Uncertainty Quantification Using Evidence Theory

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Outline of the Presentation

- Historical perspective of risk assessment
- Aleatory and epistemic uncertainty
- Mathematical structure of evidence theory
- Example using evidence theory
- Future research and cultural change

Work in collaboration with Jon Helton and Jay Johnson,

consultants with Sandia National Laboratories.

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Communities that have Developed and Used Quantitative Risk Assessment

- **Nuclear power industry:**
  - **Began development of Probabilistic Risk Assessment (PRA) in early 1970s**
  - **Focused on severe accidents of nuclear reactors**
  - **Significant improvements in procedures with NUREG-1150: "Severe Accident Risks: An Assessment for Five U.S. Nuclear Power Plants" (1990)**
- **Underground storage of nuclear waste:**
  - **Waste Isolation Pilot Plant (WIPP) for transuranic wastes**
  - **Yucca Mountain Project (YMP) for high level radioactive wastes**
- **US Nuclear weapons:**
  - **Safety and reliability assessment at Sandia**
- **NASA:**
  - **Space Shuttle**
  - **Space Station**

Quantitative Risk Assessment

- **Quantitative risk assessment (QRA) tries to answer the questions:**
  1) **What can go wrong?**
  2) **How likely is it to go wrong?**
  3) **What are the consequences of going wrong?**
  4) **What is the confidence in the answers to each of the first three questions?**
- **In answering these questions for formal QRAs:**
  - **Assumptions are clearly stated and appropriate justification is given**
  - **Initiating events, fault trees, and event trees are constructed**
  - **Likelihoods are typically quantified using probability theory**
  - **A sensitivity analysis is commonly conducted**
  - **Entire analysis is documented**
- **Some examples of successful application of QRA:**
  - **Nuclear power plants**
  - **Gas turbine engines**

Some Weaknesses in Applying QRA

- Poor understanding and characterization of what operators will do in an accident scenario
  - Example: Chernobyl and Three Mile Island
- Poor understanding of the kinds of failure modes that might be introduced due to computer hardware and software failures
  - Example: Mars rover and Space Shuttle main engine controller
- Inappropriate representation and mixing of variabilities and uncertainties
  - Example: Initial estimates of Space Shuttle loss as 1 in 100,00 flights
- Poor quantification of an organization’s “safety culture” or maintenance procedures
  - Example: Many losses of commercial airliners and loss of Columbia

Aleatory Uncertainty and Epistemic Uncertainty

- **Aleatory uncertainty is an inherent variation associated with the physical system or the environment**
  - **Also referred to as variability, irreducible uncertainty, and stochastic uncertainty, random uncertainty**
- **Examples:**
  - **Variation in atmospheric conditions and angle of attack for inlet conditions**
  - **Variation in fatigue life of compressor and turbine blades**
- **Epistemic uncertainty is an uncertainty that is due to a lack of knowledge of quantities or processes of the system or the environment**
  - **Also referred to as subjective uncertainty, reducible uncertainty, and model form uncertainty**
- **Examples:**
  - **Lack of experimental data to characterize new materials and processes**
  - **Poor understanding of coupled physics phenomena**
  - **Poor understanding of initiating events, fault trees, and event trees**

Methods for Representing Aleatory and Epistemic Uncertainties

- **Common procedure is not to separate aleatory and epistemic uncertainties:**
  - **Represent epistemic uncertainty with a uniform probability distribution**
  - **For a quantity that is a mixture of aleatory and epistemic uncertainty, use second-order probability theory**
- **It is slowly being recognized that the above procedures (especially the first) can underestimate uncertainty in:**
  - **Physical parameters**
  - **Geometry of a systems**
  - **Initial conditions**
  - **Boundary conditions**
  - **Scenarios and environments**

and can result in large underestimation of uncertainty

in system responses

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Possible Approaches to Better Representation of Epistemic Uncertainty

- **Traditional probability theory with strict separation of aleatory and epistemic uncertainty**
  - **Treat epistemic uncertainty as possible realizations with no probability associated with those realizations obtained from sampling**
- **Fuzzy set theory**
  - **Major difficulties with quantifying linguistic uncertainty**
  - **Can not combine fuzzy sets with probabilistic information**
- **Possibility theory**
  - **No clear method for combining degrees of belief and probabilistic information**
- **Evidence theory**
  - **Can correctly represent epistemic uncertainties from intervals, degrees of belief, and probabilistic information**
  - **Early criticism misdirected at Dempster's rule of aggregation of evidence**
  - **Early in development and use for complex engineering systems**

Mathematical Structure of Evidence Theory

- **Let the universal set (or sample space) be defined as**

$$\mathscr{X} = \{x : x \text{ is a possible value of the uncertain quantity}\}$$

- **Based on the information available concerning uncertain quantities, a basic probability assignment (BPA) can be defined as**

$$m(\mathscr{E}) \geq 0 \text{ for } \mathscr{E} \subset \mathscr{X}$$

$$\sum_{\mathscr{E} \subset \mathscr{X}} m(\mathscr{E}) = 1$$

- **Then the focal elements of the uncertain quantities are defined as**

$$\mathbb{X} = \{\mathscr{E} : \mathscr{E} \subset \mathscr{X}, m(\mathscr{E}) > 0\}$$

- **Then the plausibility function can be defined as**

$$Pl(\mathscr{E}) = \sum_{\mathscr{U} \cap \mathscr{E} \neq \emptyset} m(\mathscr{U})$$

- **And the belief function can be defined as**

$$Bel(\mathscr{E}) = \sum_{\mathscr{U} \subset \mathscr{E}} m(\mathscr{U})$$

Contrasts of Traditional Probability Theory and Evidence Theory

- **Traditional probability theory**

$$(\mathscr{X}, \mathbb{X}, prob_x)$$

$$prob(\mathscr{E}) + prob(\mathscr{E}^c) = 1$$

- **The least information that is typically stated for an uncertain quantity is**

$$prob_x \sim uniform\ distribution$$

- **Evidence theory**

$$(\mathscr{X}, \mathbb{X}, m_x)$$

$$prob(\mathscr{E}) + prob(\mathscr{E}^c) = 1$$

$$Pl(\mathscr{E}) + Pl(\mathscr{E}^c) \geq 1$$

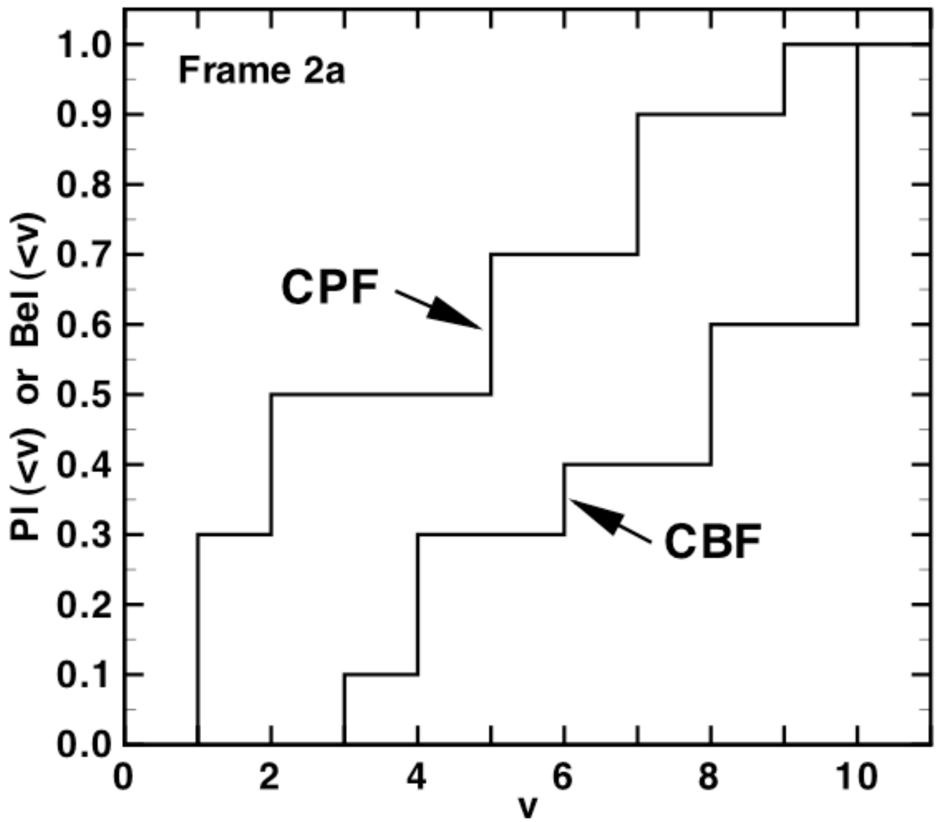
$$Bel(\mathscr{E}) + Bel(\mathscr{E}^c) \leq 1$$

- **The least information that can be stated for an uncertain quantity is**

$$m_x([a,b]) = 1$$

In evidence theory, likelihood is assigned to sets, as opposed to probability theory where likelihood is assigned to a probability density function.

Cumulative Plausibility Function and Cumulative Belief Function



- Given the same information on an uncertain quantity, in probability theory and evidence theory, it can be shown that

$$CBF(v) \leq prob(v) \leq CPF(v)$$

- CPF and CBF can be view as upper and lower probabilities of possible values

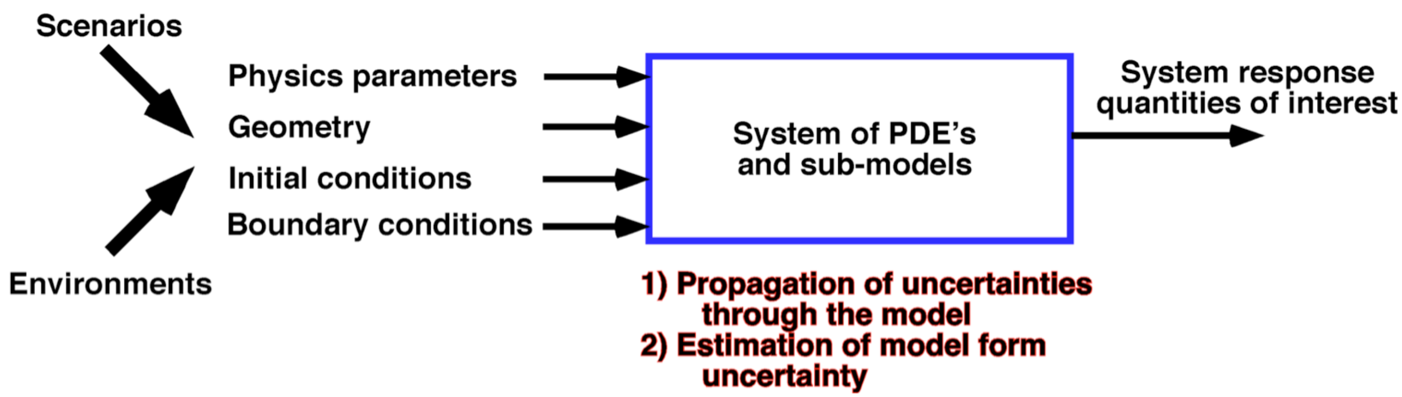
Mapping of Inputs to Outputs Through a Mathematical Model

- **The propagation of input quantities through a mathematical model to obtain outputs can be written as**

$$y = f(\vec{x})$$

  - **where $\vec{x}$ is a vector of n input quantities**
  - **f is the mathematical model describing some physical process**
  - **y is a scalar output quantity**

- **f is typically a solution of nonlinear partial differential equation that is solved numerically**



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Propagation of Uncertainty Structures Through a Model

- We are generally interested in mapping input structures to output structures

$$uncertainty\ in\ \vec{x} \rightarrow uncertainty\ in\ y$$

- The mapping is commonly done by random sampling (Monte Carlo or Latin hypercube) of the mathematical model, i.e., each sample requires a solution of the PDE for the specified quantities
- For traditional probability theory, regardless of n:
  - For the mean of y, 10 samples are typically required
  - For low probability values of y, the number of samples required is typically on the order of 1/$prob(y)$
- For evidence theory:
  - For Pl(y) and Bel(y), 10n samples are typically required
  - For low probability values of y, on the order of $n/prob(y)$ samples are required

Example Using Evidence Theory

- **Challenge Problems were constructed and published to:**
  - **Focus debate on epistemic uncertainty issues in uncertainty quantification**
  - **Better understand the effect of assumptions commonly made in uncertainty quantification analyses**
  - **Move toward agreement on the most effective ways of representing uncertainty for decision makers**
- **One set of Challenge Problems was based on the model:**

$$y = (a + b)^a$$

  - y **is the system response**
  - a **and** b **are uncertain independent parameters**
  - a **and** b **are positive, real numbers, specified over a given range**
  - **Multiple, conflicting, sources are offered for** a **and** b
  - **Sampling-based methods are suggested to estimate the uncertainty in** y

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System Response Characteristics

- **For Problem 3b:**
- **Three sources for *a*:**

$$\mathscr{A}_1 = [0.5, 1.0]$$

$$\mathscr{A}_2 = [0.2, 0.7]$$

$$\mathscr{A}_3 = [0.1, 0.6]$$

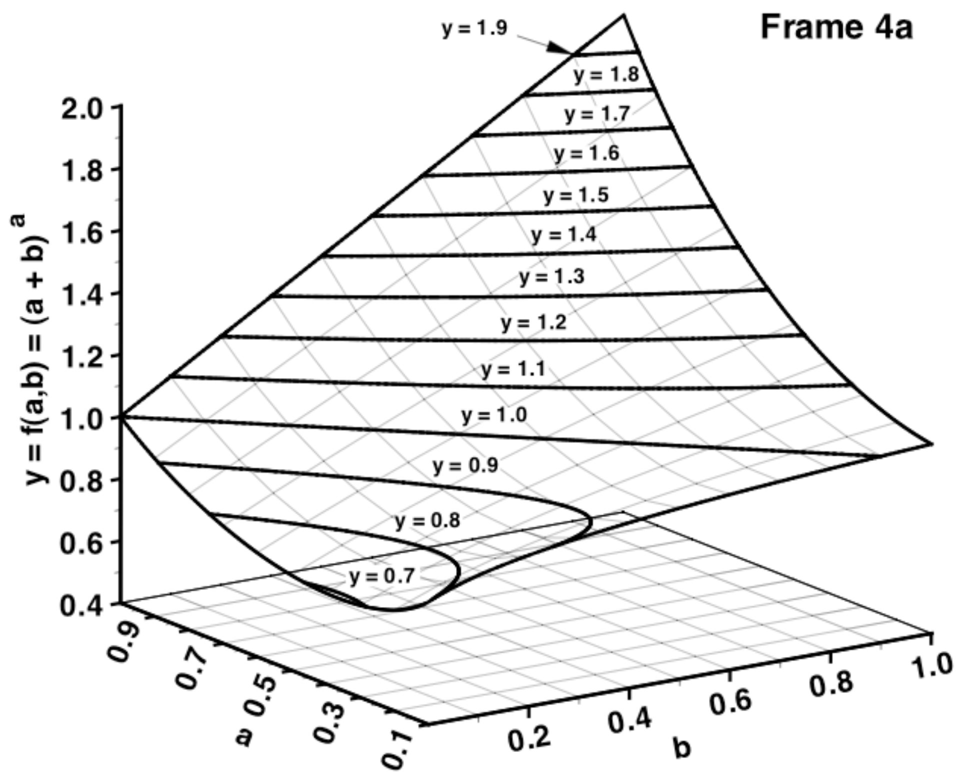
- **Four sources for *b*:**

$$\mathscr{B}_1 = [0.6, 0.6]$$

$$\mathscr{B}_2 = [0.4, 0.8]$$

$$\mathscr{B}_3 = [0.1, 0.7]$$

$$\mathscr{B}_4 = [0.0, 1.0]$$



Universal Set and Aggregation of Evidence for *a* and *b*

- **The universal set (sample space) for *a* is**

$$\mathscr{A} = \bigcup_{r=1}^{3} \mathscr{A}_r$$

- **The universal set (sample sample) for *b* is**

$$\mathscr{B} = \bigcup_{r=1}^{4} \mathscr{B}_r$$

- **Method for aggregation of conflicting evidence:**
  - **The total range of uncertainty possible from all of the sources should be preserved**
  - **All sources are weighted equally**
  - **If multiple sources agree on certain ranges of a parameter, then these ranges should have increased credibility**

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Focal Elements and Basic Probability Assignments for *a* and *b*

- **The set of focal elements of *a* is**

$$\mathbb{A} = \{\mathscr{A}_r : r = 1, 2, 3\}$$

- **The set of focal elements of *b* is**

$$\mathbb{B} = \{\mathscr{B}_s : s = 1, 2, 3, 4\}$$

- **The basic probability assignments for *a* are**

$$m_A(\mathscr{A}_1) = m_A(\mathscr{A}_2) = m_A(\mathscr{A}_3) = 1/3$$

- **The basic probability assignments for *b* are**

$$m_B(\mathscr{B}_1) = m_B(\mathscr{B}_2) = m_B(\mathscr{B}_3) = m_B(\mathscr{B}_4) = 1/4$$

Evidence Space for the Input Parameters *a* and *b*

- Let $\mathscr{X}$ be the evidence space for all of the uncertain input parameters

$$\mathscr{X} = \mathscr{A} \times \mathscr{B} = \{\vec{\mathrm{x}} : \vec{\mathrm{x}} = [a,b], a \in \mathscr{A}, b \in \mathscr{B}\}$$

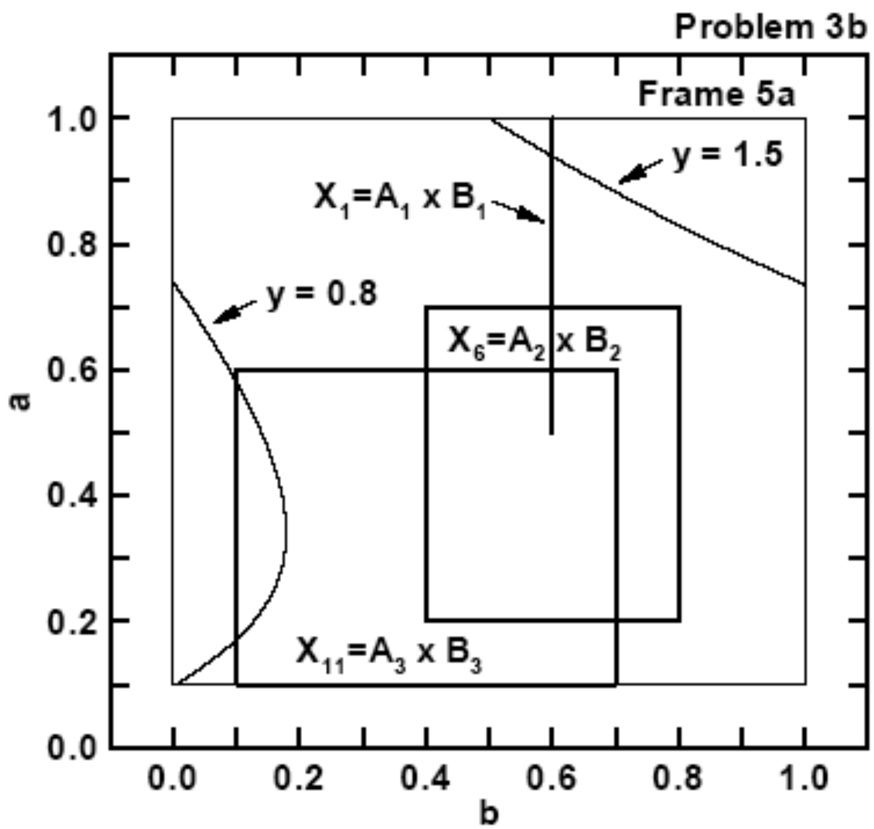
- $\mathscr{X}$ is formed by the product space of all uncertain input parameters
- Let $\mathbb{X}$ be the focal element space for all of the uncertain input parameters

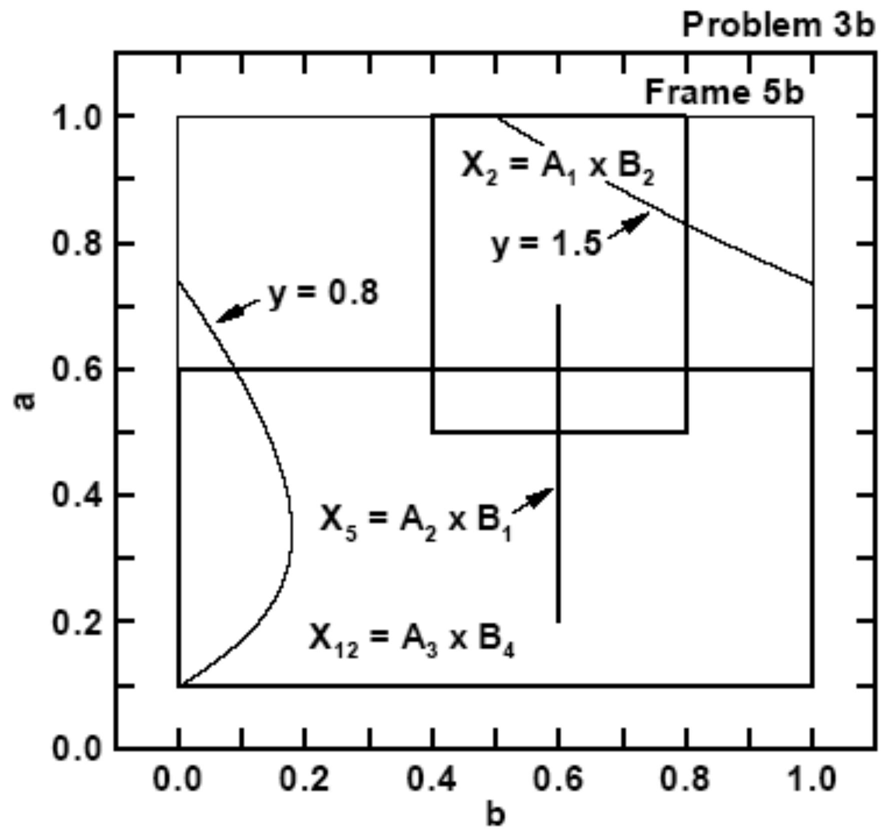
$$\mathbb{X} = \mathbb{A} \times \mathbb{B}$$

- The set $\mathbb{X}$ contains 12 sets (3 sets of $\mathbb{A}$ x 4 sets of $\mathbb{B}$)
- Similarly, the product space for the basic probability assignment is

$$m_X(\mathscr{E}) = \begin{cases} m_A(\mathscr{A}_r) m_B(\mathscr{B}_r) & for \quad \mathscr{E} = \mathscr{A}_r \times \mathscr{B}_s \in \mathbb{X} \\ 0 & otherwise \end{cases}$$

Sets $\mathscr{X}_1$, $\mathscr{X}_6$, $\mathscr{X}_{11}$ and Sets $\mathscr{X}_2$, $\mathscr{X}_5$, $\mathscr{X}_{12}$





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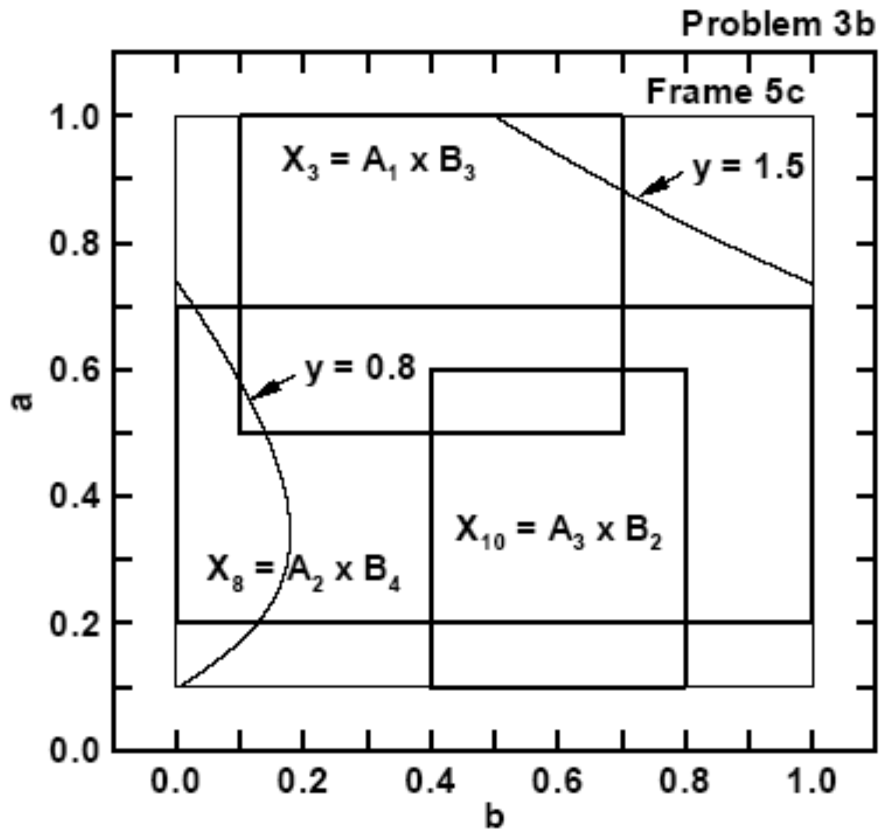
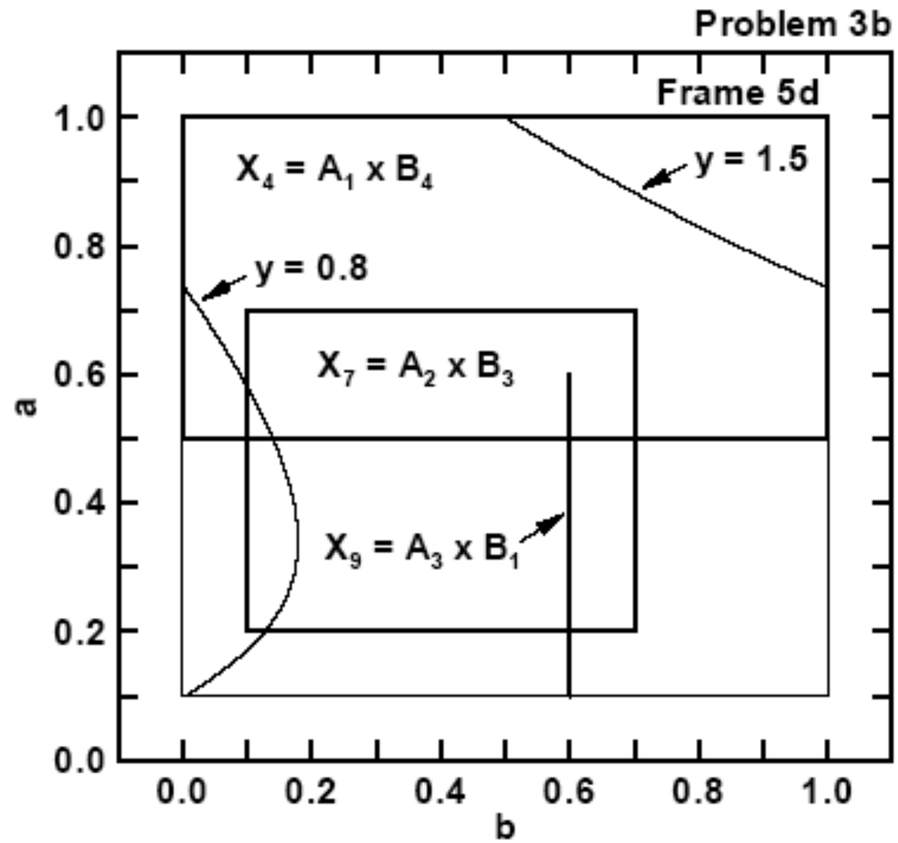
Sets $\mathcal{X}_3$, $\mathcal{X}_8$, $\mathcal{X}_{10}$ and Sets $\mathcal{X}_4$, $\mathcal{X}_7$, $\mathcal{X}_9$





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Propagation of Input Uncertainty to Output Uncertainty

- A four step numerical sampling method is used:
  1) Define a probability distribution on $\mathscr{X}$ so that input samples can be drawn (a uniform distribution over $\mathscr{X}$ is typically used.)
  2) Generate a random or Latin hypercube sample $\vec{x}_k,\ k = 1, 2, \ldots n$ from $\mathscr{X}$, using the assumed distribution from step 1.
  3) Using the specified samples, numerically compute the mapping of the input space to the output space

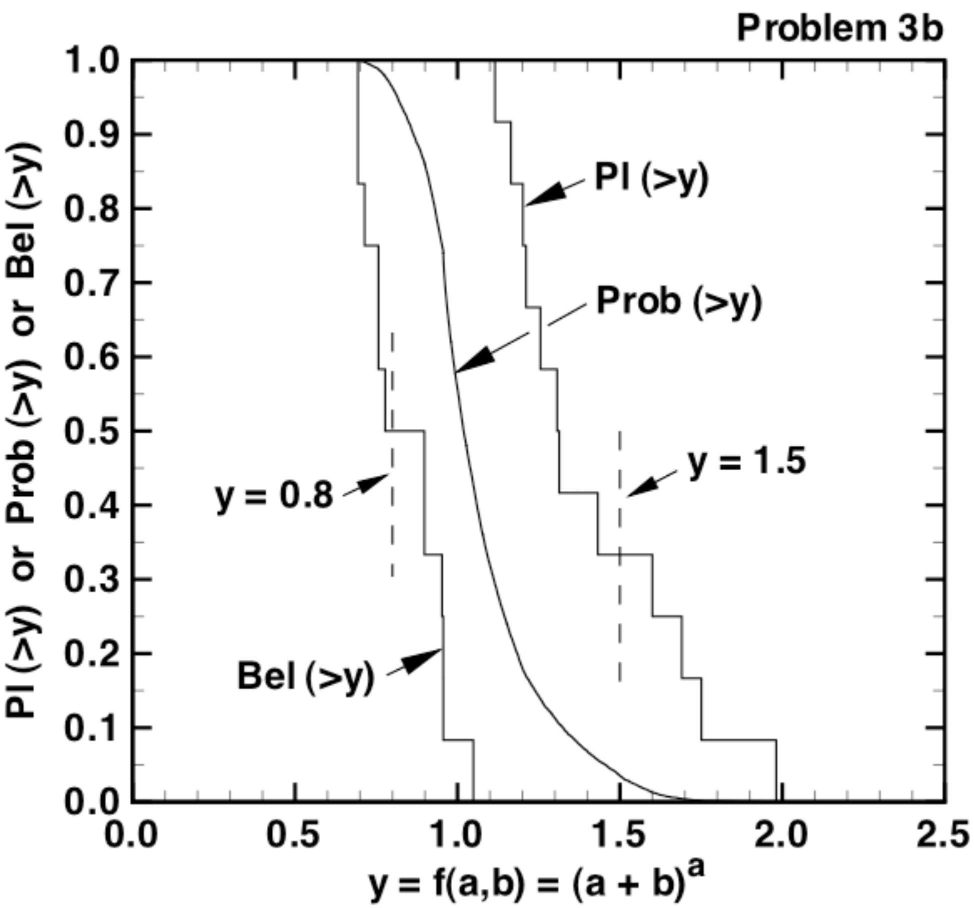
$$y_k = f(\vec{x}_k),\ k = 1, 2, \ldots n$$

  4) Compute the cumulative plausibility function (CPF) and the cumulative belief function (CBF)

$$\mathscr{CPF} \cong \left\{ \left[y_k, Pl_X(\vec{x}_j : y_j \le y_k) \right], k = 1, 2, \ldots n \right\}$$

$$\mathscr{CBF} \cong \left\{ \left[y_k, 1 - Pl_X(\vec{x}_j : y_j > y_k) \right], k = 1, 2, \ldots n \right\}$$

Complementary Cumulative Plausibility and Belief Functions



- **It can be shown that**

$$CCBF(\mathscr{Y}_v) \leq CCDF(\mathscr{Y}_v) \leq CCPF(\mathscr{Y}_v)$$

- **Suppose that for safety requirements of the system, we must have**

$$Prob(y \geq 1.5) < 0.2$$

- **What can be said about system safety?**
  - **From traditional probability**
  - **From evidence theory**

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Future Research Needs In Evidence Theory

- Improved methods for constructing basic probability assignments based on:
  - Expert opinion
  - Mixtures of experimental data and expert opinion
- Improved understanding of what aggregation methods should be used for various situations with conflicting expert opinion
- Improved understanding of types of dependence between epistemic uncertainties
- Improved sampling methods to propagate input uncertainty structures to output uncertainty structures:
  - Convergence acceleration methods using sensitivity analysis
  - Bounding methods
- Methods of conducting sensitivity analyses in evidence theory:
  - Input uncertainties can not necessarily be ordered with respect to importance on output uncertainties

Needed Changes in Engineering Culture

- Continue to move away from safety margin concepts to quantitative risk assessment concepts
- Improved recognition and segregation between aleatory and epistemic uncertainties
- Improvement in the analyst culture to address and quantify uncertainties in computational analyses
- Use system inspection and maintenance records and “close calls” to improve uncertainty quantification and system safety
- Improved separation between organizations responsible for system safety and reliability and organizations responsible for programmatic issues (cost, schedule, and performance)

“The safety organization sits right beside the person making the decisions, but behind the safety organization, there’s nothing back there. There’s no people, money, engineering expertise, analysis.” (Admiral Gehman).

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