An active-set convex QP solver based on regularized KKT systems

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Outline

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Implementations of the simplex method depend on "basis repair" to steer around near-singular basis matrices, and KKT-based QP solvers must deal with near-singular KKT systems. However, few sparse-matrix packages have the required rank-revealing features (we know of LUSOL, MA48, MA57, and HSL_MA77).

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For convex QP, we explore the idea of avoiding singular KKT systems by applying primal and dual regularization to the QP problem. A simplified single-phase active-set algorithm can then be developed. Warm starts are straightforward from any given active set, and the range of applicable KKT solvers expands.

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QPBLUR is a prototype QP solver that makes use of the block-LU KKT updates in QPBLU (Hanh Huynh's PhD dissertation, 2008) but employs regularization and the simplified active-set algorithm. The aim is to provide a new QP subproblem solver for SNOPT for problems with many degrees of freedom. Numerical results confirm the robustness of the single-phase regularized QP approach.

Supported by the Office of Naval Research and AHPCRC

Motivation (SNOPT)

Motivation: SNOPT

SNOPT: an SQP method for constrained NLP (Gill, Murray, S 2005)

SQOPT: an active-set method
Solves a sequence of convex QP problems

$$\min_{x} \quad c^T x + \tfrac{1}{2} x^T H x$$
$$Ax = b, \qquad l \le x \le u,$$

where c, H, A, b change (less and less)

Initially	H diagonal
Later	$H \leftarrow G^T H G, \quad G = (I + dv^T)$

Convex QP Solvers

Interior	Active-set	
LOQO		
HOPDM	SQOPT	reduced-Hessian
QPB	QPA	
CPLEX	QPBLU	
IPOPT	QPBLUR	

All based on KKT systems (except SQOPT)

SQOPT

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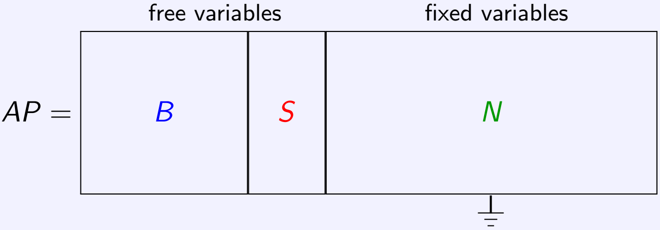
Reduced-gradient method (active-set method)



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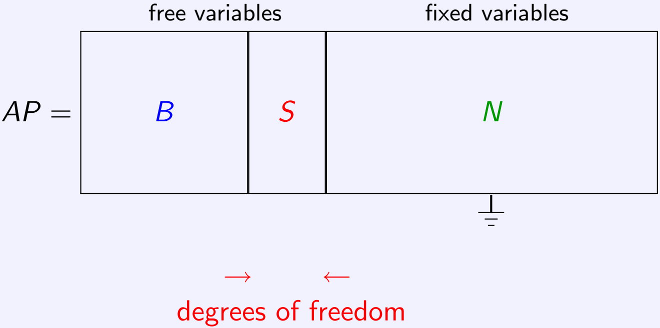

SQOPT

Any active-set method solves for search direction in Free variables:

$$\begin{pmatrix} H_F & A_F^T \\ A_F & 0 \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta y \end{pmatrix} \approx \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

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Reduced-gradient method uses a specific ordering (Gill et al. 1990):

$$A_F = \begin{pmatrix} B & S \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} H_{BB} & B^T & H_{BS} \\ B & & S \\ H_{SB} & S^T & H_{SS} \end{pmatrix}$$

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Reduced Hessian is Schur-complement of red block:

$$Z^T HZ = H_{SS} - (\dots)(\dots)^{-1}(\dots)^T$$

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Likely to be dense $\Rightarrow$ Need to work with original KKT system

SQOPT limitation

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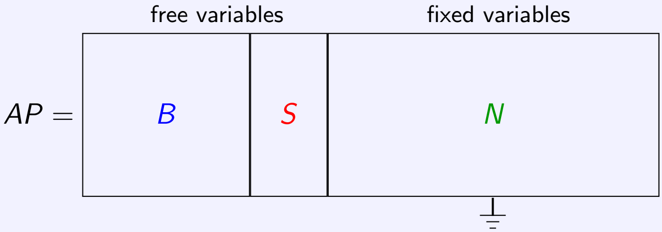

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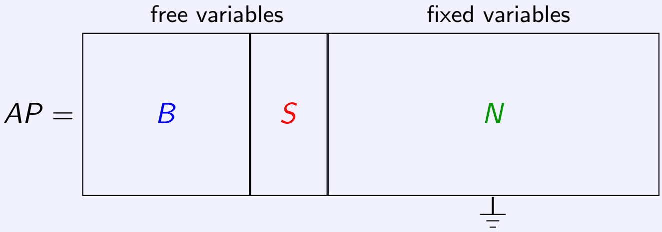

OK if 10000 2000 100000

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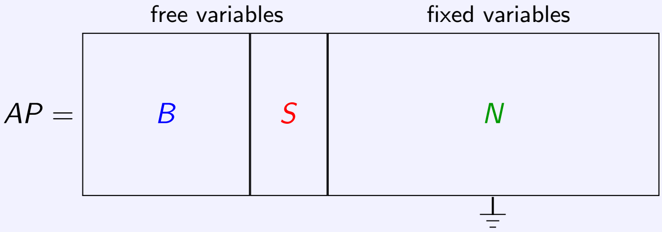

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What if	1M	1M	10M

QPBLU

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F90 convex QP solver based on block-LU updates of K

(Hanh Huynh 2008)

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- L_0, U_0 from LUSOL, MA57, PARDISO, SuperLU, UMFPACK

Singular KKTs

MA27, MA57, . . . , HSL_MA87 are rank-revealing if $u \geq 0.25$ (say)
For semidefinite QP,

$$K = \begin{pmatrix} D_1 & & A_1^T \\ & & A_2^T \\ A_1 & A_2 & \end{pmatrix}, \qquad D_1 \succ 0$$

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Theorem. K is nonsingular iff

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KKT Repair not yet implemented

Regularization

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- PDQ1 Interior method for LP (S 1996), OSL (S and Tomlin 1996)

$$\min_{x,y} \quad c^T x + \tfrac{1}{2}\gamma\|x\|^2 + \tfrac{1}{2}\delta\|y\|^2$$
$$Ax + \delta y = b, \qquad l \le x \le u$$

Indefinite $\mathrm{LDL^T}$ on KKT is stable if γ, δ sufficiently large

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- HOPDM Interior method for QP (Altman and Gondzio 1999)
  Smaller perturbation via *dynamic proximal point* terms:

$$\tfrac{1}{2}(x - x_k)^T R_p (x - x_k), \qquad \tfrac{1}{2}(y - y_k)^T R_d (y - x_y)$$

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- Can use Hanh's block-LU updates without change

Strategy

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MATLAB implementation

- Scale problem

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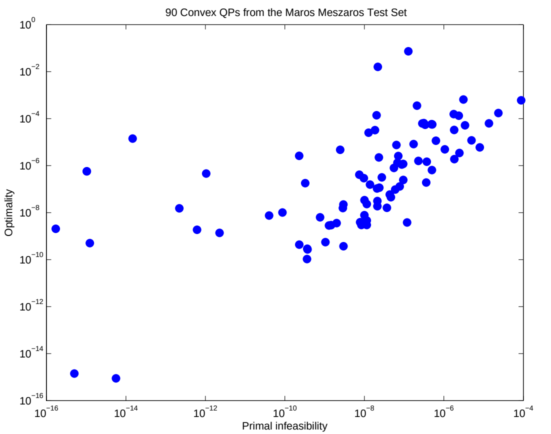
- Exit if small relative change in obj

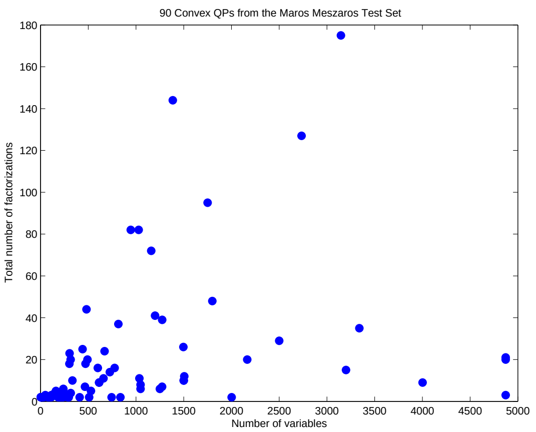
Numerical Results

Accuracy of solutions



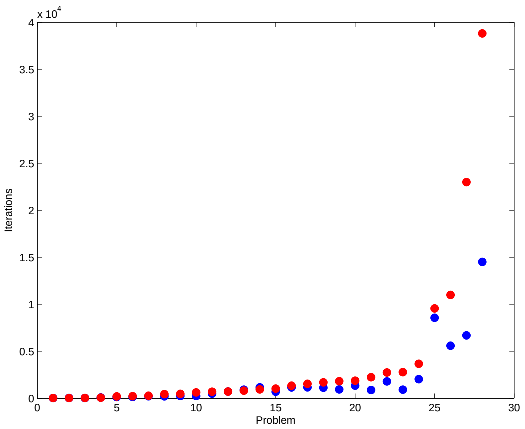
Residuals for 90 Meszaros QP test problems

KKT factorizations



No. of factorizations on 90 Meszaros QP test problems

Infeasible problems



No. of iterations on 28 infeasible LP test problems (Netlib lpi_xxx)

● $\mu \searrow 10^{-12}$ ● $\mu \equiv 1$

$$(\min\ c^Tx + \tfrac{1}{2}x^THx + \tfrac{1}{2}\delta x^Tx + \tfrac{1}{2}\mu y^Ty \quad Ax + \mu y = b,\ \ldots)$$

Conclusions

QPBLUR Pros and Cons

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Advantages

- Can start from any active set
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Disadvantages

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- Tiny regularization risks ill-conditioned KKT
  (but so far so good)

References

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Warmest thanks to organizers Howard, Chen, and Dominik