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idea of spatial filtering. By Brown's admission, her treatment does not consider the phase factor at the focal plane, but the observation plane is correctly placed at the geometrical image and reference is made to an article which includes the phase factor.[7] Neither Klein's text nor Brown's article has an extensive discussion of the result in the general case where the observation plane is not at the geometrical image.

In the future we plan to introduce the experiment on spatial filtering as before by having the students read Phillip's article. We plan this in part because the article excites interest. We also hope it will raise questions. In some cases this may require prodding, by insisting that the students understand observations made at positions other than that of the geometrical image. When questions are raised, we will refer students to Klein's text, Brown's article and to our notes. These will lead to other references, if needed.

[1] Richard A. Phillips, Amer. J. Phys. **37,** 546 (1969).
[2] Equations (1) and (2) are Eqs. (6) and (7) in Ref. 1, with a typographical error in Eq. (7) corrected.
[3] See any good optics text, e.g., John Strong, *Concepts of Classical Optics* (Freeman, San Francisco, 1958).
[4] This point of view provides another way of developing Eqs. (7) and (10). For a discussion of Fresnel diffraction when lenses are present, see Charles F. Meyer, *The Diffraction of Light, x-rays and Material Particles* (Univ. of Chicago Press, Chicago, 1934), pp. 101–106.
[5] Miles V. Klein, *Optics* (Wiley, New York, 1970).
[6] Judith C. Brown, Amer. J. Phys. **39,** 797 (1971).
[7] J. Rhodes, Amer. J. Phys. **21,** 337 (1953).

Elementary Derivation of Some of the Wigner–Witmer Rules*

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Forty-five years ago Wigner and Witmer,[1] using group theory, found the number of electronic terms of each symmetry of a diatomic molecule formed by bringing together specified terms of its separated atoms. The deepest part of their analysis concerned the assignment of g, u symmetry labels to the diatomic terms arising from like atoms in the same term. We have been unable to find an elementary (i.e., inelegant and non-group-theoretical) derivation of these rules; even Landau and Lifshitz[2] are content to quote the results. This note provides such a derivation.

We label the atomic centers a and b. On each is an n-electron atom in a state of a term with quantum numbers (l, s). We form the $(2l+1)^2(2s+1)^2$ properly antisymmetrized functions

$$[m_lm_s; m_l'm_s'] = A(m_lm_s; 1\cdots n)_a \times (m_l'm_s'; n+1\cdots 2n)_b, \quad (1)$$

where $(m_lm_s; 1\cdots n)_a$ is an atomic function, of the space-spin coordinates of electrons 1 through n, centered on a. We specify that each atomic function centered on b may be obtained from the corresponding atomic function on a by translating it from a to b. The quantization axis is chosen along ab.

Inversion of the electronic wavefunction in the midpoint of ab (denoted by I) is equivalent to inversion in one atomic center, followed by translation through an internuclear distance toward the other. Since each atomic state has the same parity under inversion through its own center, we find that

$$\begin{aligned} \mathrm{I}[m_lm_s; m_l'm_s'] &= A\mathrm{I}(m_lm_s; 1\cdots n)_a \times (m_l'm_s'; n+1\cdots 2n)_b \\ &= A(m_l'm_s'; n+1\cdots 2n)_a \times (m_lm_s; 1\cdots n)_b \\ &= (-1)^n A(m_l'm_s'; 1\cdots n)_a \times (m_lm_s; n+1\cdots 2n)_b \\ &= (-1)^{2s}[m_l'm_s'; m_lm_s]. \end{aligned} \quad (2)$$

The last equality follows because s is integral when n is even, half-integral when n is odd.

If $m_l \neq m_l'$, we can form two independent combinations $[m_lm_s; m_l'm_s'] \pm [m_l'm_s'; m_lm_s]$; Eq. (2) implies that one is g, one is u. The subspace spanned by the $(2s+1)^2$ g-functions with given m_l, m_l' is invariant under total spin **S**, as is that spanned by the $(2s+1)^2$ u-functions. Diagonalizing $\mathbf{S}^2$, we find one g state and one u state with each set of spin quantum numbers (S, M_S), $S=0, 1, \cdots, 2s$, $M_S=-S, \cdots, +S$. We do this for all sets $m_l \neq m_l'$ with fixed $|m_l+m_l'|=\Lambda$. We observe that if Λ is odd, this exhausts all states with that Λ. Therefore if Λ is odd, the number of $(\Lambda, S)_g$ terms is equal to the number of $(\Lambda, S)_u$ terms.

If Λ is even, the number of $(\Lambda, S)_g$ terms is one greater or one less than the number of $(\Lambda, S)_u$ terms depending on the symmetry under inversion of the (S, M_S) states formed from the $(2s+1)^2$ atomic functions with $m_l = m_l' = \Lambda/2$, denoted by $[m_s;\ m_s']$. We show that

$$I(S, M_S) = (-1)^S (S, M_S), \qquad (3)$$

which means that when Λ is even the "extra" (Λ, S) term is g or u as S is even or odd.

Equation (3) may be proved by induction: It is true for $S=2s$, because $(S=2s,\ M_S=2s)=[m_s=s;\ m_s'=s]$, and $I[s;\ s]=(-1)^{2s}[s;\ s]$ [Eq. (2)]. There are two states with $M_S=2s-1$; one of the combinations $[s;\ s-1]\pm[s-1;\ s]$ is even under I, the other is odd. Since I commutes with **S**, one is the $(2s,\ 2s-1)$ state, with parity $(-1)^{2s}$ under I; the other must be the $(2s-1, 2s-1)$ state which therefore has parity $(-1)^{2s-1}$ under I. There are three states with $M_S=2s-2$: $[s;\ s-2]\pm[s-2;\ s]$, and $[s-1;\ s-1]$. The first two have opposite parity; the third has parity $(-1)^{2s}$. The three states $(2s,\ 2s-2)$, $(2s-1,\ 2s-2)$, and $(2s-2,\ 2s-2)$ are formed by linear combination. As we have seen above, the first two have opposite parity; therefore $(2s-2, 2s-2)$ has parity $(-1)^{2s}\equiv(-1)^{2s-2}$, and so on.

Consider finally the Σ terms $(\Lambda=0)$. We must find the behavior under reflection in a plane containing the internuclear axis (denoted by R). Σ states are formed from the states (1) with $m_l'=-m_l$. Now

$$R[m_l m_s;\ -m_l m_s']=[-m_l m_s;\ m_l m_s'],$$

so

$$RI[m_l m_s;\ -m_l m_s']=(-1)^{2s}[m_l m_s';\ -m_l m_s].$$

That is, for fixed m_l the $(2s+1)^2$ states $[m_l m_s;\ -m_l m_s']$ behave under RI as the $(2s+1)^2$ states $[m_l m_s;\ m_l m_s']$ behave under I. Therefore RI acting on a state of a Σ term multiplies it by $(-1)^S$. If S is even, all Σ^+ terms are g and all Σ^- terms are u; if S is odd, all Σ^+ terms are u and all Σ^- terms are g.

This completes the derivation of the Wigner–Witmer correlation rules.

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† Alfred P. Sloan Fellow.

[1] E. Wigner and E. E. Witmer, Zeits. f. Physik **51**, 859 (1928).

[2] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison–Wesley, Reading, Mass., 1965), pp. 282–286.

A Corrupted Ballistic Pendulum

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The Blackwood ballistic pendulum apparatus is readily available at most schools. An effective method of modifying[1] the experiment that accompanies this piece of equipment is to remove the pendulum and use the spring gun portion in the following manner:

The ball is fired vertically. The zenith of the flight is estimated to the nearest centimeter. After approximately fifteen trials a mean height (h) and an estimate[2] of the standard deviation are computed. The students are asked to get the gun at approximately 45° and predict the point of impact. A ditto sheet is marked with concentric squares representing one, two, and three standard deviations in the value of the range. It is placed at the same evaluation as the ball to be launched with the center of the squares at the predicted range (R). The students are then challenged to place the ball on target *within three trials*.

Since the nominal predicted range value is 400 ± 4 cm, the students are apprehensive of their chances of success. They are genuinely surprised (relieved?) and pleased, however, at the effectiveness of their prediction. The pooled experimental standard deviation for a laboratory section of 12 to 16 students generally is 20% larger than the predicted value.

Of course such splendid results are deceptive and rest in part on the original experimental design. The range is given by the expression $R=2h\sin(2\theta)$ where θ is the angle of elevation of the launcher from the horizontal. The aggressive student should be encouraged to derive this formula. The other students should be shown the proof and be expected to sketch R as a function of θ. At $\theta=45°$ the range is not only a maximum, but the variation of range with angle is a minimum. The value of $\sin(2\theta)$ changes less than $\pm\frac{1}{2}\%$ for an angular difference of $\pm2°$. No protractor is necessary since a right triangle may be set up with meter sticks and the launcher accurately aligned within a fraction of a degree. The standard deviation in R, therefore, simplifies to twice the deviation in h. In practice the lateral alignment of the spring gun gives rise to a smaller scatter than the spread in the range values although a slight adjustment after the first shot is often necessary. It is important to clamp the spring launcher rigidly in position as even squeezing off a shot disturbs the alignment.

This exercise has been found effective for several reasons: