sim R_g^2(kT_H)^4\hbar^{-3}c^{-2} \sim \left( \frac{MG}{c^2} \right)^2 \left( \frac{hc^3}{MG} \right)^4 \hbar^{-3}c^{-2} = \frac{hc^6}{M^2G^2}
\]

5)

\[
\frac{dM}{dt} \sim -\frac{hc^4}{M^2G^2}
\]

One may solve this equation, or write the answer using dimensional considerations:

\[
t \sim \frac{M^3G^2}{hc^4}
\]

6) \( M \sim 4 \times 10^{12} \) kg. This is the mass of a big mountain. An accurate calculation taking into account the large number of different types of particles emitted by the black hole and all numerical coefficients that we have omitted (such as \( 15360\pi \)), would give \( 1.7 \times 10^{11} \) kg. Thus our estimate is not so bad for investigation of such a complicated situation.

The size of such a black hole would be (according to our estimate of \( M \) and of the Schwarzschild radius) \( R_g \sim 3.25 \times 10^{-15} \) m, not much different from the size of a proton.
Special Relativity, January 2008

To answer these questions, you may need some of the following information: The rest-mass energy of an electron is 511 keV. The rest-mass energy of the charged pion is 140 MeV. The average lifetime of a charged pion at rest is $2.60 \times 10^{-8}$ seconds.

1. At the Stanford Linear Accelerator Center, electrons are accelerated to energies up to 50 GeV per electron. The accelerator is 3 km long. The electrons are accelerated in "bunches" that each contain about a billion electrons; the length of a bunch is $\approx 1$ cm as measured in the laboratory frame of reference.

   (a) What is the apparent length of the accelerator as measured in the reference frame of an electron that has been accelerated to an energy of 50-GeV?

   (b) What is the apparent length of a bunch of electrons as measured in the reference frame of the 50-GeV electrons?

   (c) Is there a contradiction here? Explain.

2. In the SPEAR storage ring at the Stanford Linear Accelerator Center, electrons and positrons of equal energy collide head-on and annihilate one another to produce other particles. (Well, they did in the "old days"; now the storage ring is used as a synchrotron radiation facility.) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and positron annihilate and create a pair of back-to-back charged pions:

   $$ e^+ + e^- \rightarrow \pi^+ + \pi^- . $$

   (a) What is the average lifetime of these pions, as measured in the laboratory reference frame?

   (b) How far do the pions travel before decaying, on average, as measured in the laboratory reference frame? Is it very likely or very unlikely that the pions will decay inside the detector?

   (c) What is the speed of one pion as measured by the other pion? (Express your answer as a fraction of the speed of light.)
3. Now suppose that a 0.5 GeV electron collides head-on with a 0.5 GeV positron to produce two neutral pions:

$$e^+ + e^- \rightarrow \pi^0 + \pi^0.$$ 

Assume that the mass of the neutral pion is the same as that of the charged pion, which is approximately true. By far the most common decay mode of the $\pi^0$ is to two photons: $\pi^0 \rightarrow \gamma + \gamma$. What is the minimum possible photon energy and the maximum possible photon energy measured in the laboratory frame of reference?
Special Relativity Problem and Solutions

January 2008

To answer these questions, you may need some of the following information:
The rest-mass energy of an electron is 511 keV.
The rest-mass energy of the charged pion is 140 MeV.
The average lifetime of a charged pion at rest is $2.60 \times 10^{-8}$ seconds.

1. At the Stanford Linear Accelerator Center, electrons are accelerated
to energies up to 50 GeV per electron. The accelerator is 3 km long.
The electrons are accelerated in “bunches” that each contain about a
billion electrons; the length of a bunch is $\approx 1$ cm as measured in the
laboratory frame of reference.

(a) What is the apparent length of the accelerator as measured in
the reference frame of an electron that has been accelerated to an
energy of 50-GeV?

**SOLUTION:**

\[
\gamma = \frac{E}{m_e c^2} = \frac{50 \text{ GeV}}{511 \text{ keV}} \approx 10^5
\]

In the reference frame of the 50-GeV electrons, the length of the
accelerator is

\[
L_{\text{acc}} = L_0^{\text{acc}} = \frac{3 \text{ km}}{10^5} = 3 \text{ cm}.
\]

(b) What is the apparent length of a bunch of electrons as measured
in the reference frame of the 50-GeV electrons?

**SOLUTION:**

\[
L_{\text{bunch}} = L_0^{\text{bunch}} = \frac{L_0^{\text{bunch}}}{\gamma} = 1 \text{ cm}
\]

In the reference frame of the 50-GeV electrons, the length of a
bunch is

\[
L_0^{\text{bunch}} = \gamma L_{\text{bunch}} = 10^5 \times 1 \text{ cm} = 1 \text{ km}.
\]
(c) Is there a contradiction here? Explain.

**SOLUTION:**

The apparent contradiction is that in the reference frame of the electrons, the bunch is 1 km long and the accelerator is 3 cm long, so that the entire bunch does not fit in the accelerator. In the laboratory frame, the 1-cm bunch clearly fits entirely inside the 3-km long accelerator. The resolution to this apparent contradiction is that events that are simultaneous in one frame of reference are not simultaneous in a moving frame of references. Although the two ends of the bunch are simultaneously inside the accelerator in the laboratory frame of reference, these two events (the head of the bunch being in one location inside the accelerator while the tail of the bunch is simultaneously at a different location inside the accelerator) are not simultaneous in the frame of the reference of the electrons.

2. In the SPEAR storage ring at the Stanford Linear Accelerator Center, electrons and positrons of equal energy collide head-on and annihilate one another to produce other particles. (Well, they did in the "old days"; now the storage ring is used as a synchrotron radiation facility.) Suppose an electron with total energy of 0.5 GeV collides head-on with a positron with total energy of 0.5 GeV. The electron and positron annihilate and create a pair of back-to-back charged pions:

\[ e^+ + e^- \rightarrow \pi^+ + \pi^- . \]

(a) What is the average lifetime of these pions, as measured in the laboratory reference frame?

**SOLUTION:**

The average lifetime of the pion in the lab frame \((\tau)\) is related to the average lifetime in the pion rest frame \((\tau_0)\) by

\[ \tau = \gamma \tau_0 . \]

The energy of the pion is 500 MeV; the mass of the pion is 140 MeV. Therefore \(\gamma = \frac{E}{m_\pi c^2} = 3.57\) and

\[ \tau = 3.57 \times 2.60 \times 10^{-8} \text{ s} = 9.3 \times 10^{-8} \text{ s} . \]
(b) How far do the pions travel before decaying, on average, as measured in the laboratory reference frame? Is it very likely or very unlikely that the pions will decay inside the detector?

**SOLUTION:**
The average decay length will be \( \ell = \beta \gamma ct \). Use \( \gamma = \frac{1}{\sqrt{1-\beta^2}} \) or \( \beta = \sqrt{1 - \frac{1}{\gamma^2}} \) to find \( \beta \):

\[
\beta = \sqrt{1 - \frac{1}{3.57^2}} = 0.960.
\]

\( \ell = 0.96 \times 3.6 \times 3.0 \times 10^8 \text{ m/s} \times 2.6 \times 10^{-8} \text{ s} = 27 \text{ m}. \)

Since particle physics detectors are typically a few meters in radius, the pion is very likely to decay outside the detector.

(c) What is the speed of one pion as measured by the other pion? (Express your answer as a fraction of the speed of light.)

**SOLUTION:**

\[
\beta_1 = \frac{\beta_{1ab} + \beta_2}{1 + \beta_{1ab} \beta_2} = \frac{0.960 + 0.960}{1 + 0.960^2} = 0.999
\]

3. Now suppose that a 0.5 GeV electron collides head-on with a 0.5 GeV positron to produce two neutral pions:

\[ e^+ + e^- \rightarrow \pi^0 + \pi^0. \]

Assume that the mass of the neutral pion is the same as that of the charged pion, which is approximately true. By far the most common decay mode of the \( \pi^0 \) is to two photons: \( \pi^0 \rightarrow \gamma + \gamma \). What is the minimum possible photon energy and the maximum possible photon energy measured in the laboratory frame of reference?

**SOLUTION:**

In the \( \pi^0 \) rest frame, the energy of each photon is \( E_0^\gamma = \frac{m_\pi}{2} = 70 \text{ MeV} \) and the momentum of each photon is \( E_0^\gamma = p_0^\gamma c \). The photon energy in the lab frame will take on minimum and maximum values when the photons are emitted (back-to-back in the \( \pi^0 \) rest frame) along the
direction of the $\pi^0$ momentum in the lab frame. The energy of the photon in the lab frame is given by the Lorentz transformation:

$$E' = \gamma(E'_0 \pm \beta p'_0 c) = \gamma E'_0 (1 \pm \beta).$$

Now use $\gamma = 3.57$, $\beta = 0.960$ and $E'_0 = 70$ MeV to get

$$E'_{\text{min}} = 3.57 \times 70 \text{ MeV}(1 - 0.960) = 10 \text{ MeV}$$

and

$$E'_{\text{max}} = 3.57 \times 70 \text{ MeV}(1 + 0.960) = 490 \text{ MeV}.$$  

[The two energies sum up to the original total $\pi^0$ energy of 500 MeV as expected.]

Or, slightly more elegantly,

$$E' = \gamma E'_0 (1 \pm \beta) = E'_0 \frac{1 \pm \beta}{\sqrt{1 - \beta^2}} = E'_0 \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}.$$  

This latter form makes the formula for the relativistic Doppler shift evident:

$$\omega' = \omega'_0 \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}.$$
Quantum Mechanics I, January 2008

The Pauli matrices are given by $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) The normalized eigenfunctions of $\sigma_z$ are $\chi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues $\pm 1$. Find the eigenvalues and corresponding normalized eigenfunctions of $\sigma_z$.

(b) A particle with spin $\frac{1}{2}$ has the spin operator $S = \frac{1}{2} \hbar \sigma$. Evaluate the commutator $[\sigma_x, \sigma_y]$, and verify that it correctly gives the familiar commutation relation for $[S_x, S_y]$.

(c) For any time-independent operator $O$ and any time-independent Hamiltonian $H$, the corresponding Heisenberg operator is $O(t) = \exp(iHt/\hbar) O \exp(-iHt/\hbar)$. Show that the Heisenberg equation of motion is

$$i\hbar \frac{\partial O(t)}{\partial t} = [O(t), H].$$

The magnetic moment for a spin $S$ is $\mu = \gamma S$, where $\gamma$ is a constant (assume $\gamma > 0$). When placed in a magnetic field $B$, the Hamiltonian is $H = -\mu \cdot B = -\gamma S \cdot B$. Use the Heisenberg equation of motion for the time-dependent Heisenberg operator $S(t)$ to obtain the dynamical equation $\frac{\partial S(t)}{\partial t} = \gamma S(t) \times B$.

(d) Consider a static magnetic field $B = B_0 \hat{z}$. Use the equation of motion to find the solution for $S_x(t) + iS_y(t)$ with the initial condition $S_x(0) = \frac{1}{2} \hbar$ and $S_y(0) = 0$. Describe the subsequent motion of the spin vector.

(e) A detailed analysis shows that the equation $\frac{\partial S(t)}{\partial t} = \gamma S(t) \times B$ holds even if $B$ depends on time. Use this result to study the same spin-$\frac{1}{2}$ particle in an oscillating magnetic field $B = B_0 \cos \omega t \hat{z}$, where $B_0$ and $\omega$ are constants. For the same initial condition as in (d), solve the resulting equation of motion for $S_x(t) + iS_y(t)$ to obtain the solution

$$S_x(t) = \frac{\hbar}{2} \cos \left( \frac{\gamma B_0}{\omega} \sin \omega t \right).$$

Describe the time-dependent motion for various values of $B_0$ and fixed $\omega$. What is the minimum field $B_0$ required for a complete flip in $S_x$? For this field, how long will it take for the first spin flip to occur?
Solution for Quantum Mechanics I, January 2008

The Pauli matrices are given by \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \); \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \); \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

(a) The normalized eigenfunctions of \( \sigma_z \) are \( \chi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \chi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) with eigenvalues \( \pm 1 \). Find the eigenvalues and corresponding normalized eigenfunctions of \( \sigma_z \).

The eigenvalue equation is \( \sigma_z \chi_x = \lambda \chi_x \). Eigenvalue condition is the determinant \( |\sigma_x - \lambda| = 0 \), which gives \( \lambda = \pm 1 \). The corresponding eigenvectors are \( \chi_{x+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \chi_{x-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

(b) A particle with spin \( \frac{1}{2} \) has the spin operator \( S = \frac{1}{2} \hbar \sigma \). Evaluate the commutator \( [\sigma_x, \sigma_y] \), and verify that it correctly gives the familiar commutation relation for \( [S_x, S_y] \).

The commutator follows easily \( [\sigma_x, \sigma_y] = 2i \sigma_z \), which readily gives the expected result \( [S_x, S_y] = i \hbar S_z \) (the other two relations follow by cyclic permutation).

(c) For any time-independent operator \( O \) and any time-independent Hamiltonian \( H \), the corresponding Heisenberg operator is \( O(t) = \exp(iHt/\hbar)O\exp(-iHt/\hbar) \). Show that the Heisenberg equation of motion is

\[
\frac{i\hbar}{\partial t} \frac{\partial O(t)}{\partial t} = [O(t), H].
\]

The magnetic moment for a spin \( S \) is \( \mu = \gamma S \), where \( \gamma \) is a constant (assume \( \gamma > 0 \)). When placed in a magnetic field \( B \), the Hamiltonian is \( H = -\mu \cdot B = -\gamma S \cdot B \). Use the Heisenberg equation of motion for the time-dependent Heisenberg operator \( S(t) \) to obtain the dynamical equation \( \partial S(t)/\partial t = \gamma S(t) \times B \).

\( O \) is time-independent, so that the derivative \( \partial O(t)/\partial t \) acts only on the exponentials. Since \( H \) commutes with either exponential, direct calculation gives the stated result involving the commutator \( \exp(iHt/\hbar)[O, H]\exp(-iHt/\hbar) = [O(t), H] \).

With \( H = -\gamma B \cdot S \), the dynamical equation for \( S_j \) involves the commutator \( [S_j, S_k] \), which is proportional to \( S_l \) in cyclic order. The resulting equations can be reassembled to yield the stated result. To be explicit, \( i\hbar \dot{S}_j = -\gamma B_k [S_j, S_k] = -i\hbar \epsilon_{jkl} B_k S_l \) using the Einstein summation convention. Equivalently, \( \dot{S}_j = \gamma \epsilon_{jkl} S_l B_k \), which is the stated result written in components.

(d) Consider a static magnetic field \( B = B_0 \hat{z} \). Use the equation of motion to find the
solution for $S_x(t) + iS_y(t)$ with the initial condition $S_x(0) = \frac{1}{2}\hbar$ and $S_y(0) = 0$. Describe the subsequent motion of the spin vector.

If $B = B_0\hat{z}$, then only $S_x$ and $S_y$ are time dependent, with $\dot{S}_x = \gamma B_0 S_y$ and $\dot{S}_y = -\gamma B_0 S_x$. Combining them gives $\dot{S}_x + i\dot{S}_y = -i\gamma B_0 (S_x + iS_y)$. With the initial condition, the solution is $S_x(t) + iS_y(t) = \frac{1}{2}\hbar e^{-i\gamma B_0 t}$. The real and imaginary parts give $S_x(t) = \frac{1}{2}\hbar \cos(\gamma B_0 t)$ and $S_y(t) = -\frac{1}{2}\hbar \sin(\gamma B_0 t)$. Assuming that $\gamma > 0$, this is a clockwise precession about $\hat{z}$ with frequency $\gamma B_0$, namely the Larmor precession.

(e) A detailed analysis shows that the equation $\partial S(t)/\partial t = \gamma S(t) \times B$ holds even if $B$ depends on time. Use this result to study the same spin-$\frac{1}{2}$ particle in an oscillating magnetic field $B = B_0 \cos \omega t \hat{z}$, where $B_0$ and $\omega$ are constants. For the same initial condition as in (d), solve the resulting equation of motion for $S_x(t) + iS_y(t)$ to obtain the solution

$$S_x(t) = \frac{\hbar}{2} \cos \left( \frac{\gamma B_0}{\omega} \sin \omega t \right).$$

Describe the time-dependent motion for various values of $B_0$ and fixed $\omega$. What is the minimum field $B_0$ required for a complete flip in $S_x$? For this field, how long will it take for the first spin flip to occur?

Similar to (d), the dynamical equation now becomes $\dot{S}_x + i\dot{S}_y = -i\gamma B_0 \cos \omega t (S_x + iS_y)$. For an ordinary differential equation of the form $\dot{u} = f(t) u$, separation of variables gives the general solution $u(t) = \int^t dt' f(t')$. Since $f(t)$ involves $\cos(\omega t)$, the integral plus initial condition gives $S_x(t) + iS_y(t) = \frac{1}{2}\hbar \exp[-i(\gamma B_0/\omega) \sin \omega t]$. The real part provides the desired answer

$$S_x(t) = \frac{\hbar}{2} \cos \left( \frac{\gamma B_0}{\omega} \sin \omega t \right).$$

Note that this result reduces to that from (d) if $\omega \to 0$ (the limit is a bit singular). More generally, the motion is a precession about $\hat{z}$, but the sense reverses each half cycle of the applied field. Specifically, the precession is clockwise for $0 \leq \omega t \leq \frac{1}{2} \pi$, then counter-clockwise for the next half cycle, and so on. The solution for $S_x(t)$ is the projection of this circular motion on the $x$ axis; it is a periodic function of $t$ with period $\pi/\omega$ (since the cosine is even). For small $\gamma B_0$, the quantity $S_x(t)$ starts at $\hbar/2$ for $t = 0$ and reaches its minimum at $t_{\text{min}} = \pi/(2\omega)$, with value $(\hbar/2) \cos(\gamma B_0/\omega)$. With increasing $B_0$, this result will first yield a complete spin flip when $\gamma B_0 = \pi \omega$ (the resonance condition). The corresponding time is $t_{\text{min}} = \pi/(2\omega) = \pi^2/(2\gamma B_0)$. For still larger $B_0$, the spin will flip over and continue until the applied magnetic field becomes negative.
Consider a cylindrical container of volume $V$ separated into two volumes, $v_1$ and $v_2 = V - v_1$, by a freely movable partition. The left volume $v_1$ contains $N$ diatomic molecules of mass $m$ and the right volume $v_2$ contains $M$ monatomic molecules, also of mass $m$. (Treat the diatomic molecules as two point particles connected by a rigid rod and the monatomic molecules as point particles.) The partition allows heat to flow between the two volumes. The molecules are dilute enough that you can ignore interactions and the system can be treated classically.

1) Assume that the system is in contact with a heat bath at temperature $T$, and that the external walls conduct heat. Compute the partition function, energy, free energy, and entropy of the system as a function of $v_1$. What is the equilibrium value of $v_1$? Explain how you arrived at your answer.

2) Now assume that the external walls (but not the partition) are completely heat-insulating and that the total energy in the cylinder is $E$. What is the equilibrium value of $v_1$ in this case? Explain how you arrived at your answer.

3) If the mass of the partition is 0.1 kg and the temperature is 300 K, what is the mean square velocity of the partition?
SOLUTION: Statistical Mechanics, January 2008

The partition function is determined by assuming a diatomic (monatomic) molecule has five (three) degrees of freedom. Up to a constant:

$$\log Z = N \log v_1 + \frac{5}{2} N \log T + M \log (V - v_1) + \frac{3}{2} M \log T$$  \hspace{1cm} (1)

Defining $T = 1/\beta$ (Boltzmann const $=1$), the internal energy becomes

$$E = -\frac{\partial \log Z}{\partial \beta} = \left(\frac{5}{2} N + \frac{3}{2} M\right) T$$  \hspace{1cm} (2)

The Helmholtz free energy is

$$F = -T \log Z$$  \hspace{1cm} (3)

and the entropy is

$$S = \beta(E - F)$$  \hspace{1cm} (4)

To compute the equilibrium value of $v_1$ in part 1, use

$$\left.\frac{\partial F}{\partial v_1}\right|_T = 0$$  \hspace{1cm} (5)

This gives $v_1 = NV/(N + M)$.

In part 2 use

$$\left.\frac{\partial S}{\partial v_1}\right|_E = 0$$  \hspace{1cm} (6)

The answer is the same as in part 1.

To compute the average squared velocity of the partition, just use the equipartition of energy. Each degree of freedom gets energy $\frac{1}{2} k_BT$. Thus the velocity of the partition (call it $u$) satisfies

$$\frac{1}{2} M_{partition} u^2 = \frac{1}{2} k_BT$$  \hspace{1cm} (7)

Using $k_B = 1.4 \times 10^{-23}$ J/K and $M = 0.1$ kg gives the rms velocity

$$u = 2.05 \times 10^{-10} \text{ m/s}$$  \hspace{1cm} (8)

(namely $\approx$ an atomic diameter per second).
General Physics II, January 2008

To answer these questions, you may need some of the following information:

\[ G = 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \]

\[ m_H = 1.67 \times 10^{-27} \text{kg} \]

The mass of the sun is \( M_\odot \sim 2 \times 10^{30} \text{ kg} \).

1. Imagine a dense spherical cloud consisting of ionized hydrogen; i.e., a star of mass \( M \) and radius \( R \), shining with a luminosity (energy per unit time) \( L \). The cross-section for scattering of a photon by an electron is given by the Thomson cross section of \( \sigma_T \sim 7 \times 10^{-25} \text{cm}^2 \). What is the critical luminosity \( L_{\text{crit}} \) of the star for which the outer layer of the star starts to lift off the surface? (You should find this critical luminosity to be proportional to the mass of the star.)

2. Now imagine a neutron star of mass \( M \), with material falling onto it at a rate \( \dot{M} \). Assume that a fraction \( \epsilon \) of the rest mass energy of the accreted gas is given off as radiation before it settles on the surface of the neutron star. Assuming this luminosity is equal to the one you derived in (a), what is the accretion time-scale \( t_a = M/\dot{M} \) of this accreting neutron star?

3. Astronomers find a neutron star with twice the mass of the sun. Your advisor’s theoretical prejudice is that it formed initially with 1.41 (\( \sim \sqrt{2} \)) solar masses. Assume that it has been accreting at a rate \( \dot{M} = M/t_a \). How long (in units of \( t_a \)) has the neutron star been accreting to obtain its current mass?

4. Suppose this neutron star has a radius of 10 km and spins 30 times a second around its rotation axis. Is a spherical star a good approximation? Justify your answer.
SOLUTION: General Physics II, January 2008

1. The radiation force per particle is $\sigma_T F/c$ where the radiation flux is $F = L/(4\pi r^2)$. Equating this with the gravitational attractive force of the star per particle, $GMm_H/r^2$, one finds $L_{\text{crit}} = 4\pi GMm_Hc/\sigma_T \equiv l_{\text{crit}} M$.

2. $L_{\text{acc}} = \epsilon \dot{M}c^2$, $t_a = \frac{M}{\dot{M}} = \frac{\epsilon c^2}{l_{\text{crit}}} \sim \epsilon 1.5 \times 10^{16} \text{ s} \approx \epsilon 470 \text{ Myr}$

3. The solution of the implied ODE ($\dot{M} = M/t_a$) in b) is $M(t) = M_0 \exp(t/t_a)$. So $\Delta t = t_a \log(\sqrt{2}) \sim 0.35 t_a$.

4. Compare centrifugal forces with gravitational forces, or rotational energies with potential energies, to see that the rotation speed is only about $1\%$ of Keplerian velocity. A spherical star is a very good approximation.
Classical Particle Mechanics, January 2008

Many features of the orbits around a nonrotating black hole can be well approximated within the Newtonian equations of motion via a pseudo-Newtonian gravitational potential

$$\Psi = -\frac{GM}{(r - r_*)} \quad (r > r_*) .$$

(1)

Consider only the region outside the black hole 'horizon' \( r_* = 2GM/c^2 \), with \( M \) the mass of the 'black hole' and \( m (\ll M) \) the mass of the orbiting test particle.

1. Obtain the effective potential \( \Psi_e(\vec{l}, r) \), defined via the conserved energy

$$E = m \left[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + \Psi_e \right] ,$$

(2)

where the conserved angular momentum \( l = m\vec{l} \). Sketch the (dimensionless) function \( \Psi_e/c^2 \), for a few values of \( \alpha \equiv c\vec{l}/GM \) in the range 0–5, using the dimensionless coordinate \( x = r/r_* = rc^2/2GM \).

2. Over what range of \( r \) do circular orbits exist? For such a circular orbit, what is the corresponding angular velocity \( \Omega(r) \)?

3. Over what range of \( r \) are the circular orbits unstable?

4. For what range of \( \alpha \) will a particle dropped nearly from rest very far away \( (E = 0) \) be swallowed by the black hole?
Classical Particle Mechanics - Solutions
Robert V. Wagener

A) Let the orbital plane be \( \Theta = \pi/2 \).

- Angular momentum: \( m \vec{L} = \vec{r} \times \vec{p} = mrv_\Theta = mr^2d\phi/dt \)  

- Energy \( E = \frac{1}{2} m (v_\phi^2 + v_\Theta^2) + m \Psi \)

\[ E = m \left[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + \frac{\dot{\phi}^2}{r^2} - \frac{GM}{r-r_\odot} \right] \]

\[ \Rightarrow \Psi_e = -\frac{GM}{(r-r_\odot)} + \frac{\dot{\phi}^2}{2r^2} = \frac{a^2}{2} \left[ -\frac{1}{(x-1)^2} + \frac{a^2}{4x^2} \right] \]

Extrema:

\[ \frac{d\Psi_e}{dr} = 0 \quad \text{where} \quad \frac{x^3}{(x-1)^2} = \frac{a^2}{2} \]

The extrema merge when \( d^2\Psi_e/dr^2 \) also vanishes, which is seen to occur at \( x = 3 \) when \( a^2 = 27/2 \).
B) Circular orbit \( \Rightarrow \frac{dr}{dt} = \frac{d^2r}{dt^2} = 0 \).

\[
0 = \frac{1}{m} \frac{dE}{dt} = \frac{dr}{dt} \frac{d^2r}{dt^2} + \frac{d\psi}{dr} \frac{dr}{dt} \Rightarrow \frac{d\psi}{dr} = 0,
\]
at the positions of the dots in the plot.

Note that as \( x^2 \to \infty \), these extrema approach \( x = 1 \) and \( x \to \infty \). Thus circular orbits exist over the range \( r_* < r < \infty \).

From equation (1), \( S\ell = d\psi/dt = \tilde{I}/r^2 \).

From equation (4), \( \psi^2 = \frac{GMr^3}{(r-r_*)^2} \).

Combining gives \( S\ell^2 = \frac{GM}{r(r-r_*)^2} \).

C) Instability of circular orbits \( \Rightarrow \frac{d^2\psi}{dr^2} < 0 \).

\[
\Rightarrow \frac{x^4}{(x-1)^3} > \frac{3a^2}{4}.
\]

Employing equation (4) then gives \( x \leq 3 \text{ (r < 3r_*)} \).

D) It is seen from the plot above that the maximum \( x \) will be that for which there is an \( E = 0 \) (unstable) circular orbit. Equation (2) gives

\[
\frac{x^2}{(x-1)} = \frac{a^2}{4}.
\]

Combining with equation (4) then gives \( x = 2 \), \( a = 4 \), so the particle will be captured if \( x = c\tilde{E}/GM < 4 \).
Quantum Mechanics II, January 2008

Consider a particle of mass $m$ moving on a line subject to an attractive delta function potential $V = -\lambda \delta(x)$.

1. Using dimensional analysis, determine the characteristic energy scale for this system.

2. Find the exact bound state energy and normalized bound state wavefunction.

Now imagine that the strength of the potential oscillates with time, $V = -(\lambda_0 + \lambda_1 \cos \omega t) \delta(x)$. Assume $\lambda_1 \ll \lambda_0$ and that the particle is initially in the bound state of the system with $\lambda_1 = 0$.

3. For $\omega$ very small what is the behavior of the system?

4. At approximately what frequency $\omega$ does this behavior start to change? Describe qualitatively what starts happening.

Now suppose that, instead of oscillating, the strength of the potential changes suddenly. For $t \leq 0, V = -\lambda_0 \delta(x)$, while for $t > 0, V = -\lambda_0 \delta(x)$. Assume that for $t < 0$ the particle is in the bound state.

5. Calculate the probability that the particle remains in the bound state for $t > 0$. Plot this probability for $0 \leq \frac{\lambda}{\lambda_0} < \infty$. 
Consider a particle of mass $m$ moving on a line subject to an attractive delta function potential, $V = -\lambda \delta(x)$.

1. Using dimensional analysis determine the characteristic energy scale for this system.

$$[\lambda] = [E][L], \ [h^2/2m] = [E][L^2], \ [E] = [\lambda^2][2m/h^2]$$

2. Find the exact bound state energy and normalized bound state wavefunction.

Away from $x = 0$ the particle is free. Then by symmetry the bd state wavefunction is

$$\psi(x) = A \exp(-\kappa|x|), \ \kappa = \sqrt{(2m|E|/\hbar^2)}$$

Using $|x|^2 - 1 - 2\theta(x), \theta(x)^' = \delta(x)$, matching at $x = 0$ we get $$(\hbar^2/2m)2\kappa = \lambda$$ or $E_b = \lambda^2 m/2\hbar^2$, consistent with (1).

To normalize the wavefunction observe that $\int_{-\infty}^{\infty} dx \exp(-2\kappa|x|) = 1/\kappa$ so $\psi(x) = \sqrt{\kappa} \exp(-\kappa|x|)$.

Now imagine that the strength of the potential oscillates with time, $V = -(\lambda_0 + \lambda_1 \cos \omega t)\delta(x)$. Assume $\lambda_1 \ll \lambda_0$ and that the particle is initially in the bound state of the system with $\lambda_1 = 0$.

3. For $\omega$ very small what is the behavior of the system?

The motion is adiabatic, $\psi(x) \sim \exp(-\kappa(t)|x|)$.

4. At approximately what frequency $\omega$ does this behavior start to change? Describe qualitatively what starts happening.

When $\hbar \omega \geq |E_b|$ transitions to the continuum are possible via first order perturbation theory, Fermi's Golden Rule. (For smaller frequencies higher order perturbation theory will allow jumps but these will be higher order in $\lambda_1$.)
Now instead of oscillating suppose that the strength of the potential changes suddenly. For
\( t \leq 0, V = \lambda_b \delta(x) \), for \( t > 0, V = \lambda_a \delta(x) \). For \( t < 0 \) the particle is in the bound state.

5. Calculate the probability that the particle remains in the bound state for \( t > 0 \). Plot this probability for \( 0 < \lambda_a < \infty \).

Using the sudden approximation this probability is given by the bound state overlap squared, \( |\langle b|a \rangle|^2 \). Using the formulae above this overlap integral is \( \int dx \psi_b(x) \psi_a(x) = 2(\kappa_b \kappa_a)^{1/2}/(\kappa_b + \kappa_a) \) and the probability is \( 4\kappa_b \kappa_a/(\kappa_b + \kappa_a)^2 \). This peaks at one for \( \kappa_b = \kappa_a \) and vanishes as \( \kappa_a \to 0, \infty \).
Electricity and Magnetism, January 2008

Consider a particle of charge $e$ and mass $m$ moving in a uniform magnetic field $\vec{B}$ aligned along the $z$ axis. Throughout this problem, neglect the energy loss from radiation.

1. In the absence of an electric field, a particle with velocity $\vec{v}$ will undergo helical motion with a radius of gyration $r = v_\perp / \omega$, where $v_\perp = |\vec{v}| \sin \theta$ is the component of velocity perpendicular to the magnetic field, $\theta$ is the angle between $\vec{v}$ and the $z$ axis, and $\omega$ is the gyration angular frequency.

(a) Write out the equations of motion for the particle and verify the following relation: $|d\vec{p}/dt| = \omega |\vec{p}_\perp|$.

(b) Show that the gyration frequency for a non-relativistic ($v \ll c$) particle is $\omega_{nr} = eB/m$. How is this result changed for a relativistic particle [$\gamma = (1 - v^2/c^2)^{-1/2} \gg 1]$?

2. Now add to the magnetic field a uniform parallel electric field, $\vec{E}_\parallel = E_\parallel \hat{z}$. Find a differential equation for $\mu = |\vec{p}_\perp|/|\vec{p}|$ and its time derivative. If the particle begins with $|\vec{p}_\perp| = 0$ at $t = 0$, give the value of $\dot{\mu}$ at $t = 0$ and as $t \to \infty$. Explain qualitatively how the motion perpendicular to the $z$ axis will change over time in both the non-relativistic and the extreme relativistic ($\gamma \gg 1$) limits.

3. Now consider the effect of an added uniform perpendicular electric field with no parallel electric field. Describe the resulting motion of a non-relativistic particle. [Hint: Consider defining $\vec{v}' = \vec{v} - \vec{E} \times \vec{B}/B^2$.] Describe the difference between the motions of a positively and negatively charged particle.
SOLUTION OF ELECTRICITY AND MAGNETISM PROBLEM: 2008

The absence of an arrow implies the magnitude of a vector quantity, as in \( p = |\vec{p}| \).

1a. For helical motion:
\[
\begin{align*}
\vec{r} &= r \cos \omega t \hat{x} + r \sin \omega t \hat{y} + v_z t \hat{z} \\
\vec{p} &= \gamma m \vec{r} \\
\vec{r} &= \gamma m \vec{r} \\
p_\perp &= \gamma m r \omega (- \sin \omega t \hat{x} + \cos \omega t \hat{y}) \\
|p_\perp| &= \gamma m r \omega \\
\dot{p}_\perp &= - \gamma m r \omega^2 (\cos \omega t \hat{x} + \sin \omega t \hat{y}) \\
|\dot{p}_\perp| &= \gamma m r \omega^2 \\
|\dot{p}| &= \omega |\vec{p}_\perp|
\end{align*}
\]

1b. From the Lorentz force equation:
\[
\begin{align*}
\dot{p} &= e \vec{v} \times \vec{B} \\
|\dot{p}| &= ev_\perp B \\
|\dot{p}| &= \omega |p_\perp| \quad \text{from (1a)} \\
\omega &= \frac{ev_\perp B}{|p_\perp|} = \frac{ev_\perp B}{\gamma m v_\perp} \\
\omega &= \frac{eB}{\gamma m} \quad \text{in non-relativistic limit, } \gamma = 1
\end{align*}
\]

2.
\[
\begin{align*}
\dot{\mu} &= \frac{p \dot{\vec{p}}_\parallel - \dot{p} p_\parallel}{p^2} = \frac{\dot{\vec{p}}_\parallel}{p} - \frac{p \dot{p}_\parallel}{p^2} \\
\dot{p} &= \frac{1}{2} (p_\parallel^2 + p_\perp^2)^{1/2} 2 p_\parallel \dot{p}_\parallel = \frac{p \dot{p}_\parallel}{p} \\
\dot{\mu} &= \frac{\dot{p}_\parallel}{p} - \frac{(p \dot{p}_\parallel)}{p^2} \frac{p_\parallel}{p^2} \\
\dot{\mu} &= \frac{\dot{p}_\parallel}{p} (1 - \mu^2)
\end{align*}
\]

1
If the particle begins with $\mu = 0$, then at $t = 0$, we have:

$$\dot{\mu} = \frac{\dot{p}_\parallel}{p} = e\frac{E_\parallel}{p}$$

and at $t = \infty$, since $\mu$ approaches 1 we see that $\dot{\mu} = 0$.

For a non-relativistic particle, the perpendicular motion remains unchanged. For a relativistic particle, however, $\gamma$ increases, so $\omega_{\text{rel.}} = \omega_{\text{nr.}}/\gamma = v_\perp/r$ decreases while $v_\perp$ remains constant, so the radius $r$ of the helix must increase.

3. The perpendicular electric field, say along the $x$ axis, produces a drift along the $y$ axis. To see this, let:

$$\vec{v} = \vec{v} + \vec{E} \times \vec{B}/B^2$$

and note that since the added term is constant, we have:

$$\dot{\vec{v}} = \vec{v}$$

Substitute this in the equation for motion under the Lorentz force:

$$m\dot{\vec{v}} = m\dot{\vec{v}} = e(\vec{E} + \vec{v} \times \vec{B} + \frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B})$$

$$= e(\vec{E} \hat{\hat{x}} + \vec{v} \times \vec{B} - E\hat{\hat{x}})$$

This equation shows that the transformed velocity $\vec{v}$ traces out a helix. Thus the actual total velocity $\vec{v}$ is described by a helix plus a constant drift velocity equal to $\vec{E} \times \vec{B}/B^2$ (which is in the $-\hat{y}$ direction if $\vec{E}$ is in the $+\hat{x}$ direction).

Note that this drift is independent of the sign of the charge of the particle, but the orbits will be different because the sense of circular motion in the $x-y$ plane is opposite for $\pm$ charged particles.