General Physics I

For the following questions you may wish to use the following. Air is composed of approximately 20% $O_2$ and 80% $N_2$ where the mass number of O is 16 and that of N is 14. The mass number of He is 4.

a) Using dimensional analysis, derive the speed of sound in air up to a dimensionless constant.

b) Consider a tuning fork surrounded by a thin membrane that prevents gas exchange but that easily transmits sound with no resonances. What changes, if any, will you hear in the tone if the air contained in the membrane is replaced with helium gas? Quantify any change in frequency.

c) Now consider an open-ended pipe, as shown below. A sound can by generated by blowing air through the pipe. What changes, if any, will you hear in the tone if helium gas is blown through the pipe, instead of air? Quantify any change in frequency. (Assume that the overpressure that pushes the gas through the pipe is the same in both cases. You may assume that the pipe is small enough that the amount of gas needed to create a sound is very much smaller than the total amount of air in the room.)

d) Now take the tube to an altitude at which the normal air pressure is 60% of its sea-level value. The pressure of the air used to create the sound in the pipe is maintained at the same value that was used at sea-level. What change, if any, will you hear in the tone that the pipe produces compared to the sea-level value? Quantify any change in frequency.
General Physics I Solutions

a) Start by considering the variables that characterize an ideal gas: pressure, density and temperature.

Write down the dimensions:

- speed of sound, \( v \) – LT\(^{-1}\)
- pressure, \( p \) – ML\(^{-1}\)T\(^{-2}\)
- density, \( \rho \) – ML\(^{-3}\)
- temperature, \( T \) – K

Write \( v = p^\alpha \rho^\beta T^\gamma \)

Insert the dimensions: Write

\[
LT^{-1} = (ML^{-1}T^{-2})^\alpha (ML^{-3})^\beta K^\gamma
\]

\[
LT^{-1} = M^{\alpha+\beta} L^{-\alpha-3\beta} T^{-2\alpha} K^\gamma
\]

From this,

\[
0 = \alpha + \beta
\]
\[
1 = -\alpha - 3\beta
\]
\[
-1 = -2\alpha
\]
\[
0 = \gamma
\]

giving \( \alpha = 1/2, \beta = -1/2, \gamma = 0 \). Thus:

\[
v = C \sqrt{\frac{p}{\rho}}
\]

where \( p \) is the pressure of the gas, and \( \rho \) the density.

b) The frequency of the sound wave produced is the fundamental vibration mode of the tuning fork which depends on its geometry. Let’s call this \( f_0 \).

The wavelength of the sound wave that is generated is given by:

\[
\lambda = \frac{v_{\text{gas}}}{f_0}
\]

Now when the sound wave gets to the membrane, it sets the membrane vibrating at a frequency \( f_0 \). This sets up a sound wave in the air outside the membrane with frequency \( f_0 \), which then propagates to the ear. The sound wave sets up a vibration in the ear-drum at a frequency \( f_0 \) which you hear as the fundamental tone of the tuning fork.

Now consider what changes when the air is replaced with helium.

- The fundamental frequency of the tuning fork, \( f_0 \), is unchanged.
- The wavelength, \( \lambda \) of the sound wave within the membrane is changed since \( v_{\text{gas}} \) is different.

However, the sound we hear depends only on \( f_0 \) and so the tone is unchanged by the presence of the He.
c) The resonances of an open-ended pipe of length $L$ have a wavelength that satisfies:

$$L = n\lambda$$

where $n$ is an integer. The resonant frequencies of the pipe are then:

$$f = \frac{v_{\text{gas}}}{\lambda} = \frac{nv_{\text{gas}}}{L}$$

So the frequency depends on the length of the pipe and the speed of sound in the pipe. Now consider how this affects what we hear.

- As discussed in the previous question, it is the frequency that affects what we hear.
- When the sound wave exits the pipe, its wavelength will change if the composition of the gas changes, but its frequency will remain the same.
- Consequently, when the composition of the gas changes, the frequency of the sound generated in the pipe changes and so the sound that you hear will change in frequency.

Assume that the pressure of gas through the pipe is the same for both air and He. The density of the two gases is different though. Consequently, the ratio of the resonant frequency in helium to the resonant frequency in air is:

$$\frac{f_{\text{He}}}{f_{\text{air}}} = \frac{\rho_{\text{air}}}{\rho_{\text{He}}}$$

Now assume that both helium and air are ideal gases. Then the ratio of the densities depends only on the ratio of the molecular mass:

Therefore:

$$\frac{f_{\text{He}}}{f_{\text{air}}} = \sqrt{\frac{(0.2 \times 16 \times 2.0) + (0.8 \times 14 \times 2.0)}{4.0}} = 2.7$$

So the sound generated by the pipe gets considerably higher in frequency.

d) Again, it is the frequency of the sound that is produced that is important.

If the pressure of the gas flowing though the pipe is maintained at its sea-level value, the resonant frequency of the pipe is the same as the sea-level value.

When the air from the pipe hits the ambient air, the pressure changes, changing the speed of the sound, and thus the wavelength, but the frequency remains the same. Consequently there is no change in the tone of the pipe.
General Physics II

The figure below show results from an experiment that is designed to measure Boltzmann’s constant by observing Brownian motion of particles. Polystyrene spheres, with a diameter of 1.02 µm are suspended in a liquid with viscosity $\eta = 936$ mPa s. The spheres then exhibit random motions as they are struck by the water particles, which are themselves exhibiting thermal motions. A drop of liquid is suspended between 2 glass slides with a separation of $\sim 0.2$ mm and and viewed under a microscope. Random collisions with the liquid molecules cause the spheres to move. A sequence of snapshots are taken, and the displacements in $x$ and $y$ of each sphere, relative to its starting location, is recorded. The data are shown in the figure.

Figure 1: Mean square displacement $\langle R^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle$ measured for 107 polystyrene spheres of diameter 1.02 µm. The data are well fit by a line with slope $s = 1.84 \pm 0.02$ µm$^2$ s$^{-1}$. The figure also shows the recorded displacements $\Delta x$ and $\Delta y$ after each step. This information is not needed for this question.

a) Show that the mean-squared displacement is given by:

$$\frac{\langle R^2 \rangle}{t} = \frac{4kT}{\mu}$$  \hspace{1cm} (1)

where $T$ is the temperature of the system, and $\mu$ is a drag coefficient defined as force per unit velocity. (Hint: start by considering the equation of motion of one of the spheres. You may wish to use the identity $d(x^2)/dt = 2xdx/dt$. )

b) Without solving the fluid mechanics equations, i.e. from dimensional analysis, determine the dependence of the slope in the figure on the sphere radius, $a$, and the viscosity of the liquid $\eta$.

c) If the radius of each sphere is accurate to $\pm 0.10$ µm and the measurements are made at $T = 296 \pm 3$K, estimate the uncertainty that would be attached to the $k$ measurement from this experiment if the viscosity is perfectly known.
General Physics II: Solution

The experimental details, including the figure, are taken from Nakroshis et al., Am. J. Phys., Vol. 71, No. 6, June 2003.

a) Let’s consider motion in the $x$-direction to start with. Start by writing down the equation of motion.

- We know that there is a drag coefficient $\mu$ with units of force divided by velocity, and so the component of the drag force in the $\hat{x}$ direction must be $-\mu \frac{dx}{dt}$.
- We know that the spheres experience a impulse force, $F$, due to random strikes from water molecules. We won’t need to worry too much about the functional form of this force. Let its component in the $x$-direction be $F_x$.

Consequently we can write:

$$m \frac{d^2x}{dt^2} = F_x - \mu \frac{dx}{dt}$$

(1)

Now what we want is $\langle x^2 \rangle$, so really we want $d(x^2)/dt$. Note that:

$$\frac{d(x^2)}{dt} = 2x \frac{dx}{dt}$$

(2)

So re-write the equation of motion as:

$$mx \frac{d^2x}{dt} = xF_x - \mu x \frac{dx}{dt}$$

(3)

$$= xF_x - \frac{\mu}{2} \frac{d(x^2)}{dt}$$

(4)

Now we need to rewrite the first term as:

$$x \frac{d^2x}{dt} = x \frac{d\dot{x}}{dt} = \frac{d(x\dot{x})}{dt} - \langle \dot{x} \rangle^2$$

(5)

So:

$$mx \frac{d^2x}{dt} = m \frac{d(x\dot{x})}{dt} - m\langle \dot{x} \rangle^2 = xF_x - \frac{\mu}{2} \frac{d(x^2)}{dt}$$

(6)

Now we can take a time average:

$$m \left\langle \frac{d(x\dot{x})}{dt} \right\rangle - m \left\langle \langle \dot{x} \rangle^2 \right\rangle = \langle xF_x \rangle - \frac{\mu}{2} \left\langle \frac{d(x^2)}{dt} \right\rangle$$

(7)

This conveniently makes two of the terms disappear:

- $x(t)$ and $\dot{x}(t)$ are random functions and so their average product, and the time derivative of their average product, is zero
- Similarly, $x(t)$ and $F_x(t)$, are random functions and so their average product will be zero.

Thus:

$$m \left\langle \langle \dot{x} \rangle^2 \right\rangle = \frac{\mu}{2} \left\langle \frac{d(x^2)}{dt} \right\rangle$$

(8)

Similarly:

$$m \left\langle \langle \dot{y} \rangle^2 \right\rangle = \frac{\mu}{2} \left\langle \frac{d(y^2)}{dt} \right\rangle$$

(9)

Last updated Jan 1st 2009

5
The left hand terms is an energy term. Since this motion is essentially thermal in origin, \( m \langle (\dot{x})^2 \rangle /2 = m \langle (\dot{y})^2 \rangle /2 = kT/2 \). So now we have the answer:

\[
\frac{d}{dt} \left( \langle x^2 \rangle + \langle y^2 \rangle \right) = \frac{d}{dt} \langle R^2 \rangle = \frac{4kT}{\mu}
\]

Since the quantities on the right are time-independent:

\[
\langle R^2 \rangle = \frac{4kT}{\mu}
\]

b) To use dimensional analysis effectively, we need to include some physical knowledge and experience, as follows:

- The slope of the linear fit to the data is \( 4kT/\mu \). We can reason that the only variable in the slope that will change if the sphere radius is changed, or the viscosity is changed, will be the drag force coefficient which has units of force per unit area.
- The origin of the drag force is the viscosity, which has units of mPa s, or force per unit area per second.
- The ratio of the drag force coefficient and the viscosity \( \mu/\eta \) has units of length.
- We know from experience that in any fluid, a more streamlined object experiences a smaller drag force, so it’s reasonable to suppose that the force depends on the shape of the object, and the only parameter we have that describes the shape is the radius \( a \).

Based on this reasoning, we deduce that \( \mu/(a\eta) \) is a dimensionless group and that\(^1\):

\[
\mu = C a \eta
\]

from which the slope:

\[
s = \frac{kT}{C a \eta}
\]

where \( C \) is constant.

c) From part (a) \( k \) is related to the slope by:

\[
k = \frac{sk}{4T}
\]

from which:

\[
\left( \frac{\Delta k}{k} \right)^2 = \left( \frac{\Delta s}{s} \right)^2 + \left( \frac{\Delta \mu}{\mu} \right)^2 + \left( \frac{\Delta T}{T} \right)^2
\]

From part b,

\[
\mu = C a \eta
\]

from which:

\[
\frac{\Delta \mu}{\mu} = \frac{\Delta a}{a}
\]

for a particular sphere. Because there are 107 spheres in the measurement, the uncertainty will go down by \( \sqrt{107} \). Therefore:

\[^1\text{This is also standard result from fluid mechanics: } \mu = 6\pi a \eta\]
\[
\left( \frac{\Delta k}{k} \right)^2 = \left( \frac{\Delta s}{s} \right)^2 + \left( \frac{1}{\sqrt{107}} \frac{\Delta a}{a} \right)^2 + \left( \frac{\Delta T}{T} \right)^2
\]

\[
= \left( \frac{0.02}{1.84} \right)^2 + \left( \frac{1}{\sqrt{107}} \frac{0.1}{0.51} \right)^2 + \left( \frac{3.0}{296} \right)^2
\]

which is \( \pm 2.4\% \)
Quantum Mechanics I

Part 1: An idealized molecule can be described in first approximation by a one-dimensional harmonic oscillator with Hamiltonian,

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 \]  

(note \( h = 1 \))

a) A photon is emitted when the molecule makes a transition from the first excited state to the ground state. What is the frequency of the photon?

b) On closer examination a correction to the Hamiltonian is found that is proportional to \( x^4 \) so that

\[ H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + gx^4. \]  

To first order in the small constant \( g \) repeat part (a).

Part 2: A particle of mass \( m \) (starting from \( x < 0 \)) moves in the \( x \)-direction. There is a potential that resembles a sudden cliff. For \( x < 0 \) the potential is zero and for \( x > 0 \) it is negative.

\[ V(x) = \begin{cases} 0 & (x < 0) \\ -|V| & (x > 0) \end{cases} \]  

where \( |V| \) is a constant.

a) Classically, what is the probability that the particle falls off the edge of the cliff, i.e., continues toward the positive direction?

b) Quantum mechanically, what is the probability that the particle falls off the edge of the cliff, i.e., continues toward the positive direction?

c) In the limit that the cliff becomes infinitely high (\( |V| \to \infty \)) what is the probability that the particle falls off the edge?

d) How do the answers to parts c and a compare? How do you understand this?
Quantum Mechanics I Solutions

Part 1:

a) The energy levels of the oscillator are given by \( \omega (n + \frac{1}{2}) \). The difference of energy of the ground state and 1st excitation is \( \omega \) and that is also the frequency of the photon.

b) We have to compute the correction to the energy levels in first order perturbation theory. To do that we use oscillator operators

\[
a = \frac{y + i q}{\sqrt{2}} \quad a^+ = \frac{y - i q}{\sqrt{2}}
\]

with \( y = x \sqrt{m \omega} \) and \( q = \frac{p}{\sqrt{m \omega}} \).

The perturbation \( gx^4 \) can be written as

\[
mx^4 = \frac{g'}{4} (a + a^+)^4,
\]

with \( g' = \frac{g}{m^2 \omega^2} \).

The shift in the \( n^{th} \) level is

\[
\delta E_n = \langle n | \frac{g'}{4} (a + a^+)^4 | n \rangle
\]

Expanding \( (a + a^+)^4 \) and rearranging the terms using the following:

\[
aa^+ - a^+ a = 1 \\
a^+ a = n \\
aa^+ = n + 1
\]

one eventually finds

\[
\langle n | (a + a^+)^4 | n \rangle = 3[(n + 1)^2 + n^2]
\]

Thus the energy shifts for \( n = 0 \) and \( n = 1 \) are:

\[
\delta E_0 = \frac{3g'}{4}
\]

\[
\delta E_1 = \frac{15g'}{4}
\]

The energy difference between the two states is

\[
\Delta E = \omega + \frac{15g'}{4} - \frac{3g'}{4} = \omega + 3g'
\]

Part 2

a) Classical Probability = 1
b) The momentum of the initial particle is \( p = mv \). The wave function for negative \( x \) is

\[
\psi(x) = e^{ipx} + Re^{-ipx}
\]

(9)

where \( R \) is the reflection amplitude.

For \( x > 0 \) the momentum is \( p' \)

\[
p' = \sqrt{p^2 + 2m|V|}
\]

(10)

and the wave function is

\[
\psi(x) = Te^{ip'x}
\]

(11)

where \( T \) is the reflection amplitude.

Matching the wave function and the first derivative at \( x = 0 \) one finds

\[
T = \frac{2p}{p + p'}
\]

(12)

and the probability for transmission is

\[
T^2 = \left(\frac{2p}{p + p'}\right)^2
\]

(13)

c) As \( |V| \) becomes large so does \( p' \) and the probability for transmission goes to zero.

d) The classical approximation breaks down because the potential varies rapidly on a scale smaller than the initial wavelength.
Quantum Mechanics II

a) A particle of mass $m$ is moving on a ring of radius $R$. The position of $m$ is described by an angular coordinate $\theta$. Write down the Hamiltonian $H$ for the mass and find the energy eigenvalues and eigenstates.

b) Add a perturbation $V$ to $H$ where $V$ is given by

$$V = \lambda(1 - \cos \theta)$$

Let $\lambda$ be small. Calculate the ground state energy to second order in $\lambda$. Roughly how small must $\lambda$ be for this to be a good approximation?

c) Calculate the ground state energy and ground state wavefunction in the limit $\lambda \to \infty$. Justify whatever approximations you use.

d) Begin the system in the ground state with $\lambda$ large. At time $t = 0$ suddenly set $\lambda = 0$. Give an estimate for how long it takes the wavefunction to spread around the circle. Show your answer has the right behavior in the appropriate limiting cases.
Quantum Mechanics II Solutions

a)

\[ H = -\frac{\hbar^2}{2mR^2} \frac{d^2}{d\theta^2} \]

Eigenstates and eigenvalues:

\[ |+\rangle, n > = \frac{1}{\sqrt{2\pi}} \cos n\theta, \quad E_n = \frac{\hbar^2}{2mR^2} n^2 \]

and

\[ |-\rangle, n > = \frac{1}{\sqrt{2\pi}} \sin n\theta, \quad E_n = \frac{\hbar^2}{2mR^2} n^2 \]

b) The ground state \(|0>|\) is the constant function \( \frac{1}{\sqrt{2\pi}} \) with energy \( = 0 \). The only state with nonzero matrix element between \( V \) and \(|0>|\) is \(|+, 1>|\). \(<0|V|1>| = \lambda > 0\). So

\[ E = 0 + \lambda^2/(0 - E_1) = -\lambda^2 \frac{2mR^2}{\hbar^2} \]

This shift should be smaller than the characteristic energy splitting \( E_1 - E_0 \). So \( \lambda < E_1 = \frac{\hbar^2}{2mR^2} \).

c) When \( \lambda \rightarrow \infty \) the particle is trapped in the well at \( \theta = 0 \) and executes harmonic oscillator motion there. The potential there is approximately \( V(\theta) \sim \frac{\lambda}{2} \theta^2 \). Define \( \mu = mR^2 \). Then the Hamiltonian becomes

\[ H + V \sim -\frac{\hbar^2}{2\mu} \frac{d^2}{d\theta^2} + \frac{\lambda}{2} \theta^2 \]

So the frequency of the harmonic oscillator \( \omega = \sqrt{\lambda/\mu} \) and the ground state energy is \( E_0 \sim \frac{\hbar\omega}{\lambda} \). The ground state wavefunction of the harmonic oscillator is a gaussian \( \psi_0(\theta) \sim \exp(-\alpha\theta^2/2) \) where \( \alpha = \mu \omega / \hbar \). The width of the ground state harmonic oscillator wavefunction is \( \sim \omega^{-1/2} \), which is goes to zero as \( \lambda \rightarrow \infty \) so the harmonic oscillator approx is justified.

d) The initial wavefunction is \( \psi(\theta) \). Use an uncertainty principle argument. \( \Delta \theta \sim 1/\sqrt{\alpha} \).

\( \Delta p \sim \hbar / \Delta \theta = \hbar \sqrt{\alpha} \). The typical momentum is approximately the spread in momentum so \( p = \mu \dot{\theta} \sim \hbar \sqrt{\alpha} \). The time it takes for \( \theta \) to go \( \sim 2\pi \) is \( t \sim 2\pi / \dot{\theta} \). So \( t \sim 2\pi \sqrt{\mu / \omega \hbar} \). This gets large as \( \hbar \rightarrow 0 \) and gets small as \( \omega \rightarrow \infty \), as it should.
Special Relativity

An athlete carrying in front of him a horizontal 20 ft long wooden pole runs at such a speed that in his frame a 10 ft long room made from reinforced concrete becomes only 5 ft long. He runs into the room. At the moment when he sees that his pole touches the wall opposite to the entrance, he closes the door. If he can do it, he brings the 20 ft long pole inside the room, which, from his perspective, is 4 times shorter. Is it possible?

a) Find the velocity of the athlete.

b) Draw the space-time diagram which shows the motion of the room and the position of the pole in the reference frame of the athlete. The diagram should show world lines for the two different sides of the room and for the two ends of the pole.

c) On this diagram, show the moment when the pole touches the wall, and also the moment when the athlete observes this event.

d) What is the minimum length of the room such that when an athlete going at the speed you found in part 1 sees one end of the pole touching the wall, he and the other end of the pole are already just inside the room so he can close the door?

e) What happens to the pole in the closed room if the athlete could bring it inside? After all, the pole is longer than the room; something must be wrong here, right? [Hint: No effect can travel faster than light.]
Special Relativity Solutions

a)

\[ v = \frac{\sqrt{3}}{2}c \]

b) (and parts c and d) Refer to the attached diagram. Consider the limiting situation such that the athlete enters the room exactly at the moment \( t = 0 \) (point E) when he sees how the end of the pole reaches the wall (point F). Suppose that the length of the room in his frame is \( L \) (the original length is \( 2L \)). The distance from F to G on the diagram is equal to the pole length; let us call it P. Then from the diagram one finds

\[ L + P\frac{v}{c} = P \]

so that

\[ L = P \left( 1 - \frac{v}{c} \right) = 20 \left( 1 - \frac{\sqrt{3}}{2} \right) = 2.68 \]

This means that one can fit the pole of a size 20 ft to the room of the size \( 2L = 5.36 \) ft. Therefore 10 ft is more than sufficient.

This also means that if the size of the room is 10 ft, the pole and the athlete will be well inside it when he will see the pole touching the distant wall of the room.

c) Well, at that moment he will not see yet that the wooden pole shatters into pieces because the wave of deformation travels with speed smaller than the speed of light. But eventually the pole will be demolished: There is no way to put a long pole into a small room without destroying the pole. However, none of that will be seen by the athlete at the moment when he will be closing the door to the room: He will see the undamaged pole inside the room. Life is but an illusion...
Statistical Mechanics

A uniform ideal gas of spin-$\frac{1}{2}$ particles with mass $m$ has a number density $n = N/V$ in a box of volume $V$ with periodic boundary conditions.

a) At zero temperature, find the relation between the Fermi energy $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$, where $k_F$ a wavenumber, and the number density. Prove that the total ground-state energy is $E_0 = \frac{3}{2} \epsilon_F N$. Make an order of magnitude estimate of the interparticle spacing in (monovalent) copper to find the corresponding Fermi temperature $T_F = E_F/k_B$ and the Fermi velocity $v_F = \hbar k_F/m$.

b) Prove that the ground-state pressure is $p_F = \frac{\hbar^2}{5} \epsilon_F n$ and estimate this quantity for copper. If the temperature increases at fixed density $n$, describe qualitatively how the pressure $p(T)$ crosses over to the classical ideal-gas equation of state $p = n k_B T$.

c) Assume that each particle has a magnetic moment $\mu$. Discuss the ground state in the presence of a weak magnetic field $B$.

d) Find the induced ground-state magnetization $M$ (magnetic moment per unit volume) and the corresponding ground-state (Pauli) susceptibility $\chi_0 = M/B$.

e) Repeat part (c) for the classical Boltzmann limit and find the corresponding classical high-temperature Curie susceptibility $\chi_C$. Discuss briefly the temperature dependence of $\chi(T)$, including both high- and low-temperature limits.
Statistical Mechanics Solution

Assume a cubical box with volume \( V = L^3 \). Periodic boundary conditions give plane waves with allowed wave vectors \( \mathbf{k} = 2\pi(n_x, n_y, n_z)/L \) where \( n_j \) is any integer. Spin \( \frac{1}{2} \) means that each state is doubly degenerate unless there is a magnetic field, as in parts (c) and (d).

a) At zero temperature, a Fermi gas has all the lowest states occupied up to the Fermi energy \( \epsilon_F \). The number of particles is given by summing over all occupied states, so that
\[
N = 2 \sum_{n_x, n_y, n_z} \theta(\epsilon_F - \epsilon_k),
\]
where the factor of 2 arises from the spin degeneracy, \( \epsilon_k = h^2 k^2/(2m) \) and \( \theta(x) \) is the unit positive step function. This sum can be approximated by a threedimensional integral over the variables \( n_x, n_y, n_z \). A change of variables gives the equivalent expression
\[
N = 2V/(2\pi)^3 \int d^3 k \theta(k_F - k),
\]
where the Fermi wave number is defined by \( \epsilon_F = h^2 k_F^2/(2m) \). This integral is just the volume of a sphere of radius \( k_F \), so that \( k_F^3 = 3\pi^2 n \). The ground-state energy \( E_0 \) is just the sum of \( \epsilon_k \) over all occupied states, namely
\[
E_0 = 2 \sum_{n_x, n_y, n_z} \epsilon_k \theta(\epsilon_F - \epsilon_k).
\]
This becomes an integral over \( \mathbf{k} \) with
\[
E_0 = 2V/(2\pi)^3 \int d^3 k \epsilon_k \theta(k_F - k) \approx \frac{3}{2} \epsilon_F N.
\]
Note that \( k_F^{-1} \) is approximately the interparticle spacing. You can guess that this is of order \( 10^{-10} \) m (namely \( \sim 1 \) \( \rho A \)), so that \( k_F \approx 10^{10} \) m\(^{-1} \). Hence \( \epsilon_F \approx 5.5 \times 10^{-10} \) J or equivalently the Fermi temperature is \( T_F \approx 4 \times 10^4 \) K. Similarly, the Fermi velocity \( v_F = h k_F/m \) is \( v_F \approx 1.16 \times 10^6 \) m/s.

b) The ground-state (Fermi) pressure follows from the thermodynamic relation \( p = -(\partial E/\partial V)_N \), which gives
\[
p_F = \frac{\epsilon_F}{5} n \exp(-\epsilon_F/k_B T) \approx 2 \times 10^{10} \text{ N/m}^2,
\]
which is about \( 2 \times 10^5 \) atmospheres. For comparison, the equation of state of an ideal classical gas is \( p = nk_B T \), which holds for \( T \gg T_F \). As \( T \) falls, this linear relation holds until \( T \approx T_F \), and the curve then crosses over to a constant value \( p_F \approx nk_B T_F \).

c) If each particle has a magnetic moment \( \mu \), then the energy of an up (down) spin is shifted by \( \mp \mu B \). Thus the total number of particles is given by
\[
N = \sum_{n_x, n_y, n_z} [f(\epsilon_k - \mu B) + f(\epsilon_k + \mu B)],
\]
where \( f(\epsilon) \) is the Fermi-Dirac distribution. Because of symmetry, this quantity is unchanged to linear order in \( B \), so that \( N \) is also unchanged to this order. In contrast, the induced magnetization (magnetic moment per unit volume) is
\[
M = \mu V^{-1} \sum_{n_x, n_y, n_z} [f(\epsilon_k - \mu B) - f(\epsilon_k + \mu B)].
\]
An expansion to leading order in \( B \) yields
\[
M \approx \mu^2 B (2\pi)^{-3} \int d^3 k [-\partial f(\epsilon)/\partial \epsilon].
\]
At zero temperature, the derivative of a step function is \( \delta(\epsilon_F - \epsilon_k) \), so that the ground-state (Pauli) susceptibility is
\[
\chi_0 \approx \frac{3}{2} \mu^2 n/\epsilon_F.
\]

d) The corresponding high-temperature limit involves the Boltzmann distribution \( f_B(\epsilon) \propto \exp(-\epsilon/k_B T) \), and a detailed calculation gives the Curie susceptibility \( \chi_C \approx \mu^2 n/(k_B T) \). This high-temperature (classical) approximation holds for \( T \gg T_F \). As the temperature falls, the susceptibility crosses over to the constant Pauli susceptibility \( \chi_0 \) for \( T \lesssim T_F \).
Electromagnetism

For this question you may find the attached material (from Griffiths) and the quantities and integrals on the next page of use.

A cylindrically symmetric beam of ultra-relativistic electrons with velocity \( v \sim c \) is traveling down the center of a vacuum pipe with dimensions large enough to be ignored in this problem. The beam consists of a series of uniformly charged cylindrical regions (“bunches”) of radius \( a \) and length \( l_b \), separated by uncharged regions of length \( L \) much greater than either \( l_b \) or \( a \). The time averaged beam current is \( I_b \).

**Part 1:** Assume \( I_b \) is 0.1 A, \( L/c \) (the time between bunches) is 100 ns \( \left( 10^{-7} \text{ sec.} \right) \), \( l_b = 3 \text{ cm} \) and \( a=0.5 \text{ mm} \).

a) What is the charge of each bunch?

b) How many electrons are in each bunch?

c) What is the peak current?

**Part 2:** At time \( t_0 \) a singly charged ion with zero kinetic energy and atomic number \( A_i \) is created at a distance \( r_0 \) from the axis of symmetry.

a) What is the electric field from the ultrarelativistic electrons, as seen by the ion, as a function of time and radius?

b) Using an impulse approximation \( (l_b \to 0) \), describe (mathematically) the motion of the ion as \( N \) charged regions pass over it. Assume \( r_0 < a \).

**Part 3:** Let \( r_0 = 0 \) and let the electron beam described above be approaching a point focus a distance \( D \gg L \) from the place where the ion appears, as shown below. The ion will experience an impulse as each electron (with \( v \sim c \)) passes.
a) Will this impulse be toward or away from the focus?

b) What is the impulse from a single electron?

c) What is the impulse from a single bunch?

d) Let D be 1000 m. How long will the ion take to travel 1000 m (whether toward or away from the focus)?

Useful information, constants and integrals

- Charge on electron $e = 1.6 \times 10^{-19}$C.
- Electron mass $m_0 = 9.1 \times 10^{-31}$ kg.
- Classical proton radius $r_p = e^2/(4\pi\epsilon_0 m_p e^2) = 1.5 \times 10^{-18}$ m.
- Ratio of proton mass to electron mass $m_p/m_0 = 1836$.
- Integrals:
  \[
  \int_0^1 \frac{dx}{\sqrt{\ln(1/x)}} = \sqrt{\pi}
  \]
  \[
  \int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 + x^2}}
  \]
Classical Mechanics

In the diagram below, a small, solid, uniform density cylinder of mass, $m$, and radius, $r$, rolls without slipping inside an open larger cylindrical shell of mass, $M$, and radius, $R$, as shown. The acceleration due to gravity points “down” in the diagram, and has the usual magnitude, $g$.

![Diagram of the system](image)

a) Assuming that the larger cylinder is held fixed, write an expression for the Lagrangian appropriate to this system in terms of the variables given.

b) Find the equations of motion, and assuming that the displacement angle, $\theta$, is kept small, show that the inner cylinder experiences simple harmonic motion, and calculate the frequency of oscillation.

c) Now suppose that the outer cylinder is allowed to roll without slipping along a horizontal surface. Find the Lagrangian for that system in terms of appropriate variables.

d) Show that as long as the displacement angle, $\theta$, is still small, the motion is still simple harmonic, and find the new frequency of oscillation. Determine how the motion of the outer cylinder is coupled to that of the inner cylinder. Discuss the two limits: $m \ll M$, and $m \gg M$. 