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STOCHASTIC INCOMPLETENESS OF QUANTUM MECHANICS

I. INTRODUCTION

The purpose of this article is to bring out in as conceptually clear terms as possible what seems to be a major incompleteness in the probability theory of particles offered by classical quantum mechanics. The exact nature of this incompleteness is illustrated by consideration of some simple quantum-mechanical examples. In addition, these examples are contrasted with the fundamental assumptions of Brownian motion in classical physics on the one hand, and with a controversy of a decade ago in mathematical psychology. (The psychological example is described in detail in the appendix.) Our central claim is that classical quantum mechanics is radically incomplete in its probabilistic account of the motion of particles.

In the last part of the article we derive the time-dependent joint distribution of position and momentum of the linear harmonic oscillator, and show how the apparently physically paradoxical statistical independence of position and momentum has a natural explanation. The explanation is given within the framework of the non-quantum-mechanical stochastic theory we construct for such oscillators.

Before entering into any technical details, we shall attempt to spell out in intuitive terms our incompleteness claim. To begin with, there are two senses of completeness that we want to distinguish. One sense of completeness is that demanded by any hidden variable theories of quantum mechanics. States should be found, or be theoretically characterizable at least, that lead to precise values for all observables, that is, no observable is to have a variance greater than zero. This is the sense of completeness that derives from classical deterministic physics; it is not the sense of completeness we are concerned with here.

A second stochastic sense of completeness is that for a family of observables, or a given observable in the more restricted case, all probability questions are resolved by the theory. This of course leaves the

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matter somewhat vague, for even in the most classical settings, for good reason the probability of any subset of the sample space is not considered as having a probability, but only some appropriate σ-algebra of events, and the same is reasonable in quantum mechanics.

A good example of stochastic completeness, but nondeterministic completeness, can be illustrated by the simple case of coin flipping, where the sample space of observables is the set of all possible infinite sequences of heads and tails. Back of the sample space we postulate a state space, with the 'true' probability of the coin coming up heads on a given occasion being a number p between zero and one. A priori we may have some distribution on the unit interval for the true value of p and by experimentation we expect to determine, or to be able to estimate, its true value. Whether we can determine the exact value of p or not, given a value of p, the observables in which we are interested do not become dispersion free, but retain a variance not equal to zero. On the other hand, once the parameter p is fixed or known, all stochastic questions of probability, for example, all joint probabilities over time, are completely determined for the standard σ-algebra of events. This theory, in contrast to the quantum-mechanical examples we shall consider, is stochastically complete.

We also emphasize that we do not have in mind still another discussion of the nonexistence of joint distributions in quantum mechanics, but rather are concerned with the stochastic or temporal character of the probability distributions that can be derived in quantum mechanics and with the extent to which they can be regarded as giving a stochastically complete theory of the motion of a particle.

There is one other point worth clarifying. In the application of continuous-time, continuous-state stochastic processes, it is customary to compute only certain probabilistic quantities and not the full range of what is possible, because of the complexity and difficulty of doing so. The fact that only partial computations are ordinarily made does not disturb in any way our theoretical claim. It still remains pertinent to ask for the characterization of the stochastic process that determines a unique probability measure for any temporal sequence of events we may wish to consider.

It may be responded that one of the characteristic features of quantum mechanics is that proper stochastic processes do not arise. This is exactly the thesis of our article, but we consider it a defect and not a merit of quantum mechanics. In addition, it is our conceptual claim that the difficulties that arise from not having a fully specified stochastic process, even in the simplest cases, provide evidence of the incompleteness of the theory and the extent to which the physical grounds of the theory have not yet been fully thought out. Of course, if one were to take literally a crudely positivistic viewpoint and were prepared only to consider that which is observable, a defense could be made for the incompleteness of quantum mechanics, but we think in practice no one does this. Conceptually it is intuitively impossible to think about quantum-mechanical problems without considering the motion of particles, various constants of the motion, the position of a particle at a given instant, and so forth. Once the motion of particles is discussed and conceptually thought about, it becomes natural to ask for a characterization of the stochastic process that determines a unique measure on the possible sample paths of motion.

Four possibilities remain open. One is that by sufficient effort the quantum-mechanical theory of particles can be made stochastically complete. We conjecture that this is probably not the case, but it is not an easy matter to settle in any definitive way. A second possibility is that although the quantum-mechanical theory is incomplete, it may be completed stochastically in a wide variety of ways that are mathematically consistent with those results that can be derived from quantum mechanics. The third possibility is that quantum mechanics is stochastically incomplete, but that a proper extension to stochastic completeness is impossible and leads to both mathematical and observational inconsistencies. Undoubtedly, many think that this is the case.

Again, the issue is not at all the same as the issue concerning hidden variable theories. We do not believe that this is the situation, and we know of no proofs in the literature that suggest that this is the situation.

The fourth possibility is like the third in that proper extensions lead to mathematical inconsistencies, but different in that stochastically complete theories mathematically inconsistent with quantum mechanics but observationally equivalent to quantum mechanics can be formulated.

Thus, in our view quantum mechanics is stochastically incomplete, but it can, at least in many reasonable cases, be stochastically completed either in the sense of the second or fourth possibility.

Perhaps more fundamental and important is that qualitative methods of deriving the appropriate differential equations be found if our thesis is correct, and that these methods be used to test the reliability of a stochastic approach to quantum mechanics. The earlier work referred to later in the article has not yet sufficiently explored this avenue in our judgment, and it remains to be seen to what extent a genuine stochastic theory of quantum phenomena beginning from qualitative postulates is possible.

II. TIME-DEPENDENT PROBABILITY DISTRIBUTION OF A SINGLE OBSERVABLE

To provide a setting for the present discussion, we recall without much exposition some familiar facts; we use the familiar formalism and notation of classical quantum mechanics.

We write the time-dependent Schrödinger equation in the form of Equation (1), and we assume that the wave function Ψ is normalized to one.

$$\text{(1)} \qquad \frac{i\hbar\,\partial\Psi(x,t)}{\partial t}=H\Psi(x,t)$$

The expectation of a given operator A is given by the inner product as expressed in the following equation:

$$\text{(2)} \qquad E(A)=(\Psi, A\Psi).$$

If we replace A by $\exp(iuA)$ then we obtain the characteristic function of the probability distribution of the observable A:

$$\text{(3)} \qquad \varphi(u)=E(e^{iuA})=(\Psi, e^{iuA}\Psi).$$

Without entering into details of calculation now (see Sec. V), we obtain by these methods the time-dependent distribution of an observable for various elementary cases. For example, in the case of a one-dimensional linear harmonic oscillator, we obtain the following probability density for position x at each time t:

$$\text{(4)} \qquad f(x,t)=\frac{\alpha}{\pi^{1/2}}\,e^{-\alpha^2(x-a\cos\omega t)^2}.$$

The various constants occurring in this equation are not of particular interest and are not discussed, except to say that the constant a is related to the averaging over stationary states to obtain the 'mean' probability density. A detailed derivation of (4) is to be found in Schiff (1949, pp. 60–69).

As a different simple example, consider the probability density for the one-dimensional free particle, that is, the particle for which the potential $V=0$. It may be shown that $f(x, t)$ has the following form:

$$f(x, t)=\left(2\pi\left(\sigma_0^2+\frac{\hbar^2 t^2}{4m^2\sigma_0^2}\right)\right)^{-1/2} \exp\left(-\frac{x^2}{2(\sigma^2+\hbar^2 t^2/4m^2\sigma_0^2)}\right) \tag{5}$$

where $\sigma_0^2=\mathrm{var}(X)$ evaluated at $t=0$. In contrast to the case of the linear harmonic oscillator, the variance of whose distribution is constant, in the case of the free particle, the variance of the distribution, or, as physicists would put it, the uncertainty, increases monotonically in time from $t=0$ in both past and future directions.

What is important about Equations (4) and (5) is that they represent results that in no way depend on dubious assumptions about how to take the expectation of sums of noncommuting variables or the expectation of the characteristic function of a pair of such variables. They depend only upon a conservative interpretation of the formalism, and one that we believe would be accepted by everyone.

We thus have in classical quantum mechanics a general theory of how to get the distribution of position through time, and Equations (4) and (5) are simple examples in two manageable cases. It might therefore seem that the probabilistic theory within quantum mechanics of a single operator or observable is well established and unproblematical in character. We want to show that this is not the case.

The difficulties lie not with equations like (4) and (5), but rather with the fact that they represent the limit in a general way of what can be derived. This means that an underlying stochastic process for a single observable is not determined by the theory, and consequently, a high degree of stochastic incompleteness is a central feature of classical quantum mechanics. One intuitive way of putting the matter is that in equations like (4) and (5) we have the mean distribution through time, but nothing like the characterization of the full process.

To make our central point as forcefully as we can, we adopt three parallel lines of attack. The first is conceptually the simplest; it illustrates the situation by reference to mean learning curves in mathematical psychology. There is an exact parallel to the quantum-mechanical case, but the mathematical formulation is in every respect completely elementary. This argument is given in the appendix. The second line of attack is to contrast the probabilistic results in quantum mechanics with the classical theory of Brownian motion. Finally, the third line of attack, and in certain ways the most interesting, is to construct a classical stochastic theory of the linear harmonic oscillator and to examine its physical interpretation, especially as an extension of the incomplete quantum-mechanical theory.

III. CLASICAL THEORY OF BROWNIAN MOTION

An explicit and rigorous mathematical theory of Brownian motion has been one of the accomplishments of twentieth-century probability theory. We do not attempt to give a complete formulation, but we can convey the essential conceptual assumptions of the theory. An excellent treatment at an introductory level is found in Karlin (1966). Perhaps the best reference for a thorough study of such processes is Ito and McKean (1965).

Brownian motion as a physical phenomenon was discovered by the English botanist Brown in 1827, and, in one of his most important papers, Einstein in 1905 derived a mathematical description of the phenomenon from basic laws of physics. Let us restrict ourselves to one dimension, and so let $X(t)$ be the random variable at time t whose value is the x component of the particle in Brownian motion. Let x_0 be the position of the particle at time t_0, and let $p(x, t \mid x_0)$ be the conditional probability density of the particle at time $t+t_0$, i.e., the probability density of the random variable $X(t+t_0)$. Einstein argued from physical principles that this conditional density must satisfy the following partial differential equation.

$$\frac{\partial p(x, t \mid x_0)}{\partial t} = D \frac{\partial^2 p(x, t \mid x_0)}{\partial^2 x}, \tag{6}$$

where D is the diffusion coefficient; in the literature this equation is called the diffusion equation. By choosing an appropriate scale, we may take $D=\frac{1}{2}$, and then we can show that the following is the solution of Equation (6):

$$p(x, t \mid x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x-x_0)^2\right), \tag{7}$$

which shows that the conditional density for each time t is a standard Gaussian or normal distribution. Equation (7) is, of course, exactly the sort of equation we obtain in quantum mechanics, e.g., Equations (4) and (5). If this is all that was to be said we would not have established any difference between the classical theory of Brownian motion and quantum mechanics, but of course the probability density function given by (7) is only the beginning of the complete description of the Brownian motion process. The full theory is embodied in the following definition, which we take from Karlin (1966):

DEFINITION. *Brownian motion is a stochastic process* $\{X(t); t \geqslant 0\}$ *with the following properties:*

(a) *Every increment* $X(t+s) - X(s)$ *is normally distributed with mean* 0 *and variance* ct; $c>0$ *and* c *is a constant independent of* t;

(b) *For every pair of disjoint time intervals* $[t_1, t_2]$, $[t_3, t_4]$, *say* $t_1 < t_2 \leqslant t_3 < t_4$, *the increments* $X(t_4) - X(t_3)$ *and* $X(t_2) - X(t_1)$ *are independent random variables with distributions given in* (a), *and similarly for* n *disjoint time intervals where* n *is an arbitrary positive integer.*

From the basic assumptions of this definition, we can derive the joint density f of any finite sequence of random variables $X(t_1), X(t_2), \ldots, X(t_n)$, with $t_1 < t_2 \cdots < t_n$. In particular, it can easily be shown that

$$f(x_1, \ldots, x_n) = p(x_1, t_1)\, p(x_2 - x_1, t_2 - t_1) \ldots p(x_n - x_{n-1}, t_n - t_{n-1}), \tag{8}$$

where

$$p(x, t) = \frac{1}{\sqrt{2\pi c t}} \exp(-x^2/2ct). \tag{9}$$

It is worth noting that Equation (8) satisfies, in a direct way, the standard conditions that an indexed set of random variables be a stochastic process. Suppose for the present discussion that the indexed set is an interval of real numbers, perhaps the entire set of real numbers T, and for each t in T, there is a random variable $X(t)$. If every finite sequence of random variables indexed on T has a well-defined joint probability distribution, then the entire family of joint distribution functions defines a stochastic process $\{X(t), t \in T\}$.

It is apparent that we are extraordinarily far from realizing this condition for having a stochastic process in any of the standard quantum-mechanical cases in which we have only a mean probability density of the sort exemplified by Equations (4) and (5) for the linear harmonic oscillator and the free particle.

What is important here is not that each well-defined example of quantum mechanics must be extended to become a well-defined stochastic process, but rather the query of what is meant to be the physical interpretation of the probability densities exemplified by Equations (4) and (5). Physicists talk as if the particle has a motion; in fact, we suspect that it would be extraordinarily difficult to eliminate this talk about the motion of a particle from physical discussions of any cases of interest in quantum mechanics, and yet, the theory does not have a natural physical base in terms of the motions of the particles.

We consider this problem a more severe conceptual problem for the clear understanding of quantum mechanics than the general problem of not having joint distributions for noncommuting observables.

Some interesting attempts have been made to develop a stochastic process theory of quantum-mechanical particles as an alternative to standard quantum mechanics. Perhaps the conceptually clearest example of such attempts is provided by Nelson (1967). Nelson explores several simple, but interesting, cases and shows how the Schrödinger equation can be derived from the Markov stochastic process defined for the given physical situation, and correspondingly, how the Markov process can be derived from the Schrödinger equation.

So far as we know, however, there has been no attempt to explore the many different alternatives that are consistent with a given mean probability density, or to examine the physical plausibility of these alternatives.

We now turn to the examination of such an example.

IV. DISTINCT MODELS FOR THE SAME MEAN PROBABILITY DENSITY FUNCTION OF THE HARMONIC OSCILLATOR

We first state assumptions that should be satisfied by any stochastic process $\{X(t); -\infty < t < \infty\}$ that has as a consequence the mean Equation (4) for the linear harmonic oscillator.

(i) *The process is Markovian*, i.e., if $X_{t_1}, \ldots, X_{t_n}$ is a finite sequence of random variables with $t_1 < t_2 < \cdots < t_n$, then

$$F(X_{t_n} \mid X_{t_{n-1}}, X_{t_{n-2}}, \ldots, X_{t_1}) = F(X_{t_n} \mid X_{t_{n-1}}),$$

where F is the distribution function. (When the densities exist, we may replace F by f.)

(ii) *The process is Gaussian*, i.e., any finite sequence of random variables as defined in (i) has a multivariate normal (or Gaussian) distribution, and for each t the density $f(x, t)$ is defined by Equation (4).

(iii) *The process is mean-square continuous.*

The concept of continuity introduced in (iii) can usefully be defined in terms of the *correlation function* $\Gamma(t_1, t_2)$, which is itself defined as follows:

$$(10) \qquad \Gamma(t_1, t_2) = E(X(t_1)\, X(t_2)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, t_1; x_2, t_2)\, dx_1\, dx_2,$$

when the integrals exist. The stochastic process is mean-square continuous if for every t

$$(11) \qquad \lim_{\tau \to 0,\, \tau' \to 0} \Gamma(t+\tau, t+\tau') = \Gamma(t, t).$$

Under the strong Gaussian assumption (ii) the process is completely determined by specifying the correlation function $\Gamma(t_1, t_2)$. We have added the Markovian restriction (i) because of its intuitive physical appeal, corresponding as it does to the concept of state in classical mechanics. In similar fashion the continuity requirement is physically natural.

Our fundamental point is that many different models may be chosen that satisfy assumptions (i)–(iii). The choice reduces to a choice of the correlation function $\Gamma(t_1, t_2)$. Moreover, until the correlation function is specified, the probabilistic or stochastic aspects of the theory of the time-dependent behavior of a quantum-mechanical oscillator are incomplete. We emphasize once again that this kind of incompleteness is only uncovered by considering time-dependent phenomena – it does not arise for a fixed Hilbert space at a given time t_0. Thus, in one clear sense, the kind of stochastic incompleteness identified is distinct from that associated with quantum logic in the standard literature.

What is important and surprising, and so far as we know, not previously observed in the literature, is that a highly intuitive and quite simple combination of classical mechanics and the classical theory of stochastic processes may be used to define a stochastic process that has Equation (4) as a consequence.

We first of all recall that the differential equation of the linear harmonic oscillator in classical mechanics is

$$(12) \qquad \ddot{x} + \omega^2 x = 0$$

with one form of the general solution being

$$(13) \qquad x = a \cos \omega t,$$

which is the term that appears in (4). Let X_t now be the random variable whose value is the position of the oscillator at time t, and $\{X_t, -\infty < t < \infty\}$ the corresponding stochastic process. Let Y_t be a random variable with normal or Gaussian distribution having mean zero and variance equal to one, i.e.,

$$(15) \qquad Y_t \sim N(0, 1).$$

Then from inspection of (4) we see at once that

$$(16) \qquad X_t = a \cos \omega t + c Y_t,$$

i.e., the position random variable of the oscillator at time t is just the sum of the constant term $a \cos \omega t$ and the 'fluctuation' random variable cY_t that has a normal distribution, where the constant $c = 1/\sqrt{2}\,\alpha$. The fluctuations Y_t in the motion of the particle play the role that errors of measurement play in the classical theory of observation. It is important to emphasize, however, that we are interpreting Y_t as a physical fluctuation in the motion of the particle not arising from a measurement, but having the same conceptual basis as the fluctuations of Brownian motion. Here, of course, the variance is constant in time and there is no dispersion through time as in the case of Brownian motion.

At first glance it might seem that the stochastic process we propose for the behavior of an oscillator is completely specified by assumptions (i)–(iii) and (15) and (16), but this is not the case. The correlation function Γ is still undetermined. It is easy to show that it must be a symmetric function and in the present case depends only on the difference $\tau = t_2 - t_1$. Because for all t, Y_t has the distribution $N(0, 1)$, the stochastic process is stationary and by assumption (iii) mean-square continuous.

A fundamental theorem of Khinchine's (see Gnedenko, 1968, p. 387) states that for such a continuous stationary process the correlation function $\Gamma(\tau)$ must have the form

$$\Gamma(\tau) = \int \cos \tau x \, \mathrm{d}F(x) \tag{17}$$

where $F(x)$ is some distribution function, and for any distribution function $F(x)$ there is such a process.

So far as we can see, quantum mechanics imposes no special constraints on $\Gamma(\tau)$, and a wide variety of choices can consistently be made in accordance with (17). This possible variety of correlation functions expresses in a particularly simple way the stochastic incompleteness of quantum mechanics.

The formulation we have given of a stochastic theory for the harmonic oscillator is a direct and, we believe, relatively natural extension of what is to be found in the usual quantum-mechanical discussions of the oscillator. There is, however, another way of looking at the stochastic theory that would provide strong, if not complete, constraints on the correlation function $\Gamma(\tau)$. This is to take as fundamental the energy states of the oscillator. The transition probabilities between these energy states define the fundamental stochastic process, which is now a continuous-time discrete-state process, and the probability distribution of position is given theoretically, conditional on each of the states. As has already been indicated, Equation (4) arises in this theory by averaging over the energy states and thus over the conditional probability distributions. Amusingly enough, the contrast between the theory we have given above and the more extended theory is precisely the contrast that exists in the two forms of learning theory described in the appendix. The linear incremental model is exactly the sort of theory we have defined above, and the all-or-none model with unobservable states of conditioning corresponds to the discrete state model just sketched.

We plan subsequently to return to the detailed formulation of the discrete state theory of the oscillator, but we shall not explore it further here. The central point we have wanted to make is that the standard quantum-mechanical theory of the oscillator is stochastically incomplete, and we have tried to point out as clearly as possible the nature of the incompleteness.

V. PHYSICAL PECULIARITIES OF JOINT DISTRIBUTIONS THAT ARE PROPER

Although noncommuting observables do not for most states ψ have a proper joint distribution, there are many special cases in which they do. We want to examine one such case in some detail – that of the linear harmonic oscillator where as before we 'average' over the states – in order to bring out the central feature of the joint distribution that can be explained by the stochastic theory characterized in the previous part. What we are interested in is once again the time-dependent case. We derive the joint probability density function for position and momentum and show that the distributions of position and momentum are independent through time, which seems physically surprising in terms of ordinary physical ideas of the motion of a particle.

To begin with, it is known from the literature (Schiff, 1949, pp. 60–69) that for the case of a one-dimensional harmonic oscillator

$$\psi(x, t)=\frac{\alpha^{1/2}}{\pi^{1/4}}\exp\left[-\tfrac{1}{2}(\alpha x-\alpha a\cos\omega t)^2 - i(\tfrac{1}{2}\omega t+\alpha^2 ax\sin\omega t-\tfrac{1}{4}\alpha^2a^2\sin 2\omega t)\right],$$

where α, a, and ω are physical constants that we need not be concerned with, except that $\alpha^2=\omega/\hbar$. For convenience of calculation, we replace the momentum p by the propagation 'vector' $k=p/\hbar$. By familiar methods, we may then show, since position and momentum are canonically conjugate operators, that the joint density $f(k, x, t)$ is given by:

$$f(k, x, t)=\frac{1}{\pi}\int \psi^*(x-u/2)\, e^{-iku}\, \psi(x+u/2)\, du.$$

Using the above expression for $\psi(x, t)$, we first find that

$$\psi^*(x-u/2, t)=\frac{\alpha^{1/2}}{\pi^{1/4}}\exp[-\tfrac{1}{2}(\alpha(x-u/2)-\alpha a\cos\omega t)^2 \\ +i(\tfrac{1}{2}\omega t+\alpha^2 a(x-u/2)\sin\omega t-\tfrac{1}{4}\alpha^2 a^2\sin 2\omega t)]$$

and

$$\psi(x+u/2, t)=\frac{\alpha^{1/2}}{\pi^{1/4}}\exp[-\tfrac{1}{2}(\alpha(x+u/2)-\alpha a\cos\omega t)^2 \\ -i(\tfrac{1}{2}\omega t+\alpha^2 a(x+u/2)\sin\omega t-\tfrac{1}{4}\alpha^2 a^2\sin 2\omega t)].$$

Combining results, integrating, and simplifying, we then obtain

$$f(k, x, t)=\frac{1}{\pi}\exp[-\alpha^2x^2-k^2/\alpha^2-\alpha^2a^2 \\ +2\alpha^2xa\cos\omega t-2ak\sin\omega t],$$

but this joint density may be written in a form that shows directly the statistical independence of k and x by using the fact that $\cos^2\omega t+\sin^2\omega t=1$:

$$\begin{aligned}f(k, x, t)&=\left[\frac{1}{\sqrt{\pi}}\exp-\alpha^2(x^2-2ax\cos\omega t+a^2\cos^2\omega t)\right]\\&\quad\cdot\left[\frac{1}{\sqrt{\pi}}\exp-\left[\frac{k^2}{\alpha^2}+2ak\sin\omega t+\alpha^2a^2\sin^2\omega t\right]\right]\\&=\left[\frac{\alpha}{\sqrt{\pi}}\exp-\alpha^2(x-a\cos\omega t)^2\right]\\&\quad\cdot\left[\frac{1}{\alpha\sqrt{\pi}}\exp-\left(\frac{k}{\alpha}+\alpha a\sin\omega t\right)^2\right]\\&=f(x, t)\,f(k, t).\end{aligned}$$

We note first that as in the case of the position random variable X_t we can express the momentum – or, more exactly, the propagation – vector K_t as a sum of a constant term $\alpha a\sin\omega t$ and a random variable cZ_t where Z_t is normally distributed with mean zero and variance 1, i.e.,

$$Z_t \sim N(0, 1)$$

and

$$K_t = -\alpha^2 a \sin \omega t + \frac{\alpha}{\sqrt{2}} Z_t.$$

We also observe that the constant term of K_t is, as would be expected from classical mechanics, the derivative of the constant term of X_t divided by $\hbar$ (and recall that $\alpha^2 = \omega/\hbar$).

The statistical independence of position and momentum is now explained simply by assuming that the fluctuations in position are statistically independent of the fluctuations in velocity. In other words, for every t the normally distributed random variables Y_t and Z_t are independent. Our classical stochastic theory of the oscillator thus provides a natural and simple explanation of what might otherwise seem to be a rather puzzling quantum-mechanical result.

VI. CONCLUDING REMARKS

It might be objected that quantum mechanics should remain in a state of stochastic incompleteness because there is no evidence that a stochastically complete theory of the sort we have developed here or of the sort using diffusion processes (Nelson, 1967) will lead to any new observable phenomena, and thus the stochastic completeness is not only wasted but in a sense misleading. We think that this narrow positivistic view should be rejected. The stochastic theory of quantum phenomena gives a very easily understandable picture of the phenomena, especially, as we have emphasized here, their dynamical aspects. We are not claiming that a classical stochastic theory of all nonrelativistic quantum-mechanical phenomena can be developed, although Bartlett and Moyal (1949) have shown that this is essentially the case when the Hamiltonian is either linear or quadratic as a function of position and momentum. The exact limits of the approach combining classical mechanics and standard stochastic processes need to be examined much more thoroughly than seems yet to have been done.

We do not think it at all out of the question that the stochastic approach could lead to some new predictions that can be experimentally tested.

The second point we want to make is that the stochastic theory we have postulated for the harmonic oscillator is not crypto-deterministic, i.e., the probabilistic fluctuations do not enter only via the initial conditions. Exact knowledge of position and momentum at time t_0, if this were possible, would not lead to exact knowledge of past or future behavior of the oscillator. Even if the energy state could be observed without perturbing the system this indeterminism would still hold. This is for two reasons. First, the transition probabilities from one energy state to another are postulated as fundamentally probabilistic in the more complete stochastic theory already alluded to. Second, the conditional marginal distributions of position and momentum given the energy state are not reducible to deterministic functions, but are as fundamental to a complete classical stochastic theory of the oscillator as they are to the quantum-mechanical theory.

Our third remark concerns our view of the status of quantum logic in a stochastic theory of quantum phenomena. It should be apparent that classical logic untouched and unchanged is the appropriate logic for such stochastic theories. But this does not argue against quantum logic as the logic of observables in quantum experiments. The lattice structure of these observables is a deep-lying fact that is not disturbed by anything we have said.

On the other hand, if it were to turn out in the long run that the classical theory of stochastic processes proved to be the appropriate framework for developing the theory of quantum phenomena in a conceptually natural way, the general significance of quantum logic would almost certainly be reduced.

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APPENDIX

MEAN LEARNING CURVES IN MATHEMATICAL PSYCHOLOGY

A characteristic controversy in the psychological theory of learning is whether learning takes place on an incremental or an all-or-none basis. From the standpoint of this article, the interesting and subtle point about this controversy is that it is easy to formulate the two extremes of theory so that they both yield the same mean learning curve. The equation for the mean learning curve intuitively corresponds exactly to Equation (4) for the linear harmonic oscillator or to Equation (5) for the free particle. In this discussion and in the development of the other examples, no attempt is made to be mathematically explicit and thereby to make the rigor apparent, but it should be obvious that a mathematically precise formulation of the learning models considered in this part can easily be given.

To keep everything as simple as possible, we assume that exactly two responses are available on each trial. The experimental subject is presented a concept or a stimulus that he must learn to recognize – it does not matter what from the standpoint of theory. One of the responses – let us say response 1 – is the correct response, and the other – response 2 – is incorrect. At the beginning of the experiment the subjects, who are assumed to be homogeneous, will have a probability of responding correctly of $p_{1,1}$. In general, we use the following notation:

$$
\begin{aligned}
A_i &= \text{response } i,\ i=1,2 \\
P(A_{i,n}) &= p_{i,n} = \text{unconditional probability of response } i \text{ on trial } n \\
x_n &= \text{fixed sequence of responses through trial } n \\
P(A_{i,n} \mid x_{n-1}) &= p_{i,n}(x) = \text{conditional probability of response } i \text{ on trial } n \text{ given prior sequence of responses } x_{n-1}.
\end{aligned}
$$

The all-or-none model has the following simple formulation. An experimental subject begins in an unconditioned state, and on every trial there is a probability c that he will move from the unconditioned to the conditioned state. In addition, the process of moving from the unconditioned to the conditioned state is assumed to be a first-order Markov chain. The transition matrix as indicated has the following form, where U is the unconditioned state, and C is the conditioned state:

$$
\begin{array}{c|cc}
 & C & U \\
\hline
C & 1 & 0 \\
U & c & 1-c
\end{array}
$$

The second aspect of the all-or-none model that needs formulation is the probability of a correct response, given the state of conditioning. These assumptions are based on the following two equations:

$$P(A_{1,n} \mid U_n) = p_{1,1},$$
$$P(A_{1,n} \mid C_n) = 1.$$

The critical assumption is that the probability of a correct response, given that the subject stays in the unconditioned state, remains constant, namely, $p_{1,1}$. On the basis of these assumptions, it is easy to derive the mean learning curve for $p_{1,n}$. We first observe that the probability of being in the unconditioned state on trial n is $(1-c)^{n-1}$. Using this result, we then easily obtain the mean learning curve for incorrect responses, i.e., $p_{2,n}$.

$$\begin{aligned} p_{2,n} &= P(A_{2,n} \mid U_n)\, P(U_n) + P(A_{2,n} \mid C_n)\, P(C_n) \\ &= p_{2,1}(1-c)^{n-1} + 0. \end{aligned}$$

And thus the mean learning curve for a correct response is:

$$(18) \qquad p_{1,n} = 1 - p_{2,1}(1-c)^{n-1}.$$

Let us now look at the linear incremental model that intuitively assumes that the probability of a correct response increases on every trial. The simplest formulation of the model is in terms of a linear transformation on the probability of an incorrect response, given the preceding sequence of responses, where α is the learning parameter that, like c, lies between 0 and 1.

$$p_{2,n}(x) = \alpha p_{2,n-1}(x).$$

From this equation we derive in a direct fashion the mean learning curve for $p_{2,n}$.

$$p_{2,n} = p_{2,1}\alpha^{n-1}.$$

And thus immediately, the mean learning curve for $p_{1,n}$,

$$(19) \qquad p_{1,n} = 1 - p_{2,1}\alpha^{n-1}.$$

It is clear that Equations (18) and (19) express exactly the same mean learning curve if we set $\alpha = 1 - c$. Thus from two quite different kinds of assumptions – the assumption that learning occurs suddenly on a single trial, and the incremental assumption that learning increases on each trial, gradually approaching the asymptote of 1 but never achieving it –

we get precisely the same mean learning curves. The mean learning curve expressed by either (18) or (19) is the exact correspondence to what we have in Equations (4) and (5) for the linear harmonic oscillator and free particle.

On the other hand, the underlying assumptions of the two distinct models permit us to examine data in a decisive way to determine which is being satisfied. It is not pertinent to enter into details, but a detailed discussion can be found, for example, in Suppes and Ginsberg (1963). We can illustrate matters, however, by considering a single conditional probability. Let us suppose that $m<n$, and we assume that an error occurs on trial n; then we have the following two quite distinct conditional probabilities:

$$(20) \quad \begin{array}{ll} \text{All-or-none} & P(A_{1,m} \mid A_{2,n})=p_{1,1} \\ \text{Incremental} & P(A_{1,m} \mid A_{2,n})=1-p_{2,1}\alpha^{m-1}. \end{array}$$

In the case of the all-or-none model, notice that if an error is observed in the protocol of responses on a trial later than m the implication is that on trial m the subject was in the unconditioned state, and thus, there is no change in the probability of a correct response. In the incremental case the observation of the error on trial n does not influence the gradual increase in the probability of a correct response, and thus, the prediction for trial m is the same whether an error or a correct response occurs on trial n. The simple feature illustrated in these two conditional probabilities is central to disentangling the relative correctness of the two theories with respect to a given set of data.

From a broad conceptual standpoint, it should also be emphasized that the two distinct theories also determine uniquely the probability of any sample path. Contrary to the situation for these learning models, the determination of the sample paths for continuous-time continuous-state processes, the sort most appropriate for most of physics, is a completely nontrivial problem.

In the event of dissatisfaction with the conditioning of the past on the future expressed in Equation (20), a sharp contrast can also be drawn for an arbitrary trial n and the assumption that an error occurred on the previous trial. The all-or-none model 'starts over' at the occurrence of the error, but the incremental model is not affected. This comparison is expressed in the following pair of equations:

$$\text{All-or-none} \qquad P(A_{1,n} \mid A_{2,n-1}) = 1 - p_{2,1}(1-c)$$
$$\text{Incremental} \qquad P(A_{1,n} \mid A_{2,n-1}) = 1 - p_{2,1}\alpha^{n-1}.$$

The point being made about the two kinds of learning theories that yield the same mean learning curve can also be developed for continuous-time continuous-state learning processes, but it would take us too deeply into more intricate questions of learning than is desirable. Developments of this kind are given in Suppes and Donio (1967).

One point brought out by the continuous-time learning models that is not transparent in the above formulation concerns the matter of reinforcements. In considering the parallel to quantum mechanics, we might argue that indeed we *can* compute conditional probabilities in quantum mechanics when measurements are taken. Thus, for example, if we make an observation of an observable at a given time t_0, we can then compute the conditional distribution through time of that observable. This point is granted and it is an important part of quantum-mechanical theory, but it is not nearly enough. In the case of the continuous-time learning processes, for example, a continuous sampling of stimuli takes place independent of reinforcements, and it is the responsibility of the theory to deal with the fine structure of this sampling, just as it is the responsibility of a fully formulated, quantum-mechanical dynamical theory to deal with the behavior of the particle through time, independent of measurement interactions. It is not a proper response to say we do not know anything about the particle except when it is being measured, for it is precisely the task of the classical quantum-mechanical theory to compute various constants of the motion, constants that can be computed from the mean curves of the sort expressed by Equations (4) or (5). Furthermore, the standard conceptual way of talking about the motion of a particle implies that the kind of questions we would naturally ask, i.e., questions that correspond to the conditional probabilities in learning theory, can also be answered for quantum mechanics. And, the theory should be rich enough in its postulates to answer them.

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