BASIC MEASUREMENT THEORY

by

Patrick Suppes and Joseph L. Zinnes

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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Applied Mathematics and Statistics Laboratories
STANFORD UNIVERSITY
Stanford, California

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1. INTRODUCTION [1]

While measurement is one of the gods modern psychologists pay homage to with great regularity, the subject of measurement remains as elusive as ever. A systematic treatment of the theory is not readily found in the psychological literature. For the most part a student of the subject is confronted with an array of bewildering and conflicting catechisms, catechisms which tell him whether such and such a ritual is permissible, or, at least, whether it can be condoned. To cite just one peculiar, yet uniformly accepted example, as elementary science students we are constantly warned that it "does not make sense" (a phrase often used when no other argument is apparent) to add together numbers representing distinct properties, say, height and weight. Yet as more advanced physics students we are taught, with some effort no doubt, to multiply together numbers representing such things as velocity and time, or to divide distance numbers by time numbers. Why does multiplication make "more sense" than addition?

Rather than chart the etiology and course of these rituals, our purpose in this chapter is to build a consistent conceptual framework within which it is possible to discuss many (hopefully most) of the

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theoretical questions of measurement. At the same time, we hope to suppress as much as possible the temptation to enunciate additional dogmas or to put our stamp of approval on existing ones. Our over-riding faith, if we may call it that, is that once the various theoretical issues have been formulated in the terms set forth here, simple and specific answers having a clear logical basis can be given. To be sure, in some cases the answers to questions formulated within this framework are not so simple and may require extensive mathematical work. It may, for example, be a difficult mathematical problem to show that a given scale is (or is not) an interval scale, but this is not to suggest that the existence of an interval scale is a matter for philosophical speculation, or that it depends on the whims and fancies or even the position of the experimenter. On the contrary, the answers to questions of measurement have the same unambiguous status as the answers to mathematical questions posed in other fields of science.

2. GENERAL THEORY OF FUNDAMENTAL MEASUREMENT

A systematic approach to the subject of measurement may well begin by formulating what seem to be the two fundamental problems analysis of any procedure of measurement must consider. Briefly stated, the first problem is <u>justification of the assignment of numbers to objects or phenomena</u>. The second problem concerns <u>the specification of the degree to which this assignment is unique</u>. Each problem is taken up separately below. (The general viewpoint to be developed in this section was first articulated in Scott and Suppes (1958).)

2.1 <u>First Fundamental Problem: The Representation Theorem</u>

The early history of mathematics shows how difficult it was to divorce arithmetic from particular empirical structures. The ancient Egyptians could not think of $2 + 3$, but only of 2 bushels of wheat plus 3 bushels of wheat. Intellectually, it is a great step forward to realize that the assertion that 2 bushels of wheat plus 3 bushels of wheat equals 5 bushels of wheat involves the same mathematical considerations as the statement that 2 quarts of milk plus 3 quarts of milk equals 5 quarts of milk.

From a logical standpoint, there is just one arithmetic of numbers, not an arithmetic for bushels of wheat, and a separate arithmetic for quarts of milk. The first problem for a theory of measurement is to show how various features of this arithmetic of numbers may be applied in a variety of empirical situations. This is done by showing that certain aspects of the arithmetic of numbers have the same structure as the empirical situation investigated. The purpose of the definition of <u>isomorphism</u> to be given is to make the rough-and-ready intuitive idea of "same structure" precise. The great significance of finding such an isomorphism of structures is that we may then use many of our familiar computational methods, applied to the arithmetical structure, to infer facts about the isomorphic empirical structure.

In more complete terms we may state the first fundamental problem of an exact analysis of any procedure of measurement as follows:

> <u>Characterize the formal properties of the empirical operations and relations used in the procedure and show that they are isomorphic to appropriately chosen numerical operations and relations</u>.

Since this problem is equivalent to proving what is called a numerical representation theorem the first fundamental problem is hereafter referred to as the <u>representation problem</u> for a theory or procedure of measurement.

We may use Tarski's notion (1954) of a <u>relational system</u> to make the representational problem still more precise. A relational system is a finite sequence of the form $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$, where A is a non-empty set of elements called the <u>domain</u> of the relational system $\mathfrak{A}$, and $R_1, \ldots, R_n$ are relations on A.[2]

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[2] It is no restriction on generality to consider only relations, for from a formal standpoint operations are simply certain special relations. For example, a ternary relation T is a binary operation if whenever

$T(x,y,z)$ and $T(x,y,z')$ then $z = z'$, and we may define the binary operation symbol 'o' by the equation:

$$x \circ y = z \text{ if and only if } T(x,y,z) .$$

Moreover, there are reasons special to theories of measurement to be mentioned in Sec. 4.5 for minimizing the role of operations.

Two simple examples of relational systems are the following. Let A_1 be the set of human beings now living, and let R_1 be the binary relation on A_1 such that for all a and b in A_1, aR_1b if and only if a was born before b. $\mathfrak{A}_1 = < A_1,R_1 >$ is then a relational system in the sense just defined. Let A_2 be a set of sounds, and let D_2 be the quaternary relation representing judgment by a subject of relative magnitude of differences of pitch among the elements of A_2, i.e., for any a, b, c and d in A_2, $abDcd$ if and only if the subject judges the difference in pitch between a and b to be equal to or less than the difference between c and d. The ordered couple $\mathfrak{A}_2 = < A_2,D_2 >$ is then also a relational system.

The most important formal difference between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is that R_1 is a binary relation of ordering and D_2 is a quaternary relation for ordering of differences. It is useful to formalize this difference by defining the <u>type</u> of a relational system. If $s = < m_1,\ldots,m_n >$ is an n-termed sequence of positive integers, then a relational system

$\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ is of type s if for each $i = 1, \ldots, n$ the relation R_i is an m_i-ary relation. Thus $\mathfrak{A}_1$ is of type $\langle 2 \rangle$ and $\mathfrak{A}_2$ is of type $\langle 4 \rangle$. Note that the sequence s reduces to a single term for these two examples because each has exactly one relation. A relational system $\mathfrak{A}_3 = \langle A_3, P_3, I_3 \rangle$ where P_3 and I_3 are binary relations on A_3 is of type $\langle 2,2 \rangle$. The point of stating the type of a relational system is to make clear the most general set-theoretical features of the system. We say that two relational systems are <u>similar</u> if they are of the same type.

We now consider the important concept of isomorphism of two similar relational systems. Before stating a general definition it will perhaps be helpful to examine the definition for systems of type $\langle 2 \rangle$, i.e., systems like $\mathfrak{A}_1$. Let $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$ be two systems of type $\langle 2 \rangle$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are <u>isomorphic</u> if there is a one-one function f from A onto B such that for every a and b in A

$$aRb \text{ if and only if } f(a)Sf(b) .$$

As already stated, the intuitive idea is that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic just when they have the same structure.

For instance, let

$$A = \{1,3,5,7\}$$

$$A' = \{1,4,20,-5\}$$

$$R = \leq$$

$$R' = \geq \quad .$$

Then $\mathfrak{A} = < A,R >$ and $\mathfrak{A}' = < A',R' >$ are isomorphic. To see this, let

$$f(1) = 20 \ ,$$

$$f(3) = 4 \ ,$$

$$f(5) = 1 \ ,$$

$$f(7) = -5 \quad .$$

On the other hand, let

$$R'' = < \quad .$$

Then $\mathfrak{A} = < A,R >$ and $\mathfrak{A}'' = < A',R'' >$ are not isomorphic, for suppose they were. Then there would exist a function f such that

$$f(1)R''f(1) \ ,$$

i.e.,

$$f(1) < f(1) \ , \qquad (1)$$

because

$$1 \; R \; 1 \; ,$$

i.e.,

$$1 \leq 1 \; ,$$

but (1) is absurd.

To illustrate another point, let

$$A = \{1,2\}$$

$$A' = \{8,9,10\}$$

$$R = <$$

$$R' = > \; .$$

Then $\mathfrak{A} = \langle A, < \rangle$ and $\mathfrak{A}' = \langle A', > \rangle$ are not isomorphic just because A and A' do not have the same number of elements and thus there can be no one-one function from A to A'.

From this discussion it should be clear how the general definition of isomorphism runs. Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n \rangle$ and $\mathfrak{B} = \langle B, S_1, \ldots, S_n \rangle$ be similar relational systems. Then $\mathfrak{B}$ is an <u>isomorphic</u> image of $\mathfrak{A}$ if there is a one-one function f from A onto B such that, for each $i = 1, \ldots, n$ and for each sequence $\langle a_1, \ldots, a_{m_i} \rangle$ of elements of A, $R_i(a_1, \ldots, a_{m_i})$ if and only if $S_i(f(a_1), \ldots, f(a_{m_i}))$. Instead of saying that $\mathfrak{B}$ is an isomorphic image of $\mathfrak{A}$, we also often simply say that $\mathfrak{A}$ and $\mathfrak{B}$ are <u>isomorphic</u>.

On occasion it is too strict to require that the function f be one-one, for it may be natural in some cases to assign the same number to two distinct objects; for instance, two objects may have the same weight or length. In such cases, we weaken the definition by dropping the requirement that f be one-one and then speak of $\mathcal{B}$ being the _homomorphic_ image of $\mathcal{A}$.

The formal definitions given thus far are not special to the theory of measurement. A more direct connection is made by first distinguishing between a _numerical relational system_ and an _empirical relational system_. A numerical relational system is a relational system $< A, R_1, \ldots, R_n >$ whose domain A is a set of real numbers. Although this definition places no restrictions on the relations R_i in the numerical system, these relations in practice are limited to certain common relations obtaining between numbers. It is, however, possible to define, for example, a numerical relational system $< A, R_1 >$ in which

$$A = \{1,3,5,7\}$$

$$R_1 = \{< 1,3 > , < 5,7 > , < 7,7 >\} ,$$

but, such a numerical system will not prove very useful. The relational systems described in the preceding discussion of isomorphism are examples of more common (and useful) numerical systems. It should be obvious but nevertheless bears emphasizing, that a numerical relational system is not necessarily isomorphic to what is ordinarily called the real number system. For example, let Re be the set of all real numbers and let $<$ be the ordinary numerical relation of <u>less</u> <u>than</u>, then the numerical relational system $\langle Re, < \rangle$ is certainly not isomorphic to the usual system of real numbers employing the operations of addition and multiplication.

An empirical relational system is a relational system whose domain is a set of identifiable entities such as weights, persons, attitude statements, or sounds. If, for example, the domain A of the relational system $\langle A, R_2 \rangle$ consisted of weights, then the relation R_2 would very likely be the relation <u>is less heavy than</u>, i.e., for a and b in A aR_2b indicates that weight a is less heavy than b.

The first fundamental problem of measurement may be cast as the problem of showing that <u>any empirical relational system which purports to measure (by a simple number) a given property of the elements in the domain of the system is isomorphic (or possibly homomorphic) to an appropriately chosen numerical relational system</u>.

There are two aspects to this statement of the representation problem which perhaps need further amplification. Since, as we have emphasized, the numerical relational system does not completely characterize the real number system, the homomorphism which is required by the representation problem is <u>not</u> a homomorphism between the empirical relational system and the real number system. This does <u>not</u> mean, as is often suggested, that manipulations of the numbers in the domain of a given numerical system--to infer facts about the elements in the domain of the corresponding empirical system--must involve only those relations in the given numerical system. Relations neither contained in a given numerical system nor having a direct correspondence in the related empirical system may nevertheless be used. There are, of course, certain limitations imposed upon the manipulations of the numbers of a numerical system, but these limitations relate to certain criteria of meaningfulness of individual sentences rather than to those relations contained in a numerical system. These matters are discussed in detail in Sec. 6.

The second aspect of the representation problem needing amplification concerns the phrase "appropriately chosen numerical relational system" which appears in the last statement of the representation problem. The representation problem is not adequately solved if the isomorphism is established between a given empirical system and a numerical system employing unnatural or "pathological" relations. In fact, if the empirical system is finite or denumerable (i.e., has a finite or denumerable domain) some numerical system can always be found which is isomorphic to it. It is of no great consequence therefore merely to exhibit some numerical system which is isomorphic to an empirical system. It is of value, however, to exhibit a numerical system which is not only isomorphic to an empirical system, but which employs certain simple and familiar relations as well. A complete or precise categorization of the intuitively desirable relations is unfortunately somewhat elusive, so for this reason the statement of the representation problem refers merely to an "appropriately chosen" numerical system.

2.2 Second Fundamental Problem: Uniqueness Theorem

Solution of the representation problem for a theory of measurement does not completely lay bare the structure of the theory, for it is a banal fact of methodology that there is often a formal difference between the kind of assignment of numbers arising from different procedures of measurement. As an illustration, consider the following five statements:

(1) The number of people now in this room is 7 .

(2) Stendhal weighed 150 on September 2, 1839.

(3) The ratio of Stendhal's weight to Jane Austen's on July 3, 1814 was 1.42 .

(4) The ratio of the maximum temperature today to the maximum temperature yesterday is 1.1 .

(5) The ratio of the difference between today's and yesterday's maximum temperature to the difference between today's and tomorrow's maximum temperature will be .95 .

The empirical meaning of statements (1), (3) and (5) is clear, provided we make the natural assumptions, namely, for (3) that the same scale of weight, whether it be avoirdupois or metric, was being used, and for (5) that the same temperature scale is being used whether it be Fahrenheit or Centigrade. In contrast (2) and (4) do not have a clear empirical meaning unless the <u>particular</u> scale used for the measurement is specified. On the basis of these five statements we may formally distinguish three kinds of measurement. Counting is an example of an <u>absolute</u> scale. The number of members of a given collection of objects is determined uniquely. There is no arbitrary choice of a unit or zero available. In contrast, the usual measurement of mass or weight is an example of a ratio scale. A chemist, measuring a sample of a certain ferric salt on an equal arm balance with a standard series of metric weights, might make the statement

(6) This sample of ferric salt weighs 1.679 grams.

But this statement may be replaced by the statement:

(7) The ratio of the mass of this sample of ferric salt to the gram weight of my standard series is 1.679 , and the manufacturer of my series has certified that the ratio of my gram weight to the standard kilogram mass of platinum iridium alloy at the International Bureau of Weights and Measures, near Paris, is .0010000 .

In general, any empirical procedure for measuring mass does not determine the unit of mass. The choice of a unit is an empirically arbitrary decision made by an individual or group of individuals. Of course, once a unit of measurement has been chosen, such as the gram or pound, the numerical mass of every other object in the universe is uniquely determined. Another way of stating this is to say that the measurement of mass is unique up to multiplication by a positive constant. (The technical use of "up to" shall become clear later.) The measurement of distance is a second example of measurement of this sort. The ratio of the distance between Palo Alto and San Francisco to the distance between Washington and New York is the same whether the measurement is made in miles or yards.

The usual measurement of temperature is an example of the third formally distinct kind of measurement mentioned earlier. An empirical procedure for measuring temperature by use of a thermometer determines neither a unit nor an origin. (We are excluding from consideration here the measurement of absolute temperature whose zero point is not arbitrary.) In this sort of measurement the ratio of any two intervals is independent of the unit and zero point of measurement. For obvious reasons measurements of this kind are called _interval_ scales. Examples other than measurement of temperature are provided by the usual measurements of temporal dates, linear position, or cardinal utility.

In terms of the notion of absolute, ratio and interval scales we may formulate the second fundamental problem for any exact analysis of a procedure of measurement: _determine the scale type of the measurements resulting from the procedure_. We have termed this problem the _uniqueness_ problem for a theory of measurement. The reason for this terminology is that from a mathematical standpoint the determination of the scale type of measurements arising from a given system of empirical relations is the determination of the way in which any two numerical systems are related which use the same numerical relations and are homomorphic to the given empirical system. In the case of mass, for example, the four following statements are equivalent.

(8) The measurement of mass is on a ratio scale.

(9) The measurement of mass is unique up to multiplication by a positive number (the number corresponding to an arbitrary choice of unit).

(10) The measurement of mass is unique up to a similarity transformation (such a transformation is just multiplication by a positive number).

(11) Given any empirical system for measuring mass, then any two numerical systems that use the same numerical relations and are homomorphic to the given empirical system are related by a similarity transformation.

The validity of statement (11) will be demonstrated in Sec. 4.5 where an axiom system for measuring mass will be presented.

2.3 Formal Definition and Classification of Scales of Measurement

It is unusual to find in the literature of measurement an exact definition of scales. Within the formal framework developed in this article it is possible to give an exact characterization that seems to correspond rather closely to many of the intuitive ideas of a scale.

Two preliminary definitions are needed. We say that a numerical relational system is <u>full</u> if its domain is the set of all real numbers. Secondly, a <u>subsystem</u> of a relational system $\mathfrak{A}$ is a relational system obtained from $\mathfrak{A}$ by taking a domain which is a subset of the domain of $\mathfrak{A}$ and restricting all relations of $\mathfrak{A}$ to this subset. For example, let

$$Re = \text{the set of all real numbers}$$

$$< \;=\; \text{less than}$$

$$N = \text{the set of nonnegative integers}$$

$$<_N \;=\; \text{less than restricted to } N .$$

Then $\langle N, <_N \rangle$ is a subsystem of $\langle Re, < \rangle$, which is itself a full numerical relational system. As a second example, let

$$A = \{1,2,3\}$$

$$R = \{< 1,1 > , < 2,2 > , < 1,2 > , < 3,3 >\}$$

$$B = \{1,2\}$$

$$S_1 = \{< 1,1 > , < 2,2 > , < 1,2 >\}$$

$$S_2 = \{< 1,1 > , < 2,2 >\}$$

Then $\mathcal{B}_1 = < B,S_1 >$ is a subsystem of $\mathfrak{A} = < A,R >$, but $\mathcal{B}_2 = < B,S_2 >$ is not such a subsystem, for S_2 is not the relation R restricted to the set B.

We may now define scales. Let $\mathfrak{A}$ be an empirical relational system, let $\mathfrak{N}$ be a full numerical relational system, and let f be a function which maps $\mathfrak{A}$ homomorphically onto a subsystem of $\mathfrak{N}$. (If no two distinct objects in the domain of $\mathfrak{A}$ are assigned the same number, f is an isomorphic mapping.) We say then that the ordered triple $< \mathfrak{A}, \mathfrak{N}, f >$ is a <u>scale</u>.

As should be apparent from the discussion in Sec. 2.2, the type of scale is determined by the relative uniqueness of the numerical assignment f. We say, for instance, that a ratio scale is unique up to a similarity transformation. (A function φ from the set of real numbers to the set of real numbers is a similarity transformation if there exists a positive real number α such that for every real number x, $\varphi(x) = \alpha x$.) How may we make this uniqueness statement precise in terms of our definition of scales? The answer is reasonably simple. Let $< \mathfrak{A}, \mathfrak{N}, f >$

be a scale and g be any function having the property that $\langle \mathfrak{A}, \mathfrak{N}, g \rangle$ is also a scale. Then $\langle \mathfrak{A}, \mathfrak{N}, f \rangle$ is a <u>ratio</u> scale if there exists a similarity transformation φ such that

$$g = \varphi \circ f$$

where $\circ$ denotes the composition of functions (i.e., $(\varphi \circ f)(a) = \varphi(f(a))$). Note that in general f and g map $\mathfrak{A}$ into different subsystems although both subsystems are subsystems of the same full numerical relational system. This is necessary in order not to have different numerical interpretations of the basic empirical relations. That such different interpretations are possible even for the measurement of mass will be illustrated in Sec. 4.5.

The definition of other types of scales is analogous to the one just given for ratio scales, and we may briefly state them by giving the restriction on the transformation φ. For a given scale the transformation φ is frequently called the <u>admissible transformation</u>.

For <u>absolute</u> scales, φ must be the identity transformation, i.e., $\varphi(x) = x$, and we say that an absolute scale is unique up to the identity transformation.

For <u>interval</u> scales, φ must be a (positive) linear transformation, i.e., there is a positive real number α and a number β (positive, zero or negative) such that for every real number x, $\varphi(x) = \alpha x + \beta$. If in the measurement of temperature we wish to convert x in degrees Fahrenheit to Centigrade we use the linear transformation defined by

$\alpha = \frac{5}{9}$ and $\beta = -\frac{160}{9}$. That is,

$$y = \frac{5}{9}(x - 32) = \frac{5}{9}x - \frac{160}{9}.$$

Obviously every similarity transformation is a linear transformation with $\beta = 0$.

Another scale which is less well known but nevertheless useful is a difference scale.[3/] For this scale the function φ is a translation transformation, i.e., there is a real number β such that for every real number x, $\varphi(x) = x + \beta$. The assignment of numbers on a difference scale is unique up to an additive constant. (One example of a difference scale is discussed in Sec. 5.2.)

Still another type of scale is one which is arbitrary except for order. Moh's hardness scale, according to which minerals are ranked in regard to hardness as determined by a scratch test, and the Beaufort wind scale, whereby the strength of a wind is classified as calm, light air, light breeze, etc., are examples. We define them as follows. For ordinal scales φ must be a monotone transformation. Rather than define monotone transformations directly it will be convenient first to define monotone increasing and monotone decreasing transformations. A function φ is a monotone increasing transformation if and only if for every number x and y in the domain of φ if $x < y$ then $\varphi(x) < \varphi(y)$. Obviously

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3/ As far as we know this terminology was first suggested by Donald Davidson.

every linear transformation is a monotone increasing transformation on the set of all real numbers. The squaring function, that is, the function φ such that

$$\varphi(x) = x^2 , \tag{2}$$

is not a linear transformation but is monotone increasing on the set of nonnegative real numbers. Notice that it does not have this property on the set of all real numbers for $-5 < 4$ but

$$\varphi(-5) = 25 > 16 = \varphi(4) .$$

It is important to realize that a monotone increasing transformation need not be definable by some simple equation like (2). For example, consider the set

$$A = \{1,3,5,7\}$$

and let φ be the function defined on A such that

$$\begin{aligned} \varphi(1) &= -5 \\ \varphi(3) &= 5 \\ \varphi(5) &= 289 \\ \varphi(7) &= 993 . \end{aligned}$$

Clearly φ is monotone increasing on A , but does not satisfy any simple equation.

A function φ is a <u>monotone decreasing transformation</u> if and only if for every number x and y in the domain of φ if $x < y$ then $\varphi(x) > \varphi(y)$. Two examples of monotone decreasing transformations on the set of all real numbers are:

$$\varphi(x) = -x$$

and

$$\varphi(x) = -x^3 + 2 .$$

As another instance, consider the set A again, and let φ be defined on A such that

$$\begin{aligned} \varphi(1) &= 6 \\ \varphi(3) &= 4 \\ \varphi(5) &= 2 \\ \varphi(7) &= -10 . \end{aligned}$$

Obviously φ is monotone decreasing on A .

It will be noted that monotone transformations are simply transformations that are either monotone increasing or monotone decreasing. Although we have characterized ordinal scales in terms of monotone transformations, in practice it is often convenient to consider only monotone increasing or monotone decreasing transformations, but this restriction is mainly motivated by hallowed customs and practices rather than by considerations of empirical fact.

Numbers are also sometimes used for <u>classification</u>. For example, in some states the first number on an automobile license indicates the county in which the owner lives. The assignment of numbers in accordance with such a scale may be arbitrary except for the assignment of the same number to people in the same county and distinct numbers to people in distinct counties.

The weakest scale is one for which numbers are used simply to name an object or person. The assignment is completely arbitrary. Draft numbers and the numbers of football players are examples of this sort of measurement. Such scales are usually called nominal scales.

For classificatory and nominal scales, φ is only required to be a one-one transformation.

In addition to these classical scale types, four of which were originally proposed by Stevens (1946), another type may be mentioned: hyper-ordinal scales. These scales are similar to the ordered metric scales proposed by Coombs (1952) and are characterized by transformations (called hypermonotone) which preserve first differences. More formally, a function φ is a hypermonotone (increasing) transformation if and only if φ is a monotone transformation and for every x, y, u, and v in the domain of φ, if

$$x-y < u-v$$

then

$$\varphi(x) - \varphi(y) < \varphi(u) - \varphi(v) \quad .$$

Naturally every linear transformation is a hypermonotone increasing transformation, but the converse is not true. Consider, for example,

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$$A = \{1,2,4,8\}$$

and the function φ such that

$$\varphi(1) = 1$$

$$\varphi(2) = 2$$

$$\varphi(4) = 5$$

$$\varphi(8) = 15 .$$

Clearly φ is hypermonotone increasing but not linear on A . Various methods of measuring sensation intensities or utility yield hyperordinal scales.

There is, strictly speaking, a non-denumerable infinity of types of scales which are characterized by various groups of numerical transformations, but most of them are not of any real empirical significance. Also, it is possible to extend the notion of measurement to relational systems like lattices and partial orderings which cannot be represented numerically in any natural way. (Such an extension is urged in Coombs, Raiffa and Thrall (1954).)

By way of summary and to point out the major differences with current usage, the following aspects of the uniqueness question should be noted. Numerical assignments and scales are two different entities: the first is a function which maps an empirical system homomorphically onto a numerical system; the second is a triple one of whose terms is a numerical assignment. If only the numerical assignment is known, its scale type or degree of uniqueness cannot be determined. To determine its uniqueness we need to know the scale, which means that we need to know both an empirical relational system and a full numerical relational system.

From a knowledge of the scale we can, at least theoretically, infer precisely what the uniqueness properties of the numerical assignment are. In general, it should be noted that if the full numerical system is changed, then the numerical assignment will have quite different uniqueness properties, despite the fact that the empirical relational system may be unchanged. Therefore when we speak of the uniqueness properties of a numerical assignment or equivalently of the admissible numerical assignments, it must always be relative to an explicit or implicit scale.

Some writers of measurement theory appear to define scales in terms of the existence of certain empirical operations. Thus interval scales are described in terms of the existence of an empirical operation which permits the subject (observer or experimenter) to compare intervals and to indicate in some way whether or not they are equal. In the present formulation of scale type, no mention is made as to what kinds of "direct" observations or empirical relations exist (in the empirical relational system). Scale type is defined entirely in terms of the class of numerical assignments which map a given empirical system homomorphically onto a subsystem of the same full numerical system. If in a given instance these numerical assignments are related by a linear transformation then we have an interval scale. Precisely what empirical operations are involved in the empirical system is of no consequence. It may contain operations which permit the subject "directly" to compare intervals or the operations may be considerably more subtle.

One merit in this approach is that it takes away some of the implications generally associated with the question of scale type. For example, instead of asking how we know certain intervals are "really" equal, we ask if all the admissible numerical assignments are related by a linear transformation.

2.4 <u>Extensive and Intensive Properties</u>

Following Campbell (1919) most measurement theorists distinguish between quantities (or extensive properties) and qualities (or intensive properties) and between fundamental and derived measurement. Campbell defines these terms essentially as follows. Quantities are properties for each of which there exists an empirical operation similar to the arithmetical operation of addition. Qualities are characterized by an absence of this additive operation. Measurement is fundamental if it involves no previous measurement. (1928, p.14). If it does, it is derived. It should be added that measurement, for Campbell, implies obtaining at least an interval (or possibly, a ratio) scale.

The relationship that Campbell attributes to these two pairs of terms leads to another (implicit) definition of fundamental measurement. Only quantities, he maintains, are amenable to fundamental measurement. Frequently, therefore, Campbell implicitly defines fundamental measurement in terms of the existence of an additive operation. Thus in his discussion of temperature measurements he concludes that such measurements cannot be fundamental since "there is no physical process of addition for temperature" (Campbell, 1919, p.396). (Nine years later Campbell says, in regard to temperature, "...the temperature which is actually employed in physics is, in principle, as arbitrary and empirical as the hardness employed in mineralogy." (1928, p.119).)

More recent writers (e.g., Cohen and Nagel, 1934; Guilford, 1954) have tended to follow the essential position developed by Campbell (either that fundamental measurement <u>is</u> the measurement of extensive properties or, more precisely, that it can only be performed on extensive properties).

It is of historical interest to note that a complete and rigorous set of axioms for extensive properties was given as early as 1901 by Hoelder. His axioms specify among other things precisely what properties the addition operation is to have (See Sec. 3.5). The search for fundamental scales in psychology has frequently been identified with the search for an additive operation. (Some notable exceptions are Hempel (1952), Stevens (1951) and Torgerson (1958).) Since it is generally recognized that additive operations are so far practically non-existent in psychology it has been suggested that fundamental scales of mental tests will only occur when we have direct observation of underlying physiological phenomena (Comrey, 1951).

Since we attach no special virtue to the existence or non-existence of an additive operation we shall not attempt to give a more formal definition of the extensive-intensive distinction. In the next section we shall, however, give a definition of derived measurement. Our definition of fundamental measurement should be evident from the preceding sections. We may state it explicitly as follows. A function which maps an empirical relational system $\mathfrak{A}$ homomorphically onto a numerical relational system is said to be a <u>fundamental</u> <u>numerical</u> <u>assignment</u> for the empirical system $\mathfrak{A}$. In other words, if $\langle \mathfrak{A}, \mathfrak{N}, f \rangle$ is a scale, then the function f is a fundamental numerical assignment for the empirical system $\mathfrak{A}$. And finally, <u>fundamental measurement</u> of a set A with respect to the empirical system $\mathfrak{A}$ involves the establishment of a fundamental numerical assignment for $\mathfrak{A}$, or in other words, involves the establishment of a representation theorem for $\mathfrak{A}$. Note that to establish a fundamental numerical assignment for $\mathfrak{A}$ it is obviously necessary and sufficient to find just one numerical system homomorphic to $\mathfrak{A}$.

The fact that fundamental measurement procedures exist which are not based on an addition operation but which lead to ratio (interval or ordinal) scales will be amply demonstrated in Sec. 4. Furthermore, in Sec. 6 it will be evident (we hope) that the specification of the meaningful functional relationships between scales requires only a knowledge of the relevant scale types (the admissible transformations), but does not require a knowledge of the relations which are involved in the corresponding empirical or numerical relational systems.

3. GENERAL THEORY OF DERIVED MEASUREMENT

The preceding section has been concerned with the general theory of fundamental measurement. Fundamental measurement of a set A is always with respect to an empirical system, $\mathfrak{A}$. (A scale $< \mathfrak{A}, \mathfrak{N}, f >$ we shall henceforth call a fundamental scale.)

In contrast, derived measurement, the subject of this section, does not depend on an empirical relational system directly but upon other numerical assignments. The classic example of a derived measurement is that of density defined as the ratio of mass to volume.

The central issue for a theory of derived measurement is the status of the two basic problems of fundamental measurement: the representation and uniqueness problems. To these we now turn.

3.1 The Representation Problem.

We began the discussion of the representation problem for fundamental measurement by introducing the notion of an empirical relational system. In derived measurement the role of that concept is played by the concept of what we shall call a derived measurement system $\mathcal{B} = < B, f_1, \ldots, f_n >$ where B is a non-empty set of objects and $f_1, \ldots, f_n$ are numerical-valued functions defined on B or cartesian products of B with itself. Thus in the case of density, $\mathcal{B}$ would be the triple $< B, m, V >$, where m is the mass function and V is the volume function. In the case of pair comparison methods of scaling the derived measurement system is a couple $\mathcal{B} = < B, p >$ such that p is defined on B x B and for all a,b in B , $0 < p_{ab} < 1$, $p_{ab} + p_{ba} = 1$ and $p_{aa} = 1/2$. The usual interpretation of p_{ab} is that it is the relative frequency with which a is preferred to or is in some sense greater than, b. A number of derived measures defined in terms of $< B , p >$ have been considered in the literature; these measures are variously interpreted, sometimes as measures of utility, often as measures of response strength. We consider them in some detail later (Sec. 5).

Our approach to the representation problem is to define, in terms of derived measurement systems, derived scales. Let $\mathcal{B} = < B, f_1,\ldots,f_n >$ be a derived measurement system, let g be a numerical-valued function on B (or cartesian products of B with itself), and let R be a relation between $f_1,\ldots,f_n$ and g. We say that the triple $< \mathcal{B} , R , g >$ is a derived scale, R is the representing relation for the scale, and g is the derived numerical assignment. In most cases R will be defined by an equation.

To make the ideas clearer, let us consider two examples, beginning with density. As we have already said, $\mathcal{B} = < B , m , V >$ is the derived measurement of density and the representing relation R is defined by:

(1) $$R(m , V , d)$$

if and only if for every a in B,

$$d(a) = m(a)/V(a) .$$

The triple $< \mathcal{B}, R , d >$ is then the derived scale of density. The particularly simple form of the representing relation R for density is deceptive. The definition of (1) is an equation explicitly defining d in terms of m and V. Matters are not always this simple, as our second example will show.

Let $\mathcal{B} = < B , p >$ be a derived system for pair comparisons. We may define Bradley-Terry-Luce derived scales in the following manner. Let v be the derived numerical assignment of response strength, and let R_1 be the representing relation such that $R_1(p,v)$ if and only if for every a and b in B

$$\frac{v(a)}{v(a) + v(b)} = p_{ab} \,. \tag{2}$$

Then $< \mathcal{B} , R_1, v >$ is a Bradley-Terry-Luce derived scale of response strength. The important point for the moment is that (2) does not explicitly define the function v in terms of p. As is evident, from (2) we could only determine v up to a similarity transformation. Secondly, it is equally clear that unless some restrictions are placed on the relative frequencies p_{ab}, the function p may not stand in the relation R_1 to any v, i.e., the set of equations defined by (2) may not have a solution in terms of the unknown quantities $v(a)$, $v(b)$, etc.

From this last example, it should be clear how we may formulate the representation problem for derived measurements. Given a derived system $\mathcal{B}$ and the definition of a representing relation R, the representation problem is solved by showing that there exists a derived numerical assignment g such that $< \mathcal{B} , R , g >$ is a derived scale. In the case of density, the proof of the existence of g is trivial. For Bradley-

Terry-Luce scales such a function does not in general exist. In Sec. 5.1 we state necessary and sufficient conditions for the existence of g .

We have already pointed out that more than one representation theorem may be proved for empirical relational systems of a given kind. Similarly, different representations leading to different derived scales for a given derived system may be obtained by selecting a different representing relation R . Later (Sec. 5) we examine the possibilities for pair comparison data in some detail.

3.2 The Uniqueness Problem

In the earlier discussion of fundamental scales, we defined scale type in a relatively simple way. For instance, if $< \mathfrak{A}, \mathfrak{N}, f >$ is a fundamental scale such that for any other scale $< \mathfrak{A}, \mathfrak{N}, f' >$, f and f' are related by a similarity transformation then $< \mathfrak{A}, \mathfrak{N}, f >$ is a ratio scale.

A natural analogue of this definition may be formulated for derived scales. Ratio scales are defined as follows. Let $< \mathcal{B}, R, g >$ be a derived scale. Then it is a ratio scale in the narrow sense if for any other scale $< \mathcal{B}, R, g' >$, g and g' are related by similarity transformation. We have specified "in the narrow sense" for the following reason. According to this definition density is an absolute scale, because for a fixed $\mathcal{B} = < B, m, V >$ the function d is uniquely determined. This is not true for the response strength function v for pair comparison data, as we shall see later.

The definition of the other standard scale types is an immediate generalization of that given for ratio scales. A more important problem is to distinguish other senses of uniqueness. Density may again furnish a paradigm. It is commonly said that density is a derived ratio scale, and it is not difficult to define a second sense of ratio scale which catches this idea. The basis for the idea is, of course, that if we change measurements of mass from $m(a)$ to $\alpha m(a)$ and measurements of volume from $V(a)$ to $\beta V(a)$, then we change measurements of density from $d(a)$ to $\alpha m(a) / \beta V(a)$, and the ratio α/β defines a derived similarity transformation on the density function d.

The formal definition that corresponds to this example runs as follow. Let $< \mathcal{B} , R , g >$ be a derived scale, let $\mathcal{B}'$ result from $\mathcal{B}$ by applying admissible transformations to the numerical assignments of $\mathcal{B}$. Then $< \mathcal{B} , R , g >$ is a <u>ratio scale in the wide sense</u> if for any scale $< \mathcal{B}' , R , g' >$, g and g' are related by a similarity transformation. Obviously density is a ratio scale in the wide sense.

The reason for the separation between narrow and wide scale types is to distinguish between independent and dependent admissible transformations of derived numerical assignments. The narrow admissible transformations, those defined by the narrow scale types, can be performed without at the same time transforming one of the fundamental assignments in the derived system. For the wide admissible transformations this is not necessarily the case. These transformations of the derived numerical assignments may need to be accompanied by related transformations to certain fundamental numerical assignments. This property of wide scale types will be seen to be important in Sec. 6.

3.3 Pointer Measurement

In addition to fundamental and derived measurements a third type of measurement, called pointer measurement, may be noted. By pointer measurement we mean a numerical assignment (either fundamental or derived) which is based on the direct readings of some validated instrument. An instrument is validated if it has been shown to yield numerical values which correspond to those of some fundamental or derived numerical assignment. Consider the measurement of mass. The fundamental measurement of mass is a long and tedious operation (See Sec. 4.5). However, once this has been accomplished there is no need to go through this procedure to determine the mass (or weight which is proportional to it) of some particular object such as a steak. As every housewife knows the weight of a steak is determined by placing it on a measuring instrument (a "scale") and then noting the deflection of the pointer. The housewife assumes, however, that the stamp of approval on the scale (by, say, the department of weights and measures) means that someone has taken the trouble to verify that the deflections of the pointer under certain "standard" conditions do indeed correspond to the values of a given fundamental or derived numerical assignment. In other words, the housewife assumes the instrument to have been validated.

To construct an instrument which will provide direct or at least quick measurement of some fundamental or derived scale it is generally necessary to utilize some established empirical law or theory involving the fundamental or derived scale in question. In the case of mass an instrument is frequently built based on Hooke's law (the extension of a spring is proportional to the force acting on the spring) and on the law of gravity (the force exerted by an object is proportional to its mass). Once a spring has been selected which satisfies Hooke's law within the accuracy desired (under "standard" conditions of temperature, humidity, etc.) the next step is to calibrate the spring, that is, to determine what amount of mass would be required to produce each possible extension of the spring, or equivalently, each possible deflection of the pointer attached to the spring. Generally the calibration is performed by selecting two known weights, say 32 and 212 kg. and then spacing off 180 equal divisions between the two deflections corresponding to the two weights. If higher accuracy is required and if it is possible, further division of the "scale" is carried out.

Although the problem of constructing and using a pointer instrument is obviously an important practical problem, it is not our purpose to treat this problem in detail here. There are first two aspects of pointer measurements which are of theoretical interest, and these pertain to the two fundamental problems of measurement theory: the problem of justifying the numerical assignment, in this case the readings of the instrument, and the problem of specifying the uniqueness of the numerical assignment. The answer to the first problem for pointer measurement has already been given; the readings are justified by comparing them to the appropriate fundamental or derived numerical assignment. All too often in the behavioral sciences a direct reading instrument is available (and used) despite the fact that its readings are not justified; the readings do not correspond to any known fundamental or derived numerical assignment. On the surface it would appear that such pseudo-pointer instruments would be useless and their readings meaningless. Their prevalence in psychology (e.g., mental tests, questionnaires, indices) however, suggests that this conclusion may be too strong. The difficulty with rejecting out of hand pseudo-pointer instruments is that they may be converted (all too easily) into a fundamental measurement procedure yielding an absolute scale. This can be done by merely asserting first that the instrument is not intended to be a pointer instrument; it is not intended to give readings corresponding to some known numerical assignment. And second, that the readings are based entirely on the counting operation, the readings merely referring to the number of divisions which are to the left (or right) of the pointer after each deflection. Since the counting operation can always yield a fundamental absolute scale there can be no logical quarrel with anyone who uses this procedure to convert what would appear to be a pseudo-pointer instrument into a fundamental measuring instrument. One can, however, raise the question as to what is accomplished by the use of such an instrument. Generally speaking the answer seems to be that the instrument may be able to predict some future event which is of practical importance. A mental test score, for example, based on the number of correct answers may be able to predict success in college or in a job. The justification in the use of such instruments would then lie solely in the degree to which they are able to predict significant events, not as with most "normal" fundamental measures, in the homomorphism between an empirical system and a numerical system.

The answer to the second problem, the uniqueness problem, for pointer measurement has been the source of some confusion. The uniqueness of the readings is determined by the uniqueness of the corresponding fundamental or derived numerical assignment, not as might appear, by the method of calibrating the pointer instrument. The fact that the pointer instrument for measuring mass gives a ratio scale is not due to the equal spacing of the divisions on the dial; it would be quite possible to use a non-linear spring and have the divisions of the dial unequally spaced without altering the scale type. On the other hand, neither is it due to the fact that two points (e.g., 32 kg and 212 kg) are generally fixed or determined by the calibration procedure. Suppose, for example, the fundamental measurement of mass only yielded an ordinal scale, and the extension of the spring were montonically related to mass. Then while precisely the same calibration procedure could be carried out, that is, two points could be fixed and the dial divided into equally spaced divisions, the resulting readings of the instrument would nevertheless be on an ordinal scale only. Thus the scale type of a pointer instrument derives directly from the scale type of the corresponding fundamental or derived numerical assignment. As a corollary to this it follows that merely inspecting the divisions on the dial of a pointer instrument, or even observing the calibration proceedings of the instrument does not enable one to infer the scale type of the measurements obtained from the instrument.

4. EXAMPLES OF FUNDAMENTAL MEASUREMENT

In this section some concrete instances of different empirical systems are given and in each case solutions to the representation and uniqueness problems are exhibited. The empirical systems themselves are not necessarily of interest. Many are too restrictive in one way or another to have direct application to significant psychological situations. These systems, however, will permit bringing into focus some of the essential aspects of measurement.

The proofs establishing representation and uniqueness theorems are generally long and involved. For most empirical systems therefore we shall have to content ourselves with simply stating the results. This means that for each empirical system described a full numerical system will be stated, and then, more particularly, at least one subsystem will be given which is a homomorphic image of the empirical system. The uniqueness problem is answered by giving the relationship between any two subsystems (of the full system) which are homomorphic images of the empirical system. Two proofs are given in Sec. 4.1 and 4.2. The first proof (Sec. 4.1) is relatively simple, and it will serve to introduce some of the necessary terms and ideas which are generally encountered. The second proof (Sec. 4.2) is considerably more difficult so that it will give some notion of the complexity of the representation problem.

There is, as was indicated previously (Sec. 2.1), some degree of arbitrariness in the selection of a full numerical system for a given empirical system. In each case other full numerical systems could have been chosen and a different representation theorem established. Thus, it should be emphasized that although the representation problem is solved for each empirical system by exhibiting one homomorphic numerical system having certain reasonably natural and simple properties, the existence of other, perhaps equally desirable, numerical systems is not ruled out.

4.1 Quasi-Series

Nominal and classificatory scales are somewhat trivial examples of measurements so that we shall consider first an empirical system leading to an ordinal scale. The empirical system to be described is one that might, for example, be applicable to a lifted weight experiment in which subjects are presented the weights pairwise and are instructed to indicate either which weight of each pair seems the heavier or whether they seem equally heavy.

Some definitions which will be useful are as follows. A relational system $\mathfrak{A} = < A,R >$ consisting of a set A and a single binary relation R is called a binary system. A binary system $< A,I >$ is called a classificatory system if and only if I is an equivalence relation on A, i.e., if and only if I has the following properties

1) reflexive: if $a \in A$, then $a I a$;

2) symmetric: if $a, b \in A$ and $a I b$ then $b I a$;

3) transitive: if $a, b, c \in A$ and if $a I b$, $b I c$, then $a I c$,

where $a \in A$ means that a is a member of the set A. A common example of an equivalence relation is the identity relation $=$. In psychological contexts it is convenient to think of the indifference relation or the "seems alike" relation as an equivalence relation although there are many cases in which the transitivity property fails to hold.

We can now define a quasi-series (Hempel, 1952).

Definition 1. The relational system $\mathfrak{A} = \langle A, I, P \rangle$ is a quasi-series if and only if:

1) $\langle A, I \rangle$ is a classificatory system;

2) P is a binary, transitive relation;

3) If $a, b \in A$, then exactly one of the following holds:

aPb, bPa, aIb.

If, for example, A were a set of persons, I the relation the same height as and P the relation shorter in height than, then the relational system $\langle A, I, P \rangle$ would be a quasi-series. An important feature of a quasi-series is that the subject may be permitted to express his indifference when two stimuli seem alike.

Let us assume that the quasi-series $\langle A, I, P \rangle$ is an empirical relational system, the set A being a certain set of stimuli and the relations I and P being observable or empirical binary relations. Solving the representation problem for this empirical system will mean finding a (simple) homomorphic, rather than isomorphic, numerical system. If two elements a and b in A are related by I, i.e., if aIb, we can reasonably expect that it will be necessary to assign the same number to both a and b. Hence, the numerical assignment cannot be one-one.

One way of proceeding to establish a representation theorem for a quasi-series and at the same time to deal with the easier notion of isomorphism is to group or partition the elements of A in certain subsets. If the subsets have been selected judiciously enough then it may be possible to establish an isomorphism between these subsets and a numerical system. All the elements within a given subset could then be assigned the same number while the numerical assignment defined on the subsets would be one-one. That is, each subset would correspond to a distinct number. To accomplish this end we introduce the notion of I-equivalence classes.

If a is in A then the I-<u>equivalence class of which</u> a <u>is a member</u> is the set of all elements b in A such that $a I b$. This equivalence class is denoted by $[a]$. In symbols

$$[a] = \{b | b \in A \ \& \ a I b\} .$$

The set of all I-equivalence classes which are obtainable from A is denoted by A/I. As an example, let A be the set of persons born in the U.S. If I is the relation of equivalence such that $a I b$ if and only if a and b are born in the same state, then [Abraham Lincoln] is the set of all persons born in Kentucky and [Robert Taft] the set of all persons born in Ohio. In this example there are approximately 400 million elements in A and 50 elements (corresponding to the 50 states) in the set A/I.

One of the properties of equivalence classes that is important is their property of partitioning the set A into disjoint subsets or classes. Each element of A then belongs to one and only one equivalence class. We state this property as a theorem.

Theorem 1. If $[a]$, $[b] \in A/I$ then either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Proof: Assume that $[a] \cap [b] \neq \emptyset$. Let c be an element in both $[a]$ and $[b]$ and let a' be an arbitrary element of $[a]$. We have then $a\,I\,c$, $b\,I\,c$ and $a\,I\,a'$ from the definition of equivalence classes. From the symmetry and transitivity of I, we infer $b\,I\,a'$, whence $[a] \subseteq [b]$. By an exactly similar argument $[b] \subseteq [a]$, whence $[a] = [b]$, and the theorem is proved.

Thus if each element in A/I is assigned a unique number--and this is our aim--every element of A will be associated with exactly one number. The net effect then of obtaining a numerical system isomorphic to an empirical system having A/I as its domain is to obtain the desired homomorphic numerical system for the system $\langle A,I,P \rangle$.

There is another useful property of equivalence classes which should also be apparent. If $a, b \in A$ and if $a\,I\,b$, then the equivalence class of a equals the equivalence class of b, and conversely. This property will also be stated as a theorem.

Theorem 2. If $[a]$, $[b] \in A/I$, then $[a] = [b]$ if and only if $a\,I\,b$.

Proof: First, assume $a\,I\,b$. Let a' be an arbitrary element in $[a]$. Then $a\,I\,a'$ by the definition of $[a]$, and $b\,I\,a$ from the symmetry of I and hence $b\,I\,a'$. Consequently by definition of $[b]$, $a' \in [b]$. Since a' is an arbitrary element of $[a]$, we have then established that $[a] \subseteq [b]$. By a similar argument, it is easy to show that $[b]$ is a subset of $[a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we conclude that $[a] = [b]$.

Next, assume: $[a] = [b]$. Since $b \in [b]$, using our assumption we infer at once that $b \in [a]$, and hence $a I b$. Q.E.D.

We next want to define a relation on the elements of A/I. So far only the relations I and P have been defined on the elements of A and obviously to establish the desired isomorphism we shall have to have a relation defined on A/I. This relationship should of course correspond in some way to the relation P. Accordingly we define the binary relation P^* as follows.

Definition 2. $[a] P^* [b]$ *if and only if* $a P b$.

This definition needs some preliminary justification because it could lead to contradictions. We must rule out the possibility that for some element $a' \in [a]$, and for some element $b' \in [b]$, both $b' P a'$ and $a P b$ hold. The following two theorems are therefore required before we can safely proceed to use Definition 2. The first theorem asserts that if $a P b$, then every element in $[a]$ may be substituted for a.

Theorem 3. If $a P b$ and $a' I a$, then $a' P b$.

Proof: Assume $a' I a$ and $a P b$. From assumption 3 of a quasi-series exactly one of the following must hold: $a' I b$, $b P a'$, and $a' P b$. However, we cannot have $a' I b$ since by the transitivity and symmetry property of I, $a' I b$ and $a' I a$ imply $a I b$ which contradicts our initial assumption. Furthermore we cannot have $b P a'$ since by the transitivity property of P, $a P b$ and $b P a'$ imply $a P a'$ which also contradicts our initial assumption. Hence we must have $a' P b$.

Q.E.D.

Theorem 4. If $a\,P\,b$ and $b'\,I\,b$, then $a\,P\,b'$.

The proof is similar to the proof of Theorem 3.

Both theorems taken together imply that if $a\,P\,b$ then for any element $a' \in [a]$ and for any element $b' \in [b]$, $a'\,P\,b'$.

From Theorems 3 and 4 it follows that the relation P^* will not lead to inconsistencies and hence we define the relational system $\mathfrak{A}/I = \langle A/I, P^* \rangle$ which is obtained from the quasi-series $\mathfrak{A} = \langle A, I, P \rangle$. The representation problem can then be solved by establishing an isomorphism between $\langle A/I, P^* \rangle$ and an appropriate numerical relational system. The numerical relational system to be used for this purpose is called a numerical series. (It is not a series of numerical terms.) Consider first the definition of a series. A binary system $\langle A, R \rangle$ is a series if R has the following properties:

1) R is asymmetric in A, i.e., if $a\,R\,b$ then not $b\,R\,a$;

2) R is transitive in A;

3) R is connected in A, i.e., if $a \neq b$, then either $a\,R\,b$ or $b\,R\,a$.

By a numerical series we mean a binary numerical relational system $\langle N, R \rangle$ in which N is a set of real numbers and R is either the arithmetical relation less than or the arithmetical relation greater than restricted to the set N. Clearly R satisfies the properties of asymmetry, transitivity and connectedness so that a numerical series is a series. As would be expected, we also have the following theorem.

Theorem 5. If $\langle A, I, P \rangle$ is a quasi-series then $\langle A/I, P^* \rangle$ is a series, that is, P^* is asymmetric, transitive and connected in A/I.

Theorem 5 follows directly from the definition of P^* in Definition 2 and from the properties of P and I given in the definition of the quasi-series $< A,I,P >$.

The representation theorem for a finite or denumerable quasi-series can now be stated. Some additional restrictions are needed for the non-denumerable case.

Theorem 6 (Representation Theorem). Let the relational system $\mathfrak{A} = < A,I,P >$ be a quasi-series where A/I is a finite or denumerable set. Then there exists a numerical series isomorphic to $< A/I,P^* >$.

Proof: We give the proof for the denumerable case. The proof for A/I finite is much simpler and is essentially a special case.

Because A/I is a denumerable set, its elements may be enumerated as $a_1, a_2, \ldots, a_n, \ldots$. (It is important to note that this is in general not the ordering of A/I under P^*, but is rather an ordering we know exists under the hypothesis that A/I is denumerable.) We now define by induction the appropriate isomorphism function f--the induction being on the enumeration $a_1, a_2, \ldots, a_n, \ldots$. We first set

$$f(a_1) = 0 \quad ,$$

and then consider a_n. There are three cases, the first two of which are very simple.

Case 1. $a_i P^* a_n$, for $i = 1, 2, \ldots, n-1$. Then set

$$f(a_n) = n \quad .$$

Case 2. $a_n P^* a_i$, for $i = 1, 2, \ldots, n-1$. Here set

$$f(a_n) = -n \quad .$$

Case 3. There are integers i and j less than n such that $a_i P^* a_n P^* a_j$. Define

$$a_n^* = \max \{a_i | a_i P^* a_n \ \& \ i < n\}$$

$$b_n^* = \min \{a_j | a_n P^* a_j \ \& \ j < n\} .$$

The maximum and minimum are with respect to the ordering P^*, i.e., for example, a_n^* is such that $a_n^* P^* a_n$ and for every $i < n$ if $a_i \neq a_n^*$ then $a_i P^* a_n^*$. The existence of a unique greatest lower bound a_n^* just before a_n under P^* and somewhere before a_n in the enumeration $a_1, a_2, \ldots, a_n, \ldots,$ and the similar existence of the unique least upper bound b_n^* depends on all the axioms for a series as well as the fact that the number of elements of A/I before a_n in the enumeration $a_1, a_2, \ldots, a_n, \ldots$ is finite. If P^* is not connected, there could be two elements a_n^* and a_n^{**} satisfying the condition that they both preceded a_n in the ordering P^* and nothing else was between each of them and a_n under the ordering P^*. Similar difficulties could ensue if either transitivity or asymmetry were dropped as conditions on P^*. On the other hand, the finiteness of the number of elements a_i, for $i < n$ is necessary to establish that at least one maximal a_n^* and minimal b_n^* exist.

At this point we need to use the fact that the rational numbers are also denumerable, and thus may be enumerated as $r_1, r_2, \ldots, r_n, \ldots$. We are defining $f(a_n)$ to be a rational number for each a_n, and more particularly we define for Case 3, $f(a_n)$ as the first r_i between $f(a_n^*)$ and $f(b_n^*)$ in our enumeration of the rational numbers, if

$f(a_n^*) < f(b_n^*)$, and if $f(a_n^*) \geq f(b_n^*)$, we set $f(a_n) = 0$. (The existence of r_i for $f(a_n^*) < f(b_n^*)$ follows immediately from the fact that between any two rational numbers there exists another rational number.)

We want to show that this second possibility--$f(a_n^*) \geq f(b_n^*)$--leads to absurdity and thereby establish at the same time that

$$a_i P^* a_j \quad \text{if and only if} \quad f(a_i) < f(a_j) \ . \tag{1}$$

Let a_n be the first element in our enumeration for which for $i, j < n$, (1) does not hold. Thus (1) holds for $i, j < n - 1$, and the failure must be due to $i = n - 1$ or $j = n - 1$. Now a_{n-1} must follow under one of the three cases. Clearly if it falls under Case 1 or Case 2, (1) is satisfied. Consider now Case 3. Because a_{n-1}^* and b_{n-1}^* precede a_{n-1} in the enumeration, and because by definition of a_{n-1}^* and b_{n-1}^* , together with the transitivity of P^* , we must have $a_{n-1}^* P^* b_{n-1}^*$, it follows from (1) that $f(a_{n-1}^*) < f(b_{n-1}^*)$. But then by definition of $f(a_{n-1})$, we have

$$f(a_{n-1}^*) < f(a_{n-1}) < f(b_{n-1}^*) \quad ,$$

and also $a_{n-1}^* P^* a_{n-1} P^* b_{n-1}^*$, and contrary to our supposition (1) holds for a_{n-1} . Thus (1) holds for every n , and our theorem is established. (Note that the one-one character of f follows at once from (1).)

To complete the solution of the representation problem for quasi-series, two definitions are needed to characterize necessary and sufficient conditions for an infinite quasi-series which is not finite or denumerable to be numerically representable. The classical example to show that additional restrictions are needed is the lexicographic ordering of the set of all ordered pairs of real numbers. The ordering P is defined as follows for x, y, u and v in real numbers.

$$\langle x,y \rangle P \langle u,v \rangle \text{ if and only if } x < u \text{ or } x = u \text{ and } y < v .$$

The proof that the set of pairs of real numbers under the ordering P cannot be represented as a numerical series we leave as an exercise.

One definition is needed. Let the binary system $\mathfrak{A} = \langle A,R \rangle$ be a series and let B be a subset of A. Then B is <u>order-dense in</u> $\mathfrak{A}$ if and only if for every a and b in A and not in B there is a c in B such that aRc and cRb. Speaking loosely in terms of sets rather than relational systems, the set of rational numbers is a subset which is order-dense in the real numbers with respect to the natural ordering less than. Observe that the notions just defined could have been defined for arbitrary relational systems $\mathfrak{A} = \langle A,R \rangle$ which are not necessarily series, but then certain relational systems which are not dense in any intuitive sense would turn out to be dense under the definition. A simple example is the system $\langle N,\leq \rangle$ where N is the set of positive integers.

<u>Theorem</u> 7. (<u>Representation Theorem</u>). <u>Let the structure</u> $\mathfrak{A} = \langle A,I,P \rangle$ <u>be a quasi-series where</u> A/I <u>is an infinite set. Then a necessary and</u>

sufficient condition for the existence of a numerical series isomorphic to $\langle A/I, P^* \rangle$ *is the existence of a denumerable subset* B *of* A/I *which is order dense in* $\langle A/I, P^* \rangle$.

The proof of this theorem is omitted. A proof may be found in Birkhoff (1948, p. 32). It may be remarked that the proof of necessity requires the axiom of choice. Economists and others interested in applications of theorems like Theorem 7 to utility theory or demand analysis are often concerned with questions of continuity concerning the isomorphism function. Various sufficient topological conditions are given in Debreu (1954). He does not treat necessary conditions which would require an extremely difficult topological classification of quasi-series.

Theorems 6 and 7 together give necessary and sufficient conditions for any quasi-series to be representable by a numerical series. We now turn to the simple solution of the uniqueness problem for quasi-series.

Theorem 8. (*Uniqueness Theorem*). *Let* $\mathfrak{A} = \langle A, I, P \rangle$ *be a quasi-series. Then any two numerical series isomorphic to* $\mathfrak{A}/I$ *are related by a monotone transformation.*

Proof: Let $\langle N_1, R_1 \rangle$ and $\langle N_2, R_2 \rangle$ be two numerical series isomorphic to $\mathfrak{A}/I$. (R_1 and R_2 may each be either the relation $<$ or the relation $>$.) We want to find a function φ such that the domain of φ is N_1 and the range is N_2, i.e.,

$$D(\varphi) = N_1$$

$$R(\varphi) = N_2$$

and for every x,y in N_1, if $x\, R_1\, y$ then $\varphi(x) R_2 \varphi(y)$, i.e., we want a monotonic function which maps N_1 onto N_2.

Let f_1 and f_2 be two functions satisfying the hypothesis, that is, f_1 maps $< A/I, P^* >$ isomorphically onto the numerical series $< N_1, R_1 >$ and f_2 maps $< A/I, P^* >$ isomorphically onto the numerical series $< N_2, R_2 >$. Consider the domains and ranges of f_1 and f_2 and their inverses:

$$D(f_1) = D(f_2) = R(f_1^{-1}) = R(f_2^{-1}) = A/I$$

$$R(f_1) = D(f_1^{-1}) = N_1$$

$$R(f_2) = D(f_2^{-1}) = N_2 \; .$$

Consider then the function $f_2 \circ f_1^{-1}$. Does it have the desired properties?

$$D(f_2 \circ f_1^{-1}) = N_1$$

$$R(f_2 \circ f_1^{-1}) = N_2$$

Suppose $x, y \in N_1$ and $x\, R_1\, y$. Then $f_1^{-1}(x)\, P^*\, f_1^{-1}(y)$ and hence from the definition of f_2, $f_2(f_1^{-1}(x)) R_2 f_2(f_1^{-1}(y))$, i.e., $(f_2 \circ f_1^{-1})(x) R_2 (f_2 \circ f_1^{-1})(y)$, which completes the proof.

We have shown that given any two isomorphic numerical series they are related by a monotone transformation. That in general the numerical series are not related by any stronger transformation can be easily proved by a counter-example.

4.2 Semi-orders

In Luce (1956) the concept of a semi-order is introduced as a natural and realistic generalization of quasi-series. The intuitive idea is that in many situations judgments of indifference concerning some attribute of stimuli, like the pitch or loudness of tones, or the utility of economic goods, is not transitive. Thus a subject may judge tone a to be just as loud as tone b, and tone b to be just as loud as tone c, but find to his surprise that he judges tone a definitely louder than tone c.

In his original paper Luce uses a system consisting of two binary relations, that is, the kind of system used above for quasi-series. In Scott and Suppes (1958) Luce's axioms are simplified and only a binary system is used. The latter analysis will be considered here.

Definition 3. A semi-order is a binary system $\mathfrak{A} = \langle A, P \rangle$ in which the following three axioms are satisfied for all a, b, c, d in A:

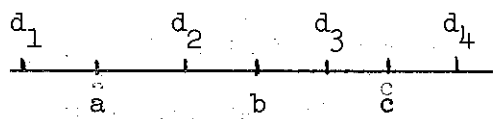
1) Not aPa;

2) If aPb and cPd then either aPd or cPb;

3) If aPb and bPc then either aPd or dPc.

In the case of loudness P is interpreted as definitely louder than. To make the last two axioms more intuitive, we may illustrate (3) by a simple geometrical picture. We place a, b and c on the line such that they are separated by at least one j.n.d. The axiom asserts that for any element d it is then the case that either aPd or dPc. The four different kinds of positions d can have are shown as d_1,

d_2, d_3 and d_4 in the drawing below. It is evident that for d_1 and d_2 we have $d_1 P_c$ and $d_2 P_c$; and for d_3 and d_4 the other alternative holds, namely, $a P d_3$ and $a P d_4$.



The indifference relation I can be defined in terms of P as follows.

Definition 4. $a I b$ if and only if not $a P b$ and not $b P a$.

In contrast to a quasi-series, the indifference relation I in a semi-order is not an equivalence relation. It lacks the transitivity property. However, we may define a relation E in terms of I which is an equivalence relation.

Definition 5. $a E b$ if and only if: for every c in A , $a I c$ if and only if $b I c$.

The fact that E is an equivalence relation, that it is reflexive, symmetric and transitive can easily be verified. As in the case of a quasi-series we introduce a relation P^{**} corresponding to P defined on the E-equivalence classes of A , i.e., on the elements of A/E .

Definition 6. $[a] P^{**} [b]$ if and only if $a P b$.

To justify Definition 6 theorems corresponding to Theorems 3 and 4 are needed. They are easily proved.

Theorem 9. If $a P c$ and $a E b$ then $b P c$.

Theorem 10. If $c P a$ and $a E b$ then $c P b$.

Unlike the relation P^* in Definition 2, the relation P^{**} does not have the connectedness property. In fact we may define the relation I^* as follows.

Definition 7. $[a]\, I^*\, [b]$ *if and only if* $a\, I\, b$.

It may be seen that $[a] I^* [b]$ does *not* imply that $[a] = [b]$ so that P^{**} is, in fact, not connected. This means that the relational system $< A/E, P^{**} >$ is not a series and even worse, P^{**} does not order the elements of A/E as was the case for the relation P^* and the set A/I of a quasi-series. A relation R which does order the elements of A/E can be defined in terms of P^{**} as follows.

Definition 8. $[a]\, R\, [b]$ *if and only if for all* $[c]$ *in* A/E , *if* $[c] P^{**} [a]$ *then* $[c] P^{**} [b]$, *and if* $[b] P^{**} [c]$ *then* $[a] P^{**} [c]$.

The relation R , it can be verified, is a *simple order*, i.e., it is reflexive, anti-symmetric ($[a]\, R\, [b]$ and $[b]\, R\, [a]$ implies $[a] = [b]$) , transitive and connected in A/E . The connection between P^{**} and R is clearer if one notices that $[a] P^{**} [b]$ implies $[a]\, R\, [b]$ but not conversely. The simple ordering property of R will be useful in the proof establishing the representation theorem. The representation problem consists of establishing an isomorphism between the relational system $\mathfrak{A}/E = < A/E, P^{**} >$ and an appropriate numerical relational system.

In Luce (1956) no representation theorem in our sense is proved for semi-orders, since a just noticeable difference function is introduced which varies with the individual elements of A , i.e., the j.n.d. function is defined on A , and no fixed numerical interpretation of P

and I is given which holds for all elements of A. Actually it would be intuitively more desirable if Luce's results were the strongest possible for semi-orders. Unfortunately, a stronger result than his can be proved, namely a numerical interpretation of P can be found, which has as a consequence that the j.n.d. function is constant for all elements of A.

We turn now to the formal solution of the representation problem for semi-orders. The proof is that given in Scott and Suppes (1958).

The numerical relational system to be selected for the representation theorem is called a _numerical semi-order_. A binary system $< N, \gg_\delta >$ is a _numerical semi-order_ if and only if N is a set of real numbers, and the relation $\gg_\delta$ is the binary relation having the property that for all x and y in N $x \gg_\delta y$ if and only if $x > y + \delta$. The number δ is the numerical measure of the j.n.d. It is easily checked that the relation $\gg_\delta$ satisfies the axioms for a semi-order and thus any numerical semi-order is a semi-order. Furthermore it is an immediate consequence of $\gg_\delta$ that δ is positive.[3]

The representation theorem for finite sets is as follows.

Theorem 11 (Representation Theorem). Let the binary system $\mathfrak{A} = < A, P >$ be a semi-order and let A/E be a finite set. Then $< A/E, P^{**} >$ is isomorphic to some numerical semi-order.

---

[3] It is a technical point worth noting that it would not be correct to define a numerical semi-order as a triple $< N, \gg, \delta >$ for there is nothing in $< A/E, P^{**} >$ of which δ is the isomorphic image. Taking the course we do makes δ part of the definition of $\gg$.

Proof: Under the relation R , A/E is simply ordered. Let $A/E = \{a_0, a_1, \ldots, a_n\}$ where $a_i R a_{i-1}$ and $a_i \neq a_{i-1}$. To simplify the notation of the proof, we set $\delta = 1$ and write $\gg$ instead of $\gg_1$. (The proof shows in fact that we may always take $\delta = 1$ if we so desire.) Define the function f as follows:

$$f(a_i) = x_i \ , \ i = 0, 1, \ldots n$$

where x_i is determined uniquely by the following two conditions:

1) If $a_i I^* a_0$, then $x_i = \frac{i}{i+1}$.

2) If $a_i I^* a_j$ and $a_i P^{**} a_{j-1}$ where $j > 0$, then

$$x_i = \frac{i}{i+1} x_j + \frac{1}{i+1} x_{j-1} + 1 \ .$$

Condition (1) holds when a_i and a_0 are separated by less than a j.n.d. , but x_i is defined so that $x_{i-1} < x_i$. Similar remarks apply to condition (2). Notice that in (2) the hypothesis implies that $j \leq i$. Notice further that every element a_i comes either under (1) or (2). If for no j , $a_i P^{**} a_{j-1}$ then $a_i I^* a_0$ and (1) applies. Also if $a_i I^{**} a_j$ and $a_i I^{**} a_{j-1}$, we find an earlier a_j in the ordering such that $a_i P^{**} a_{j-1}$.

To show that the numerical semi-order $< \{x_i\}, \gg >$ is an isomorphic image of $< A/E, P^{**} >$ we must show that f is one-one and $a_i P^{**} a_j$ if and only if $x_i \gg x_j$.

The one-one property of f can be shown by proving that $x_i > x_{i-1}$. This we do by induction on i . To simplify the presentation, we give an explicit breakdown of cases.

Case 1. $a_i \, I^* \, a_0$. Then also $a_{i-1} \, I^* \, a_0$, and

$$x_i = \frac{i}{i+1} > x_{i-1} = \frac{i-1}{i} \quad .$$

Case 2. $a_i \, I^* \, a_j$ and $a_i \, P^{**} a_{j-1}$ for some j .

2a. $a_{i-1} \, I^* \, a_0$. Then $x_{i-1} < 1$ and since $x_{j-1} > x_0 = 0$, from (2), $x_i > 1$.

2b. $a_{i-1} P^{**} a_0$. Let a_k be the first element such that $a_{i-1} I^* a_k$ and $a_{i-1} P^{**} a_{k-1}$. By definition

$$x_{i-1} = \frac{i-1}{i} x_k + \frac{1}{i} x_{k-1} + 1 \quad .$$

We then have two subcases of subcase 2b to consider.

2b1. $j = i$. Then by virtue of (2)

$$x_i = \frac{i}{i+1} x_i + \frac{1}{i+1} x_{i-1} + 1 \quad ,$$

whence simplifying

$$x_i = x_{i-1} + i + 1 \quad ,$$

and thus

$$x_i > x_{i-1} \quad .$$

2b2. $j < i$. It is easily shown that by selection of k , $k \leq j$.

We know that for this case $a_i \, R \, a_{i-1}$, $a_{i-1} \, R \, a_j$ and $a_i \, I^* \, a_j$, whence $a_{i-1} I^* a_j$ because a_{i-1} is "between" a_i and a_j (with possibly $a_j = a_{i-1}$). If $k > j$ then from definition of R $a_{i-1} P^{**} a_{j-1}$, which contradicts the assumption that k is the first element (in the ordering generated by R) such that $a_{i-1} I^* a_k$ and $a_{i-1} P^{**} a_{k-1}$, and so we conclude $k \leq j$.

If $k = j$, we have at once from (2)

$$x_{i-1} = x_k - \frac{1}{i}(x_k - x_{k-1}) + 1$$

$$x_i = x_k - \frac{1}{i+1}(x_k - x_{k-1}) + 1 ,$$

and since $\frac{1}{i+1} < \frac{1}{i}$, we infer that $x_i > x_{i-1}$.

If $k < j$, the argument is slightly more complex. By our inductive hypothesis $x_k < x_j$ and $x_{k-1} < x_{j-1}$, whence $x_k \leq x_{j-1}$. Now from (2)

$$x_{i-1} < x_k + 1$$

$$x_i > x_{j-1} + 1 ,$$

whence $x_i > x_{j-1} + 1 \geq x_k + 1 > x_{i-1}$, and the proof that $x_{i-1} < x_i$ is complete for all cases.

The next step is to prove that, if $a_i P^{**} a_k$, then $x_i > x_k + 1$. Let a_j be the first element such that $a_i I^* a_j$ and $a_i P^{**} a_{j-1}$. We have $j - 1 \geq k$, and, in view of the preceding argument, $x_{j-1} \geq x_k$. But $x_{j-1} + 1 < x_i$, whence $x_i > x_k + 1$.

Conversely we must show that, if $x_i > x_k + 1$, then $a_i P^{**} a_k$. The hypothesis of course implies $i > k$. Assume by way of contradiction that not $a_i P^{**} a_k$. It follows that $a_i I^* a_k$. Let a_j be the first element such that $a_i I^* a_j$; then $k \geq j$ and $x_k \geq x_j$. If $j = 0$, then $a_i I^* a_0$ and $a_k I^* a_0$, because $a_i R a_k$. But then $0 \leq x_i < 1$

and $0 \leq x_k < 1$, which contradicts the inequality $x_i > x_k + 1$. We conclude that $j > 0$. Now $x_i < x_j + 1$, but $x_k \geq x_j$, and thus $x_i < x_k + 1$, which again is a contradiction. Q.E.D.

The proof just given is not necessarily valid for the denumerable case, and it is an open problem as to what is the strongest representation theorem that may be proved when A/E is an infinite set.

The uniqueness problem for semi-orders is complicated and appears to have no simple solution.

4.3 Infinite Difference Systems

A relational system $< A,D >$ is called a quaternary system if D is a quaternary relation. In this section and the one following quaternary systems leading to interval scales are considered.

The notion behind the quaternary relation D is that $ab\,D\,cd$ holds when the subjective (algebraic) difference between a and b is equal to or less than that between c and d . In the case of utility or value, the set A would be a set of alternatives consisting of events, objects, experiences, etc. The interpretation $ab\,D\,cd$ is that the difference in preference between a and b is not greater than the difference in preference between c and d . Such an interpretation could be made for example if a subject, having in his possession objects a and c decides that he will not pay more money to replace a by b than he will to replace c by d , or if he does not prefer the pair a and to the pair b and c . Similar interpretations of utility differences can be made using as alternatives gambles or probability mixtures. (A detailed analysis of a probabilistic interpretation of quaternary systems is to be found in Davidson, Suppes and Siegel (1957).) If the set A consisted of color chips the interpretation of $ab\,D\,cd$ could be that stimuli a and b are at least as similar to each other as are stimuli c and d.

The empirical relational system to be considered here is a quaternary system which is an <u>infinite</u> <u>difference</u> <u>system</u> (abbreviated as i.d. system). To define an i.d. system it will be convenient to introduce certain relations defined in terms of the quaternary relation D.

<u>Definition</u> 9. $a\,P\,b$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> <u>not</u> $ab\,D\,aa$.

For the case of utility measurement, the relation P will be interpreted as a strict preference relation, a relation that is transitive and asymmetric in A.

<u>Definition</u> 10. $A\,I\,b$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> $ab\,D\,ba$ <u>and</u> $ba\,D\,ab$.

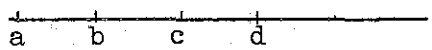
The relation I is the familiar relation of indifference. Note, of course, that the expected properties (like transitivity) of the binary relations P and I cannot be proved merely on the basis of these definitions. For that purpose the axioms to be given in the definition of the i.d. system are needed.

<u>Definition</u> 11. $ab\,M\,cd$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> $ab\,D\,cd$, $cd\,D\,ab$ <u>and</u> $b\,I\,c$.

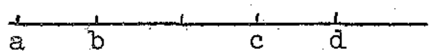
If we think of a, b, c and d as points on a line then $ab\,M\,cd$ implies that the interval (a,b) equals the interval (c,d) and the points b and c coincide. Higher powers of the relation M are defined recursively.

Definition 12. abM^1cd *if and only if* $abMcd$; $abM^{n+1}cd$ *if and only if there exists* e *and* f *in* A *such that* $abM^n ef$ *and* $efMcd$. Again letting a, b, c, and d be points on a line the relation $abM^n cd$ implies that the intervals (a,b) and (c,d) are of the same length and that there are $(n-1)$ intervals of this length between b and c . More particularly consider the following diagrams.

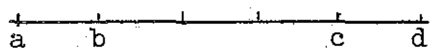
1) abM^2cd



2) abM^3cd



3) abM^4cd



The interval (a,d) under (1) is three times the length of the interval (a,b) ; under (2) it is four times, etc. Thus from M and its powers we may infer specific length relations. Later, when we discuss the Archimedean axiom, we shall see that the relation M enables us to establish commensurability of all differences with each other.

We are now in position to define an infinite difference system.

Definition 13. *A quaternary system* $\mathfrak{A} = \langle A, D \rangle$ *is an* infinite difference system *if and only if the following seven axioms are satisfied for every* a, b, c, d, e *and* f *in* A :

1. *If* $abDcd$ *and* $cdDef$, *then* $abDef$;
2. $abDcd$ *or* $cdDab$;
3. *If* $abDcd$, *then* $acDbd$;
4. *If* $abDcd$, *then* $dcDba$;
5. *There is a* c *in* A *such that* $acDcb$ *and* $cbDac$;

6. *If* $a\,P\,b$ *and not* $ab\,D\,cd$, *then there is an* e *in* A *such that* $a\,P\,e$, $e\,P\,b$ *and* $cd\,D\,ae$;

7. *If* $a\,P\,b$ *and* $ab\,D\,cd$, *then there are* e,f *in* A *and an* n *such that* $ceM^n fd$ *and* $ce\,D\,ab$.[4/]

These axioms are essentially those given in Suppes and Winet (1955). The first four axioms establish some of the elementary properties of the relation D . Axiom 1 indicates that D is transitive and Axiom 2 that it is strongly connected in A .

The last three axioms are existence axioms and are basic to the proof of the representation and uniqueness theorems. Axiom 5 may be interpreted to mean that between any two elements in A , there exists a third element in A which is a midpoint. A direct consequence of this axiom is that the set A is infinite (in all non-trivial cases). Axiom 6 postulates a kind of continuity condition. And Axiom 7 is the Archimedean axiom. The general Archimedean principle may be formulated as follows. Let L_1 be a distance no matter how large, and let L_2 be a distance no matter how small. Then there is a positive integer n such that an n^{th} part of L_1 is smaller than L_2 . On the other hand, there is a positive integer m such that if we lay off L_2 m times on a line, the resulting distance or length will be greater than L_1 .

4/ We are indebted to Michael Levine for showing that the following axiom is a consequence of Axioms 1-3 and thus may be eliminated: If $b\,I\,a$ or $b\,P\,a$, and $bc\,D\,ef$ then $ac\,D\,ef$.

In other words, any two quantities in an Archimedean system are comparable in measurement. Every system of measurement that leads to an interval or ratio scale must satisfy the Archimedean principle in some form in order for a numerical representation theorem to be proved. Axiom 7 is one appropriate formulation for the system at hand.

The numerical relational system which we shall use to establish the representation theorem for an i.d. system is called a <u>numerical infinite difference system</u> or, more simply, a <u>numerical i.d. system</u>. This numerical relational system is defined as follows. Let N be a non-empty set of real numbers closed under the formation of midpoints, i.e., if x and y are in N then $(x+y)/2$ is in N. Let Δ be the quaternary relation restricted to N such that for any real numbers x, y, z, w in N

$$xy \Delta zw \text{ if and only if } x-y \leq z-w .$$

Then the quaternary system $\mathfrak{N} = \langle N, \Delta \rangle$ is a <u>numerical i.d. system</u>.

As usual we may state the representation theorem either in terms of a homomorphism or in terms of an isomorphism between empirical and numerical relational systems. To utilize the latter alternative we need merely introduce the relational system $\mathfrak{A}/I = \langle A/I, D^* \rangle$ where the set A/I consists of the I-equivalence classes of A and the relation D^* is defined as follows.

<u>Definition</u> 14. $[a][b]\, D^* [c][d]$ <u>if and only if</u> $ab\, D\, cd$.

Thus using the isomorphism concept the representation theorem for an i.d. system may now be stated.

<u>Theorem</u> 12. (<u>Representation</u> <u>Theorem</u>). <u>If a quaternary system</u> $\mathfrak{A} = \langle A, D \rangle$ <u>is an i.d. system, then</u> $\mathfrak{A}/I = \langle A/I, D^* \rangle$ <u>is isomorphic to a numerical i.d. system</u>.

The proof is omitted. (See Suppes and Winet (1955); also for proof of the next theorem.) For the uniqueness problem we have the following theorem.

<u>Theorem</u> 13. (<u>Uniqueness Theorem</u>). <u>If a quaternary system</u> $\mathfrak{A} = \langle A, D \rangle$ <u>is an i.d. system, then any two numerical i.d. systems isomorphic to</u> $\mathfrak{A}/I = \langle A/I, D^* \rangle$ <u>are related by a linear transformation</u>.

The proof of Theorem 13 is also omitted. However, we shall give the much simpler proof of a related theorem. The point of this related theorem is that as long as we restrict ourselves to the primitive and defined notions of quaternary systems it is not possible to do better than obtain measurement unique up to a linear transformation. Thus Theorem 13 cannot be improved by adding additional axioms to those given in the definition of the i.d. system. Since the proof does not depend on any of the axioms of an i.d. system, we may state it for arbitrary quaternary systems. Generalizing the numerical i.d. system slightly, a quaternary system $\mathfrak{N} = \langle N, \Delta \rangle$ is a <u>numerical difference system</u> if N is a set of real numbers and Δ is the numerical quaternary relation defined previously. We may then formulate our result.

<u>Theorem</u> 14. <u>Let a quaternary system</u> $\mathfrak{A} = \langle A, D \rangle$ <u>be isomorphic to a numerical difference system</u> $\mathfrak{N} = \langle N, \Delta \rangle$, <u>and let</u> $\mathfrak{N}' = \langle N', \Delta' \rangle$ <u>be a numerical difference system related to</u> $\mathfrak{N}$ <u>by a linear transformation</u>. <u>Then</u> $\mathfrak{A}$ <u>is isomorphic to</u> $\mathfrak{N}'$.

Proof: The proof of the theorem is very simple; it hinges upon the purely set-theoretical, axiom-free character of the definition of isomorphism. Since the relation of being isomorphic is transitive, to show $\mathfrak{A}$ and $\mathfrak{N}'$ are isomorphic it will suffice to show that $\mathfrak{N}$ and $\mathfrak{N}'$ are isomorphic.

Let f be the linear transformation from N to N'. It is clear that f is the appropriate isomorphism function, for it is one-one, and if for every x in N

$$f(x) = \alpha x + \beta \ , \qquad \alpha > 0$$

we have the following equivalences for any x, y, u and v in N :

$$xy \,\Delta\, uv$$

if and only if

$$x-y \leq u-v$$

if and only if

$$(\alpha x + \beta) - (\alpha y + \beta) \leq (\alpha u + \beta) - (\alpha v + \beta)$$

if and only if

$$f(x) - f(y) \leq f(u) - f(v)$$

if and only if

$$f(x)f(y) \,\Delta'\, f(u)f(v) \quad .$$

Q.E.D.

One interpretation of infinite difference systems is of sufficiently general importance to be emphasized. This interpretation is closely related to classical scaling methods for pair comparisons, about which more is said in Sec. 5.1. Subjects are asked to choose between alternatives or stimuli, and they are asked to make this choice a number of times. There are many situations--from judging the hue of colors to preference among economic bundles--in which subjects vacillate in their choices. The probability p_{ab} that a will be chosen over b may be estimated from the relative frequency with which a is so chosen. From inequalities of the form $p_{ab} \leq p_{cd}$ we then obtain an interpretation of the quaternary relation $ab\, D\, cd$. Thus the representation and uniqueness theorems proved here have direct application to pair comparison methods.

An important problem for infinite difference systems is the idealization involved in the transitivity of the indifference relation I, which is a consequence of the first four axioms. The question naturally arises can infinite difference systems be generalized in the way that semi-orders generalized series or simple orderings. Surprisingly enough, mathematical work on this problem goes back to an early paper of Norbert Wiener (1921). Unfortunately Wiener's paper is extremely difficult to read: it is written in the notation of the latter two volumes of Whitehead and Russell's <u>Principia Mathematica</u>;[5] no clear

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[5] It is of some historical interest to note that a rather elaborate theory of measurement is given in vol. 3 of <u>Principia Mathematica</u>, but as far as we know it has had little impact on the theory of measurement--actually with good reason, for the developments there are more closely

connected with classical mathematical topics like Eudoxus' theory of proportion and the construction of the real numbers than with any formal questions which arise in an empirical context.

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axioms are formulated; and no proofs are given. On the other hand, the focus of the paper is the important problem of explicitly considering non-transitivities that arise from subliminal phenomena. A discussion of similar problems in economic contexts is to be found in the interesting series of papers of W. E. Armstrong (1939), (1948) and (1951). An exact axiomatic reconstruction of Wiener's ideas is to be found in the dissertation of Muriel Wood Gerlach (1957); her axioms are too complicated to state here. Moreover, they suffer from not making distinguishability of stimuli a probabilistic concept, although a probabilistic interpretation similar to that just given for the quaternary relation D is also possible for her primitive concepts.

4.4 Finite Equal Difference Systems

Since the infinite difference systems of the preceding section are not easily realized in many empirical situations, it is desirable to have at hand a finite empirical relational system which yields the same measurement results. To this end we now develop briefly the theory of finite equal difference systems (abbreviated as f.d. systems). (See Suppes, 1957, Chapter 12.) The intuitive idea is that we select a finite set of stimuli such that when we order the stimuli according to some characteristic such as hue, pitch or utility two stimuli adjacent in the ordering have the same difference in intensity as any two other such adjacent stimuli. It is to be emphasized that no sort of underlying physical scale need be assumed to apply this theory. It is not a psychophysical theory of measurement.

One new elementary definition is needed.

Definition 15. aJb if and only if aPb and for all c in A if aPc then either bIc or bPc.

The indifference relation I and the strict preference relation P appearing in Definition 15 are defined in Definitions 9 and 10. The interpretation of J is that aJb holds when a is an immediate predecessor of b with respect to the relation P. In the following definition of an f.d. system the final axiom referring to the relation J replaces the three existence axioms which were used to characterize an infinite difference system. Note that the first four axioms below are just the same as those for i.d. systems.

Definition 16. A finite equal difference system is a quaternary system $\mathfrak{A} = \langle A, D \rangle$ in which A is a finite set and for every a, b, c, d, e and f in A the following five axioms are satisfied.

1. If $abDcd$ and $cdDef$ then $abDef$;
2. $abDcd$ or $cdDab$;
3. If $abDcd$ then $acDbd$;
4. If $abDcd$ then $dcDba$;
5. If aJb and cJd, then $abDcd$ and $cdDab$.

In Axiom 5 the equal spacing assumption is imposed.

For the corresponding numerical relational system we have a numerical f.d. system which we define as follows. Let N be a finite, non-empty set of numbers such that differences between numbers adjacent under the natural ordering $<$ are equal, and let Δ be the numerical quaternary relation already defined restricted to N . Then the quaternary system $\mathfrak{N} = \langle N, \Delta \rangle$ is a numerical f.d. system.

We may now state representation and uniqueness theorems for finite equal difference systems.

Theorem 15 (Representation Theorem). If a quaternary system $\mathfrak{A} = \langle A, D \rangle$ is an f.d. system then $\mathfrak{A}/I = \langle A/I, D^* \rangle$ is isomorphic to a numerical f.d. system.

Theorem 16 (Uniqueness Theorem). If a quaternary system $\mathfrak{A} = \langle A, D \rangle$ is an f.d. system, then any two numerical f.d. systems isomorphic to $\mathfrak{A}/I$ are related by a linear transformation.

One might be tempted to conjecture that the first four axioms of Definition 16 would characterize all finite difference systems for which a numerical representation could be found (the representations of a given system would not necessarily be related by a linear transformation). The resulting theory would then represent one formalization of Coombs' ordered metric scale. However, Scott and Suppes (1958) have proved that the theory of all representable finite difference systems is not characterized by these four axioms, and worse still, cannot be characterized by any simple finite list of axioms.

The f.d. systems are not as artificial or as impractical as they may seem. One theory for approximating these systems is to be found in Davidson, Suppes and Siegel (1957, Chapter 2). However, these systems can have more general usefulness if they are used to establish a "standard set" of stimuli. In the case of tones, for example, a set of tones may be selected in a successive manner so that the set satisfies Axiom 5. If this standard set of tones satisfies the remaining five axioms, then we know from Theorem 15 that the set of tones may be assigned numbers that are on an interval scale. Arbitrary tones which are not in the standard set but which satisfy the first four axioms may then be located within intervals bounded by adjacent tones in the standard set. This means that by decreasing the spacing between the standard tones any arbitrary tone may be measured within any desired degree of accuracy. This is in fact what a chemist does in using a standard set of weights and an equal arm balance to determine the weight of an unknown object. His accuracy of measurement is limited by the size of the smallest interval between the standard weights, or if he also uses a rider, by the gradations on the rider.

Other relational systems closely related to f.d. systems may appropriately be mentioned at this point. Among the simplest and most appealing are the <u>bisection</u> systems $\mathfrak{A} = \langle A, B \rangle$, where B is a ternary relation on the set A with the interpretation that $B(a,b,c)$ if and only if b is the midpoint of the interval between a and c. The method of bisection, which consists in finding the midpoint b, has a long history in psychophysics. The formal criticism of many experiments in which it has been used is that the variety of checks necessary to guarantee isomorphism with an appropriate numerical system are not usually performed. For example, if $B(a,b,c)$ implies that $a\,P\,b$ and $b\,P\,c$, where P is the usual ordering relation, then from the fact that

$B(a,b,c)$ and $B(c,d,e)$ we should be able to infer $B(a,c,e)$. But the explicit test of this inference is too seldom made. Without it there is no real guarantee that a subjective scale for a stimulus dimension has been constructed by the method of bisection.

Because of the large number of axiomatic analyses already given in the section we shall not give axioms for bisection systems. The axioms in any case are rather similar to those of Definition 16, and the formal connection between the difference relation D and the ternary bisection relation B should be obvious:

$$B(a,b,c) \quad \text{if and only if} \quad ab\,D\,bc \quad \text{and} \quad bc\,D\,ab \ .$$

As an alternative to giving general axioms for bisection systems it may be of some interest to look at the problem of characterizing bisection systems in a somewhat different manner, namely, by simply listing for a given number n of stimuli what relations must hold. In perusing this list it should be kept in mind that we assume that bisection systems have the same property of equal spacing possessed by f.d. systems. As examples, let us consider the cases of $n = 5$ and $n = 6$.

For $n = 5$, let $A = \{a,b,c,d,e\}$ with the ordering $aPbPcPdPe$. We then have exactly four instances of the bisection relation, namely, $B(a,b,c)$, $B(b,c,d)$, $B(c,d,e)$ and $B(a,c,e)$.

For $n = 6$, we may add the element f to A with the ordering $aPbPcPdPePf$. To the four instances of the bisection relation for $n = 5$, we now add two more, namely, $B(d,e,f)$ and $B(b,d,f)$. We may proceed in this manner for any n to characterize completely the bisection system with n stimuli, none of which are equivalent with respect to the property being studied. Establishing the representation and uniqueness theorems is then a trivial task. The disadvantages of this approach to characterizing those relational systems for which numerical representation theorems exist are twofold. In the first place, in contrast to the statement of general axioms, the listing of instances does not give us general insight into the structure of the systems. Secondly, for systems of measurement that have a more complicated or less sharply defined structure than do bisection systems, the listing of instances can become tedious and awkward--semi-orders provide a good example.

4.5 Extensive Systems

We consider next a relational system leading to a ratio scale. Since this relational system contains an operation o that corresponds to an addition operation we may justifiably call this system an *extensive system* (see Sec. 2.4). The axioms which we shall use to define an extensive system (Suppes, 1951) are similar to those first developed by Hölder (1901). Hölder's axioms however are more restrictive than necessary in that they require the homomorphic numerical relational systems to be non-denumerable (and non-finite). The present set of axioms apply both to denumerable and non-denumerable but infinite relational systems.

Definition 17. *An* extensive system $\langle A,R,o \rangle$ *is a relational system consisting of the binary relation* R *, the binary operation* o *from* $A \times A$ *to* A *and satisfying the following six axioms for* a,b,c *in* A .

1. *If* $a R b$ *and* $b R c$, *then* $a R c$;

2. $(a \circ b) \circ c R a \circ (b \circ c)$;

3. *If* $a R b$ *then* $a \circ c R c \circ b$;

4. *If not* $a R b$, *then there is a* c *in* A *such that* $a R b \circ c$ *and* $b \circ c R a$;

5. *Not* $a \circ b R a$;

6. *If* $a R b$, *then there is a number* n *such that* $b R na$ *where the notation* na *is defined recursively as follows:*

   $1a = a$ *and* $na = (n - 1)a \circ a$.

It can be shown that the relation R is a weak ordering (it is transitive and strongly connected) of the elements of A . If A is a set of weights then the interpretation of $a R b$ is that a is either less heavy than b or equal in heaviness to b . The interpretation of $a \circ b$ for weights is simply the weight obtained by combining the two weights a and b , e.g., by placing both on the same side of an equal arm balance. Axiom 2 establishes the associativity property of the operation $\circ$. Axiom 5 implies that mass, for example, is always positive. This axiom together with the order properties of R and the definition of $\circ$ as an operation from $A \times A$ to A imply that the set A is infinite. Axiom 6 is another form of the Archimedean principle mentioned earlier.

Again we introduce the indifference relation I so that A may be paritioned into equivalence classes.

Definition 18. $a I b$ *if and only if* $a R b$ *and* $b R a$.

Corresponding to R and o we define R* and o* which are defined for the elements of A/I .

<u>Definition</u> 19. [a]R*[b] <u>if and only if</u> aRb .

<u>Definition</u> 20. [a]o*[b] = [a o b] .

For the representation theorem we seek now a numerical relational system isomorphic to $< A/I, R^*, o^* >$. The numerical system we shall use for this purpose is defined as follows. Let $< N, \leq, + >$ be a numerical relational system in which N is a non-empty set of positive real numbers closed under addition and subtraction of smaller numbers from larger numbers, i.e., if $x, y \in N$ and $x > y$ then $(x + y) \in N$ and $(x - y) \in N$. Let $\leq$ be the usual numerical binary relation and + the usual numerical binary operation of addition, both relations restricted to the set N . Then $< N, \leq, + >$ is a <u>numerical extensive system</u>. An example of a numerical extensive system is the system consisting of the set of positive integers (together with $\leq$ and +) .

The representation and uniqueness theorems can now be expressed as follows.

<u>Theorem</u> 17 (<u>Representation Theorem</u>). <u>If a relational system</u> $\mathfrak{A} = < A, R, o >$ <u>is an extensive system, then</u> $\mathfrak{A}/I = < A/I, R^*, o^* >$ <u>is isomorphic to a numerical extensive system.</u>

The proof of this theorem which we omit (see Suppes, 1951) consists in defining the numerical assignment f as follows:

$$f([a]) = \text{the greatest lower bound of } S([a],[e])$$

where S([a],[e]) , a set of rational numbers, is given by,

$$S([a],[e]) = \{ \tfrac{m}{n} \mid n[a]R^*m[e] \text{ , n, m positive integers} \}$$

where e is an arbitrarily chosen element from A and where $n[a]$ is defined recursively: $1[a] = [a]$ and $n[a] = (n - 1)[a] \circ^* [a]$. Since $f([e]) = 1$, the choice of $[e]$ corresponds to the choice of a unit. The remainder of the proof consists in showing that f has the required properties, namely, that

1. $[a]R^*[b]$ if and only if $f([a]) \leq f([b])$;
2. $f([a]\circ^*[b]) = f([a]) + f([b])$;
3. If $[a] \neq [b]$, then $f([a]) \neq f([b])$, i.e., f is one-one.

<u>Theorem</u> 18 (<u>Uniqueness Theorem</u>). <u>If a relational system $\mathfrak{A} = \langle A,R,o \rangle$ is an extensive system, then any two numerical extensive systems isomorphic to $\mathfrak{A}/I = \langle A/I,R^*,o^* \rangle$ are related by a similarity transformation.</u>

Proof: Let g be any numerical assignment establishing an isomorphism between the system $\langle A/I,R^*,o^* \rangle$ and some numerical extensive system. It will suffice to show that g is related by a similarity transformation to the function f defined above. Let $g([e]) = \alpha$. We show by a reductio ad absurdum that for every a in A

$$g([a]) = \alpha f([a]) \quad . \tag{1}$$

Suppose now that for some a in A

$$g([a]) < \alpha f([a]) \quad . \tag{2}$$

From (2) it follows that a rational number $\frac{m}{n}$ exists such that

$$\frac{g([a])}{\alpha} < \frac{m}{n} < f([a]) \text{ ,} \tag{3}$$

which from the definition of f implies that

$$m[e]R^*n[a] \quad . \tag{4}$$

However, by our initial assumption g is also a numerical assignment which establishes the desired isomorphism. Hence from (4) we have

$$mg([e]) \leq ng([a]) \quad , \tag{5}$$

which, because $g([e]) = \alpha$, can be written as

$$\frac{m}{n} \leq \frac{g([a])}{\alpha} \quad . \tag{6}$$

But (6) contradicts (3). Similarly, by assuming there exists an a in A such that $\alpha f([a]) < g([a])$, we may also arrive at a contradiction. Hence (1) is established. Q.E.D.

Although Theorem 18 asserts that extensive systems lead to ratio scales this should not be construed as implying, as some have suggested (see 2.4), that only extensive systems will yield these scales. As a brief example of a non-extensive system (a system not containing the operation o) leading to a ratio scale let us construct a system along the lines of an i.d. or f.d. system $< A,D >$, but with the following modifications. Let B be a set of elements drawn from $A \times A$, i.e., if $e = (a,b)$ is in B then a and b are in A. Let S be the binary relation on B corresponding to the relation D, i.e., if $e = (a,b)$ and $f = (c,d)$ are in B then eSf if and only if $ab\,D\,cd$. By using a set of axioms corresponding to those of an i.d. or f.d. system, we may conclude that the relational system $< B,S >$ will yield a ratio scale. This follows from the fact that infinite difference and finite difference systems lead to interval scales and the intervals of such a scale lie on a ratio scale.

There are two remarks we want to make about extensive systems to conclude this brief analysis of them. The first concerns the necessity of interpreting the operation $\circ$ as numerical addition. That this is not necessary is shown by the fact that it is a simple matter to construct another representation theorem in which the operation $\circ$ corresponds to the multiplication operation $\cdot$. One simple way of establishing the existence of a numerical system $\mathfrak{N}^* = \langle N^*, \leq, \cdot \rangle$ homomorphic to the extensive system $\mathfrak{A} = \langle A, R, \leq \rangle$ is to apply an exponential transformation to $\mathfrak{N} = \langle N, \leq, + \rangle$. That is, let

$$N^* = \{y \mid y = e^x \text{ for some } x \text{ in } N\} \quad .$$

Obviously $\mathfrak{N}$ is isomorphic to $\mathfrak{N}^*$, and since $\mathfrak{A}$ is homomorphic to $\mathfrak{N}$ it is therefore homomorphic to $\mathfrak{N}^*$ as well. From a mathematical standpoint the representation theorem based on $\mathfrak{N}^*$ is as valid and useful as the one based on $\mathfrak{N}$ so there is no basis for interpreting the operation $\circ$ as intrinsically an addition operation, rather than, say, a multiplication operation.

Which representation theorem you choose does of course affect the uniqueness properties of the numerical assignment. The numerical assignment $f(a)$ which maps $\mathfrak{A}$ onto $\mathfrak{N}$ we know from Theorem 18 is determined up to a similarity transformation. But since $y = \exp(x)$, if x is transformed to kx then y will transform to y^k . Thus the numerical assignment $f'(a)$ which maps $\mathfrak{A}$ onto $\mathfrak{N}^*$ is determined up to a power transformation, not a similarity transformation.

The second remark concerns the fact that for any extensive system $\mathfrak{A} = \langle A, R, o \rangle$, the set A must be infinite. It is the most patent fact of empirical measurement that to determine the weight or length of a physical object it is sufficient to consider only a finite number of objects. The difficulty with Definition 17 is a too slavish imitation of the number system. The essential point is that the empirical ternary relation of combination that is meant to correspond to the arithmetical operation of addition should not actually have all the formal properties of numerical addition. In particular, in order to avoid the infinity of A, it is simplest to drop the closure requirement on the operation o in Definition 18 and replace it by a ternary relation that is technically not a binary operation on A. With this change we can then construct a theory of finite extensive systems which is very similar to the theory of finite difference systems. Finite extensive systems thus constructed correspond very closely in structure to standard series of weights and measures commonly used in physical and chemical laboratories. We do not pursue in this chapter the axiomatic analysis of finite extensive systems because they are more pertinent to physics than to psychology. The important methodological point is that from the standpoint of fundamental measurement there is no difference between difference systems and extensive systems, finite or infinite. One system is just as good a methodological example as the other of fundamental measurement.

4.6 Single and Multidimensional Coombs Systems

Up to this point the requirements of fundamental measurement have been formulated in terms of a homomorphism between empirical and numerical relational systems. There are several ways this definition of fundamental measurement can be extended.

One extension can be obtained by broadening the notion of an empirical relational system to encompass more than one domain, that is, more than one set of physical entities to be measured. In many empirical situations, subjects and stimuli have different, non-interchangeable roles, so that it is often convenient to group them separately. For example, if a is a subject, α and β pictures, and T the relation $T(a,\alpha,\beta)$ if and only if a likes α at least as much as β, then we can be certain that $T(\alpha,a,\beta)$ does not obtain. In fact, if A_1 is a set of subjects and A_2 a set of stimuli and T is defined as above, then we can expect $T(a,\alpha,\beta)$ to imply that $a \in A_1$ and $\alpha, \beta \in A_2$. Thus if it were desirable to measure both subjects and stimuli simultaneously, it would be natural to define the empirical system $\mathfrak{A}^* = \langle A_1, A_2, T \rangle$ consisting of the two domains A_1 and A_2 and the ternary relation T. Although it is clear how one could then proceed to obtain separate numerical assignments for A_1 and A_2, another approach can be taken which, while somewhat less natural, is more appropriate to the general framework we have constructed.

Let $A = A_1 \cup A_2$, i.e., let A be the union of the sets A_1 and A_2. Using the fact that a subset of A is a one-place relation, the system $\mathfrak{A} = \langle A, A_1, A_2, T \rangle$ is nothing more than a relational system.

Hence the usual notions of homomorphism, etc., are directly relevant. Let us carry this example a bit further so as to illustrate how Coombs' unfolding technique (1950) may be formalized in terms of fundamental measurement.

Let A, A_1 and A_2 be sets such that $A = A_1 \cup A_2$, and T a ternary relation such that $T(a,\alpha,\beta)$ implies $a \in A_1$, and $\alpha, \beta \in A_2$. Then the relational system $< A, A_1, A_2, T >$ is a <u>preferential system</u>. By a <u>numerical preferential system</u> we shall mean a numerical relational system $\mathcal{N} = < N, N_1, N_2, S >$ in which N, N_1 and N_2 are sets of real numbers, $N = N_1 \cup N_2$ and S is a ternary numerical relation such that for all $x \in N_1$ and $\zeta, \omega \in N_2$

$$S(x,\zeta,\omega) \text{ if and only if } |x - \zeta| \leq |x - \omega| .$$

To map a preferential system $< A, A_1, A_2, T >$ homomorphically onto a numerical preferential system $< N, N_1, N_2, S >$ we desire a function f such that for all $a \in A_1$ and $\alpha, \beta \in A_2$

$$T(a,\alpha,\beta) \text{ if and only if } |f(a) - f(\alpha)| \leq |f(a) - f(\beta)| .$$

If such a function exists for the preferential system $< A, A_1, A_2, T >$ then the system can be called a <u>Coombs system</u>. (What we here call a Coombs system corresponds to Quadrant Ia data in Coombs, 1960.) Obviously if a preferential system is a Coombs system then a representation theorem involving a numerical preferential system can be readily established. While there do not appear to be any simple necessary and sufficient conditions which indicate when a given preferential system is a Coombs system, some necessary conditions can easily be stated. For example: for all a in A_1, and α,β,γ in A_2

1) If $T(a,\alpha,\beta)$ and $T(a,\beta,\gamma)$ then $T(a,\alpha,\gamma)$;

2) $T(a,\alpha,\beta)$ or $T(a,\beta,\alpha)$.

Define for each subject a , a binary relation Q_a as follows on A_2 :

$$\alpha Q_a \beta \text{ if and only if } T(a,\alpha,\beta) \quad .$$

It follows at once from (1) and (2) that

3) If $\alpha Q_a \beta$ and $\beta Q_a \gamma$ then $\alpha Q_a \gamma$;

4) $\alpha Q_a \beta$ or $\beta Q_a \alpha$.

From (3) and (4) we may derive for each subject a a simple ordering of the stimuli, or, in Coombs' terminology, obtain a set of I-scales. Although assumptions (1) and (2) do not guarantee the existence of a Coombs system, in actual practice when the number of stimuli is not too large, it is a relatively simple matter to determine from inspection of the I-scales whether a given preferential system is a Coombs system. The uniqueness question or scale type of a Coombs system also does not have a simple general solution, but again for any particular instance, certain statements can be made about the order relation between some of the numerical intervals.

A second modification of fundamental measurement can be made to include multidimensional scaling methods. The extension to multidimensional methods is actually quite simple and direct.

Let us define an r-dimensional numerical vector relational system $< N_r, S_1, \ldots, S_N >$ as follows: N_r is a set of r-dimensional vectors

$\mathbf{x} = (x_1, x_2, \ldots, x_r)$ where each x_i is a real number and $S_1, \ldots, S_N$ are relations on the vectors in N_r. Then the definition of an r-<u>dimensional</u> <u>homomorphism</u> (or isomorphism) between an empirical relational system $< A, R_1, \ldots, R_N >$ and an r-dimensional numerical relational system $< N_r, S_1, \ldots, S_N >$ is an obvious extension of the one dimensional case. The difference is merely this: the range of the numerical assignment f is now a set of vectors or r-tuples rather than a set of real numbers. An r-<u>dimensional</u> <u>representation</u> <u>theorem</u> can be defined in terms of the establishment of an r-dimensional homomorphism.

As an illustration of a multidimensional measurement theory, let us consider further the theory of preferential systems. To obtain an r-dimensional representation theorem for a given preferential system $< A, A_1, A_2, T >$ we wish to show that the preferential system is an r-<u>dimensional</u> <u>Coombs</u> <u>system</u>. This means that the preferential system is homomorphic to an r-<u>dimensional</u> <u>numerical</u> <u>preferential</u> <u>system</u> $< N_r, N_1, N_2, S >$ in which N_r, N_1, N_2 are sets of r-dimensional vectors, $N_1 \cup N_2 = N_r$ and S is a numerical relation on N_r such that for all $\mathbf{x} \in N_1$, and $\boldsymbol{\xi}$, $\boldsymbol{\omega}$ N_2

$$S(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\omega}) \text{ if and only if } |\mathbf{x} - \boldsymbol{\xi}| \leq |\mathbf{x} - \boldsymbol{\xi}| \,,$$

the notation $|\mathbf{x}|$ denoting the magnitude of vector $\mathbf{x}$. Using the notation $\mathbf{x} \cdot \mathbf{x}$ or $\mathbf{x}^2$ to indicate the scalar product of vector $\mathbf{x}$ with itself, the relation S may also be defined as

$$S(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\omega}) \text{ if and only if } (\mathbf{x} - \boldsymbol{\xi})^2 \leq (\mathbf{x} - \boldsymbol{\omega})^2 \,.$$

Letting $\underset{\sim}{x} = (x_1, x_2, \ldots, x_r)$, $\underset{\sim}{\xi} = (\xi_1, \xi_2, \ldots, \xi_r)$ and $\underset{\sim}{\omega} = (\omega_1, \omega_2, \ldots, \omega_r)$, the relation S can also be expressed in terms of the components of the vectors:

$$S(\underset{\sim}{x}, \underset{\sim}{\xi}, \underset{\sim}{\omega}) \text{ if and only if } (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + \cdots (x_r - \xi_r)^2$$
$$\leq (x_1 - \omega_1)^2 + (x_2 - \omega_2)^2 + \cdots + (x_r - \omega_r)^2 .$$

The restrictions to be imposed on a preferential system to guarantee the existence of an r-dimensional Coombs system (and hence an r-dimensional representation theorem) are not easy to state. Some general statements can be made if the preferential system satisfies the two assumptions described above. If, for example, the number of stimuli in A_1 is n , then an $(n-1)$ dimensional representation theorem will always exist. Generally, however, it is desirable to prove an r-dimensional representation theorem in which r , the dimensionality of the space, is minimal. Conditions which establish a lower bound on the value of r for which an r-dimensional representation theorem can be proved are given by Bennett and Hays (1960).

5. EXAMPLES OF DERIVED MEASUREMENT

To explore in some depth the formal issues that arise in derived measurement we mainly concentrate in this section on pair comparison methods.

5.1 Bradley-Terry-Luce Systems for Pair Comparisons

We have already sketched the formal properties of these systems in Sec. 3, but we shall recapitulate briefly here. The theory is based on Bradley and Terry (1953); Bradley (1954a, 1954b, 1955); Luce (1959).

Let B be a non-empty, finite set of objects or stimuli and p be a numerical assignment on $B \times B$ such that for all a, b in B, $0 < p_{ab} < 1$, $p_{ab} + p_{ba} = 1$ and $p_{aa} = 1/2$. Then a derived measurement system $\mathcal{B} = \langle B, p \rangle$ will be called a pair comparison system. Generally p_{ab} will be the relative frequency with which a is preferred to, or is greater in some sense, than b. Note that a pair comparison system is not a relational system, because the values of the function p do not form a subset of B, and hence at the outset it should be clear that we are not involved in the present discussion with fundamental measurement.

We recall that a derived scale is a triple $\langle \mathcal{B}, R, g \rangle$ where $\mathcal{B}$ is a derived measurement system, R is a representing relation and g is a numerical assignment defined on B. In order for there to be numerical assignments satisfying the usual representing relations for the Bradley-Terry-Luce (or B.T.L.) theory it is necessary to impose a restriction on pair comparison systems.

Definition 21. A pair comparison system $\mathcal{B} = < B, p >$ is a B.T.L. system if and only if the multiplication condition holds, i.e., if and only if for all a, b, c in B

$$\left(\frac{p_{ab}}{p_{ba}}\right)\left(\frac{p_{bc}}{p_{cb}}\right) = \frac{p_{ac}}{p_{ca}} .$$

The standard representing relation for B.T.L. systems is the following, which we label R_1, and, following notation in the literature, let v_1 be the derived numerical assignment.

Definition 22. $R_1(p, v_1)$ if and only if for all a, b in B

$$\frac{v_{1a}}{v_{1a} + v_{1b}} = p_{ab} .$$

(Here and subsequently we often write v_a instead of $v(a)$ for the value of the numerical assignment v.)

We may easily prove the following representation theorem.

Theorem 19 (Representation Theorem I). If a pair comparison system $\mathcal{B} = < B, p >$ is a B.T.L. system, then there exists a numerical assignment v_1 such that $< \mathcal{B}, R_1, v_1 >$ is a derived scale.

Proof: We define a function v on B as follows (for convenience of notation we omit the subscript on v). Let a be any element of B and let v_a be any positive real number. For every b in B we then define

$$v_b = \frac{p_{ba}}{p_{ab}} v_a .$$

We then have for any b and c in B

$$\frac{v_b}{v_b + v_c} = \frac{\frac{p_{ba}}{p_{ab}} v_a}{\frac{p_{ba}}{p_{ab}} v_a + \frac{p_{ca}}{p_{ac}} v_a} .$$

Now by virtue of the multiplication rule of Definition 22

$$\frac{p_{ca}}{p_{ac}} = \left(\frac{p_{cb}}{p_{bc}}\right)\left(\frac{p_{ba}}{p_{ab}}\right) .$$

Substituting this result in above and canceling v_a, we obtain

$$\frac{v_b}{v_b + v_c} = \frac{\frac{p_{ba}}{p_{ab}}}{\frac{p_{ba}}{p_{ab}} + \left(\frac{p_{cb}}{p_{bc}}\right)\left(\frac{p_{ba}}{p_{ab}}\right)}$$

$$= \frac{1}{1 + \frac{p_{cb}}{p_{bc}}} = \frac{p_{bc}}{p_{bc} + (1 - p_{bc})} = p_{bc}$$

the desired result.

It can also be readily established that the existence of a derived scale $< \mathcal{B}, R_1, v >$ is a sufficient condition for the pair comparison system $\mathcal{B} = < B, p >$ to be a B.T.L. system.

When the numerical assignments in the derived measurement system $\mathcal{B}$ are all on an absolute scale it is evident that the narrow and wide senses of scale type defined in Sec. 3 coincide. This is the situation for pair comparison systems because the relative frequency function p_{ab} is on an absolute scale. We may easily prove that any two derived assignments v_1 and v_1' for a B.T.L. system are related by a positive or negative similarity transformation. Let us call a scale for which an admissible similarity transformation may be either positive or negative a generalized ratio scale.

Theorem 20 (Uniqueness Theorem I). *If* $\mathcal{B} = \langle B, p \rangle$ *is a* B.T.L. *system, then a derived scale* $\langle \mathcal{B}, R_1, v_1 \rangle$ *is a generalized ratio scale in both the narrow and wide senses.*

Proof: Let $\langle \mathcal{B}, R_1, v \rangle$ and $\langle \mathcal{B}, R_1, v' \rangle$ be two derived scales. We show that v and v' are related by a positive or negative similarity transformation. First we observe that for all a in B, $v_a \neq 0$ and $v_a' \neq 0$, for otherwise we would not have $p_{aa} = 1/2$. Let a be any element of B. Then there must be a k such that either $k > 0$ or $k < 0$ and

$$v_a' = k\, v_a \quad .$$

Suppose now there is a b in A such that

$$v_b' \neq k\, v_b \quad .$$

Let then

$$v_b' = (k + \epsilon)\, v_b \ ,$$

with $\epsilon \neq 0$. Then on the basis of Definition 22

$$\frac{v_a}{v_a + v_b} = p_{ab} = \frac{v_a'}{v_a' + v_b'} \ , \tag{1}$$

but

$$\frac{v'_a}{v'_a + v'_b} = \frac{kv_a}{kv_a + kv_b + \epsilon v_b} \quad . \tag{2}$$

Combining (1) and (2), cross-multiplying and then simplifying, we have at once that

$$\epsilon v_a \ v_b = 0 \quad ,$$

which is absurd on the basis of our supposition. This establishes that in fact $< \mathcal{B}, R_1, v >$ is a generalized ratio scale.

We may require that v be positive by modifying Definition 22.

Definition 23. $R_2(p, v_2)$ *if and only if for all* a, b *in* B

$$\frac{v_{2a}}{v_{2a} + v_{2b}} = p_{ab} \ , \ v_{2a} > 0 \quad .$$

The existence of a function v_2 follows immediately from the proof of Theorem 19.

Theorem 21 (Representation Theorem II). *If* $\mathcal{B} = < B, p >$ *is a* B.T.L. *system, then there exists a numerical assignment* v_2 *such that* $< \mathcal{B}, R_2, v_2 >$ *is a derived scale.*

By the same argument used to establish Theorem 20 we also have:

Theorem 22 (Uniqueness Theorem II). *If* $\mathcal{B} = < B, p >$ *is a* B.T.L. *system, then a derived scale* $< \mathcal{B}, R_2, v_2 >$ *is a ratio scale in both the narrow and wide senses.*

As far as we can see the choice between representing relations R_1 and

R_2 is essentially arbitrary, although it is perhaps intuitively more satisfactory to have $v_a > 0$.

However, R_1 and R_2 do not exhaust the possibilities, as we now wish to show. First, we define:

Definition 24. $R_3^{(k)}(p, v_3)$ if and only if for all a, b in B

$$\frac{v_{3a}^k}{v_{3a}^k + v_{3b}^k} = p_{ab}\ ,\ v_{3a} > 0\ ,\ k > 0\ .$$

In this case, v_a is raised to a power k and k is a parameter of the relation $R_3^{(k)}$. We have as before:

Theorem 23 (Representation Theorem III). If $\mathcal{B} = < B, p >$ is a B.T.L. system, then there exists a function v_3 such that $< \mathcal{B}, R_3^{(k)}, v_3 >$ is a derived scale.

Proof: Use the function v_1 defined in the proof of Theorem 19 and take the k^{th} root of v_{1a} for each a in B to obtain the numerical assignment needed for the present theorem.

For fixed k, i.e., for the parameter k of R_3, it is easily seen that we have a ratio scale.

Theorem 24 (Uniqueness Theorem III). If $\mathcal{B} = < B, p >$ is a B.T.L. system, then a derived scale $< \mathcal{B}, R_3^{(k)}, v_3 >$ is a ratio scale in both the narrow and wide senses.

On the other hand, by letting k change from one derived assignment to another we obtain a log interval scale, i.e., there exist α, $\beta > 0$ such that for all x,

$$\varphi(x) = \alpha x^\beta$$

where φ is an admissible transformation of the scale. The definition of R_4 is as follows:

Definition 25. $R_4(p, v_4)$ if and only if there is a $k > 0$ such that for all a and b in A

$$\frac{v_{4a}^k}{v_{4a}^k + v_{4b}^k} = p_{ab} \;, \; v_{4a} > 0 \;.$$

Since the descriptions of alternative representation theorems parallel Theorem 19 and since their proofs also follow directly from this theorem, we shall henceforth omit further explicit descriptions of these representation theorems. For the uniqueness theorem corresponding to $R_4(p, v_4)$ we have:

Theorem 25 (Uniqueness Theorem IV). If $\mathcal{B} = < B, p >$ is a B.T.L. system then a derived scale $< \mathcal{B}, R_4, v_4 >$ is a log interval scale in both the narrow and wide senses.

Proof: We first observe that if $< \mathcal{B}, R_4, v >$ is a derived scale for $\mathcal{B}$, (again omitting the subscript for convenience) then if we define for a in B

$$v'_a = \alpha v_a^\beta$$

with α , $\beta > 0$, then $< \mathcal{B}, R_4, v' >$ is also a derived scale for $\mathcal{B}$. For

$$v_a = \left(\frac{v'_a}{\alpha}\right)^{1/\beta} \;,$$

and thus from

$$\frac{v_a^k}{v_a^k + v_b^k} = p_{ab}$$

we infer

$$\frac{\left(\frac{v'_a}{\alpha}\right)^{k/\beta}}{\left(\frac{v_a}{\alpha}\right)^{k/\beta} + \left(\frac{v'_b}{\alpha}\right)^{k/\beta}} = p_{ab} \quad ,$$

but from the right-hand side of the last equation we may cancel $\frac{1}{\alpha}^{k/\beta}$ and letting $k' = k/\beta$, we have

$$\frac{{v'_a}^{k'}}{{v'_a}^{k'} + {v'_b}^{k'}} = p_{ab} \quad .$$

On the other hand, let $< \mathscr{B}, R_4, v >$ and $< \mathscr{B}, R_4, v' >$ be two derived scales for $\mathscr{B}$. We show that they are related by a power transformation $\varphi(x) = \alpha x^\beta$. Let k' and k be the powers associated with v' and v by (4). As above, we then let $\beta = k/k'$. Let now a be any element of B. There must be an $\alpha > 0$ such that

$$v'_a = \alpha v_a^\beta \quad .$$

Suppose now, contrary to the theorem that there is a b in B such that

$$v'_b \neq \alpha v_b^\beta \quad .$$

Let

$$v'_b = \alpha v_b^{\beta} + \epsilon \quad .$$

Then on the basis of (4)

$$\frac{{v'_a}^{k'}}{{v'_a}^{k'} + {v'_b}^{k'}} = \frac{v_a^k}{v_a^k + v_b^k} = p_{ab} \quad .$$

Substituting and cross multiplying we have:

$$\alpha v_a^{\beta k'+k} + \alpha v_a^{\beta k'} v_b^k = \alpha v_a^{\beta k'+k} + v_a^k \, (\alpha v_b^{\beta} + \epsilon)^{k'} \quad .$$

Remembering that $\beta = k/k'$, we see that this equation can only hold if $\epsilon = 0$ contrary to our supposition, which establishes the theorem.

Still another representing relation for B.T.L. systems is $R_5(p, v_5)$ defined as follows:

Definition 26. $R_5(p, v_5)$ if and only if there is a $k > 0$ such that for all a and b in A

$$\frac{v_{5a} + k}{v_{5a} + v_{5b} + 2k} = p_{ab} \quad .$$

The representation theorem for B.T.L. systems based on $R_5(p, v_5)$ is an immediate consequence of Theorem 19. For the uniqueness theorem we have:

Theorem 26 (Uniqueness Theorem V). If $\mathscr{B} = \langle B, p \rangle$ is a B.T.L. system then $\langle \mathscr{B}, R_5, v_5 \rangle$ is an interval scale in both the narrow and wide senses.

The proof of Theorem 26 depends upon the observation that if the function v_{5a} satisfies Definition 26, then the function $v'_{5a} = \alpha v_a + \beta$, α, $\beta > 0$, will also satisfy Definition 26. That is,

$$\frac{v'_{5a} + k'}{v'_{5a} + v'_{5b} + 2k'} = p_{ab}$$

where $k' = k\alpha - \beta$.

Finally, we note that by appropriate choice of the representing relation we may also obtain a difference scale for B.T.L. systems. We define:

Definition 27. $R_6(p, v_6)$ if and only if for all a, b in B

$$\frac{1}{1 + e^{v_{6b} - v_{6a}}} = p_{ab} \quad .$$

To prove the representation theorem based on $R_6(p, v_6)$ it is simplest to transform the function v_1 given in Definition 1 by a log transformation. The following uniqueness theorem is also easily proved.

Theorem 27 (Uniqueness Theorem VI). If $\mathcal{B} = < B, p >$ is a B.T.L. system then $< \mathcal{B}, R_6, v_6 >$ is a difference scale in both the narrow and wide senses.

In the discussion of that most classical of all cases of measurement, extensive systems, in the previous section, we saw that radically different representation theorems could be proved with resulting variations in scale type. Precisely the same thing obtains in derived measurement, as we have shown in the present section by exhibiting six different representing relations for B.T.L. systems. The choice among them, it seems to us, can objectively only be based on considerations of computational convenience. It is not possible to claim that a B.T.L. system has as its only correct numerical assignment one that lies on a ratio, log interval or difference scale. Any one of the three, and others as well, are acceptable.

It may be observed, however, that although there is no strict mathematical argument for choosing a representation, say R_2, that yields a ratio scale rather than one, say R_4, that yields a log interval scale, it does not seem scientifically sensible to introduce an additional parameter like k that is not needed and has no obvious psychological interpretation. If a representation like R_4 is used, it is hard to see why a representation introducing additional parameters may not also be considered. For example:

Definition 28. $R_7(p, v)$ if and only if there are positive numbers k_1, k_2, and k_3 such that for all a, b in B

$$\frac{k_1 v_a^{k2} + k_3}{k_1 v_a^{k2} + k_3 + k_1 v_b^{k2} + k_3} = p_{ab} .$$

But once this line of development is begun it has no end. When a mathematically unnecessary parameter is introduced, there should be very strong psychological arguments to justify it.

5.2 Thurstone Systems for Pair Comparisons

In contrast to a B.T.L. system a Thurstone system imposes somewhat different restrictions on a pair comparison system. We shall consider here just two of the five versions or cases described by Thurstone. Although these cases can be described using less restrictive assumptions than those originally proposed by Thurstone (see Torgerson, 1958), for the purpose of simplicity we shall follow Thurstone's treatment.

Although it would be desirable to define a Thurstone system entirely in terms of p_{ab} , p_{bc} , p_{ac} , etc., as we did in Definition 21 for a B.T.L. system, there is unfortunately no simple analogue of the multiplication theorem for Thurstone systems. That is, there is no simple equation which places the appropriate restriction on a pair comparison system. This means that we shall be forced to define a Thurstone system in terms of the existence of one of the representing relations. Consequently, the representation theorem based on this relation will be a trivial consequence of the definition.

Let $N(x)$ be the unit normal cumulative, i.e., the cumulative distribution function with mean zero and unit variance. We then have the following definitions of case III and case V-Thurstone systems.

Definition 29. A pair comparison system $\mathcal{B} = \langle B, p \rangle$ is a case III Thurstone system if and only if there are functions μ_a and σ_a^2 such that for all a , b in B

$$p_{ab} = N\left(\frac{\mu_a - \mu_b}{\sqrt{\sigma_a^2 + \sigma_b^2}}\right) .$$

If, moreover, for all a and b in B , $\sigma_a^2 = \sigma_b^2$, $\mathcal{B}$ is a case V - Thurstone system.

The additional restriction for case V is equality of the variances for the normal distributions corresponding to elements of B . The first representing relation we shall consider for case V - Thurstone systems is the obvious one:

(1) $S_1(p, \mu)$ if and only if there is a $\sigma > 0$ such that for all a and b in B

$$p_{ab} = N\left(\frac{\mu_a - \mu_b}{\sqrt{2}\,\sigma}\right) .$$

The proof of the representation theorem for case V - Thurstone systems based on $S_1(p, \mu)$ is as we indicated an obvious consequence of Definition 29. We shall nevertheless state the theorem to show the analogue of Theorem 19 for B.T.L. systems.

Theorem 27 (Representation Theorem I). If a pair comparison system $\mathcal{B} = < B, p >$ is a case V - Thurstone system, then there exists a function μ such that $< \mathcal{B}, S_1, \mu >$ is a derived scale.

The uniqueness theorem corresponding to this representation theorem is as follows.

Theorem 28 (Uniqueness Theorem I). If $\mathcal{B} = < B, p >$ is a case V - Thurstone system, then $< \mathcal{B}, S_1, \mu >$ is an interval scale in both the narrow and wide senses.

We shall omit detailed proofs of the uniqueness theorems in this section except to point out that the admissible transformations described by the uniqueness theorems do lead to numerical assignments which also satisfy the appropriate representing relation. Here it may be observed that if μ satisfies $S_1(p, \mu)$ then $\mu' = \delta\mu + \beta$, $\delta > 0$, will satisfy $S_1(p, \mu')$ when $\sigma' = \delta\sigma$, since then

$$\frac{\mu_a - \mu_b}{\sqrt{2}\ \sigma} = \frac{\mu_a' - \mu_b'}{\sqrt{2}\ \sigma'} .$$

In other words, if $< \mathcal{B}, S_1, \mu >$ is a derived scale for $\mathcal{B}$, then so is $< \mathcal{B}, S_1, \mu' >$. Other representation theorems can be proved for case V - Thurstone systems in which σ plays somewhat different roles. We consider two such possibilities.

(2) $S_2^{(\sigma)}(p, \mu)$ if and only if for all a, b in B

$$p_{ab} = N\left(\frac{\mu_a - \mu_b}{\sqrt{2}\ \sigma}\right) .$$

Here σ is a parameter of the relation $S_2^{(\sigma)}(p, \mu)$. The representation theorem for $S_2^{(\sigma)}(p, \mu)$ corresponds, mutatis mutandis, to Theorem 27. For a fixed σ we obviously have a difference scale, rather than an interval scale. That is, if $\mu' = \mu + \beta$, then if $< \mathcal{B}, S_2^{(\sigma)}, \mu >$ is a derived scale so is $< \mathcal{B}, S_2^{(\sigma)}, \mu' >$.

<u>Theorem</u> 29 (<u>Uniqueness</u> <u>Theorem</u> <u>II</u>). <u>If $\mathcal{B}$ is a case V - Thurstone system, then $< \mathcal{B}, S_2^{(\sigma)}, \mu >$ is a difference scale in both the narrow and wide senses.</u>

Another treatment of the σ parameter is to suppress it entirely by defining the numerical assignment $\mu_a' = \dfrac{\mu_a}{\sqrt{2}\ \sigma}$. In terms of representing relations we have:

(3) $S_3(p, \mu)$ if and only if for all a, b in B

$$p_{ab} = N(\mu_a - \mu_b) \quad .$$

Clearly $< \mathcal{B}, S_3, \mu >$ is a difference scale also so that we shall omit explicit description of both the representation and uniqueness theorems based on S_3.

To demonstrate the possibility of obtaining ratio scales for case V - Thurstone systems we introduce the following representing relation:

(4) $S_4(p, \mu)$ if and only if for all a, b in B

$$p_{ab} = N\left(\log \frac{\mu_a}{\mu_b}\right) .$$

The proof of the representation theorem based on S_4 follows from Theorem 27, for if we let

$$\mu_a = \exp \frac{\mu'_a}{\sqrt{2}\,\sigma}$$

then if μ' satisfies $S_1(p, \mu')$, μ will satisfy $S_4(p, \mu)$.

For the uniqueness theorem we have.

<u>Theorem</u> 30 (<u>Uniqueness Theorem IV</u>). <u>If $\mathcal{B}$ is a case V - Thurstone system, then $< \mathcal{B}, S_4, \mu >$ is a generalized ratio scale in both the narrow and wide senses</u>.

We observe that the similarity transformation is certainly an admissible transformation since if $\mu' = \delta\mu$,

$$\log \frac{\mu_a}{\mu_b} = \log \frac{\mu'_a}{\mu'_b} \quad .$$

To obtain a log interval scale for case V - Thurstone systems we define:

(5) $S_5(p, \mu)$ if and only if there exists a k such that for all a, b in B

$$p_{ab} = N\left(k \log \frac{\mu_a}{\mu_b}\right) .$$

The corresponding representation theorem can easily be proved from the previous (unstated) representation theorem by defining the function μ as the k^{th} root of the function μ' which satisfies $S_4(p, \mu')$. It may be verified that if the function μ exists which satisfies $S_5(p, \mu)$, the function $\mu' = \delta\mu^\beta$ satisfies $S_5(p, \mu')$ when $k' = k/\beta$ since then

$$k \log \frac{\mu_a}{\mu_b} = k' \log \frac{\mu'_a}{\mu'_b} .$$

Again we merely state the uniqueness theorem without proof.

<u>Theorem</u> 31 (<u>Uniqueness Theorem V</u>). <u>If $\mathcal{B}$ is a case V - Thurstone system, then $< \mathcal{B}, S_5, \mu >$ is a generalized log interval scale in both the narrow and wide senses</u>.

For case III - Thurstone systems, we mention just one representation relation:

(6) $S_6(p, \mu)$ if and only if for every a and b in A there exist positive numbers σ_a^2 and σ_b^2 such that

$$p_{ab} = N\left(\frac{\mu_a - \mu_b}{\sqrt{\sigma_a^2 + \sigma_b^2}}\right).$$

As in previous examples, we omit the obvious representation theorem. The uniqueness theorem, on the other hand, does not appear to be simple, and as far as we know, the exact solution is not known. The following simple counterexample shows that the result must be something weaker than an interval scale. Let

$$\mu_a = 1 \qquad \sigma_a^2 = 1$$

$$\mu_b = 2 \qquad \sigma_b^2 = 2$$

$$\mu_c = 4 \qquad \sigma_c^2 = 3 \quad .$$

We now transform to a new function μ' without changing p_{ab} :

$$\mu'_a = 1 \qquad {\sigma'_a}^2 = \frac{5}{12}$$

$$\mu'_b = 2 \qquad {\sigma'_b}^2 = \frac{31}{12}$$

$$\mu'_c = 5 \qquad {\sigma'_c}^2 = \frac{59}{12} \quad .$$

It is at once evident that no linear transformation can relate μ and μ' for

$$\frac{\mu_b - \mu_a}{\mu_c - \mu_a} \neq \frac{\mu'_b - \mu'_a}{\mu'_c - \mu'_a} \quad .$$

The fact that both B.T.L. and Thurstone systems yield either ratio or difference scales in a natural way raises the question if a given pair comparison system can be both a B.T.L. and Thurstone system. In the trivial case that $p_{ab} = 1/2$ for all a and b in A this is indeed the case, but it is easy to construct pair comparison systems which are B.T.L. but not Thurstone - case V systems or Thurstone - case V but not B.T.L. systems. Interesting open problems are (i) the complete characterization of the pair comparison systems which are both B.T.L. and Thurstone - case V systems, and (ii) detailed analysis of the relation between B.T.L. and Thurstone - case III systems (for some further discussion see (Luce, 1959, pp. 54-58)).

5.3 Monotone Systems for Pair Comparisons

Some important similarities between B.T.L. and Thurstone systems can be brought out by considering further at this point the infinite difference systems of Sec. 4.3. It was indicated in Sec. 4.3 that the quaternary relation D could be interpreted as follows:

$$abDcd \text{ if and only if } p_{ab} \leq p_{cd} \quad . \tag{1}$$

If the quaternary system $\mathcal{O} = \langle A, D \rangle$ with D interpreted as in (1) satisfies the assumptions of an infinite difference system then from Theorem 12 we know that a numerical assignment f_1 will exist such that for a,b,c,d in B

$$p_{ab} \leq p_{cd} \text{ if and only if } f_1(a) - f_1(b) \leq f_1(c) - f_1(d) \quad . \tag{2}$$

And from Theorem 13 it follows that the numerical assignment f is unique up to a linear transformation. Following Adams and Messick (1957) we use (2) to define monotone systems.

Definition 30. *Let* $\mathcal{B} = \langle B, p \rangle$ *be a pair comparison system. Then* $\mathcal{B}$ *is a* monotone system *if and only if there is a numerical assignment* f_1 *defined on* A *such that for* a,b,c *and* d *in* A

$$p_{ab} \leq p_{cd} \text{ if and only if } f_1(a) - f_1(b) \leq f_1(c) - f_1(d) \quad .$$

The representation theorem corresponding to this definition is obvious and need not be stated.

It is at once obvious that any B.T.L. or case V - Thurstone system is a monotone system; the converse is of course not true. Also any pair comparison system that is an infinite difference or finite difference system under the interpretation given by (1) is also a monotone system. The first four axioms are the same for both i.d. and f.d. systems and are satisfied by any monotone system; the theory of measurement would be a much simpler subject if conversely any system that satisfied these four axioms were also a monotone system. That this converse is not true is shown in Scott and Suppes (1958).

In the context of the discussion of pair comparisons, it is worth noting that under the interpretation given by (1), Axiom 3 for f.d. and i.d. systems expresses what Davidson and Marschak (1959) call the _quadruple condition_. Axioms 2 and 5 have as a consequence the principle of _strong stochastic transitivity_, i.e., that if $p_{ab} \geq \frac{1}{2}$ and $p_{bc} \geq \frac{1}{2}$ then $p_{ac} \geq p_{ab}$ and $p_{ac} \geq p_{bc}$. It follows from the remarks in the preceding paragraph that any B.T.L. system or case V - Thurstone system satisfies the quadruple condition and the principle of strong stochastic transitivity.

Finally we remark that other representation theorems may be proved for monotone systems. We may for example define a representing relation $R(p, f_2)$ involving ratios rather than differences as follows:

$R(p, f_2)$ if and only if for all a,b,c and d in B $p_{ab} \leq p_{cd}$ if and only if

$$\frac{f_2(a)}{f_2(b)} \leq \frac{f_2(c)}{f_2(d)} .$$

By letting

$$f_3(a) = e^{f_1(a)}$$

the representation theorem for monotone systems based on $R(p, f_2)$ follows directly from Definition 30.

The general conclusion is that for all monotone systems, as well as those that are B.T.L. or Thurstone systems, there is no intrinsic reason for choosing derived numerical assignments expressing the observable relations of pair comparison probabilities in terms of numerical differences rather than numerical ratios.

5.4 Multidimensional Thurstone Systems

To illustrate a multidimensional derived assignment, we shall consider a multidimensional extension of a Thurstone system - case V.

Let A be a non-empty set of stimuli and q a real valued function from $A \times A \times A \times A$ to the open interval $(0,1)$ satisfying the following properties for a, b, c, d in A : $q_{ab,cd} = 1 - q_{cd,ab}$, $q_{ab,cd} = q_{ab,dc} = q_{ba,cd}$, and $q_{ab,ab} = 1/2$. Then $\mathfrak{A} = \langle A, q \rangle$ is a quadruple system. The usual interpretation of $q_{ab,cd}$ is that it is the relative frequency with which stimuli (a,b) are judged more similar to each other than the stimuli (c,d). To achieve an r-dimensional derived assignment for $\mathfrak{A}$, we wish to define an r-dimensional vector $\underset{\sim}{x}_a$ in terms of the function q for each element a in A.

The multidimensional extension of a Thurstone system in (Torgerson, 1958) involves assuming that the distance d_{ab} between the r-dimensional vectors $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ is normally distributed and that

$$q_{ab,cd} = P[d_{ab} < d_{cd}] \quad . \tag{1}$$

The difficulty with this approach is that since d_{ab} is the magnitude of the distance between x_a and x_b, it cannot be normally distributed because it cannot take on negative values. A more consistent and direct extension of a Thurstone system would be to assume that for each vector $\underset{\sim}{x}_a$ associated with stimulus a in A, the <u>projection</u> of the vector $\underset{\sim}{x}_a$ on the i^{th} axis, x_{ia}, is normally and independently distributed with mean μ_{ia} and variance σ^2_{ia}. The distribution of $x_{ia} - x_{ib}$, the projection of the vector $(\underset{\sim}{x}_a - \underset{\sim}{x}_b)$ on the i^{th} axis, is then also normally distributed with a mean $\mu_{ia} - \mu_{ib}$ and variance $\sigma^2_{ia} + \sigma^2_{ib}$. Making the case V assumptions, we may set $\sigma^2_{ia} + \sigma^2_{ib}$ equal to a constant, and for simplicity we set it equal to one. A weaker multi-dimensional interpretation of the case V assumptions is to let $\sigma^2_{ia} + \sigma^2_{ib} = \sigma^2_i$ for all a, b in A. With some minor modifications the following equation will incorporate this weaker assumption as well.

If y_{ab} is the random variable corresponding to the distance between $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$, then as Hefner (1958) has pointed out, since

$$y^2_{ab} = \sum_{i=1}^{r} (x_{ia} - x_{ib})^2 \quad , \tag{2}$$

y^2_{ab} is distributed as a non-central χ^2 with r degrees of freedom and with non-centrality parameter D^2_{ab} equal to

$$D^2_{ab} = \sum_{i=1}^{r} (\mu_{ia} - \mu_{ib})^2 \quad . \tag{3}$$

For the case $a = b$, two interpretations are possible. We can assume that y_{aa} is identically equal to zero or that it represents the distance between two vectors $\underset{\sim}{x}_a$ and $\underset{\sim}{x}'_a$ independently sampled from the same distribution. Which interpretation is better would depend on how the experiment was conducted, whether for example, $q_{aa,bc}$, $(b \neq c)$ is necessarily equal to one or whether it may be less than one. To minimize the mathematical discussion we shall assume that $y_{aa} = |\underset{\sim}{x}_a - \underset{\sim}{x}'_a|$ so that y^2_{aa} is then distributed as a central χ^2.

We shall also assume that

$$q_{ab,cd} = P[y^2_{ab} \leq y^2_{cd}] \tag{4}$$

or equivalently, that

$$q_{ab,cd} = P\left[\frac{y^2_{ab}}{y^2_{cd}} \leq 1\right] \quad . \tag{5}$$

Hence letting $w_{ab,cd} = y^2_{ab}/y^2_{cd}$ and letting the probability density of $w_{ab,cd}$ be $f(w)$, then

$$q_{ab,cd} = \int_0^1 f_{ab,cd}(w)dw \quad . \tag{6}$$

To specify the nature of the density $f(w)$ we need to be clear about whether y^2_{ab} and y^2_{cd} are independently distributed for all a,b,c,d in A. Again one special case arises when $a = c$ or $b = d$. The interpretation for this case would generally depend on whether $q_{ab,ac}$ is derived from trials in which only three stimuli are presented or from trials in which four stimuli (two of which are equivalent) are presented. We shall make the latter assumption here so that it will be reasonable to assume that y^2_{ab} and y^2_{ac} are independently distributed for all a,b,c in A and therefore that in general y^2_{ab} and y^2_{cd} are independent random variables.

Since w is then the ratio of two independent non-central χ^2 variates, $f(w)$ is a non-central F density with r degrees of freedom and with non-centrality parameters D^2_{ab} and D^2_{cd}. With this in mind we can define an r-dimensional Thurstone system as follows.

Definition 31. If $\mathfrak{A} = \langle A, q \rangle$ is a quadruple system, then it is an r-dimensional Thurstone system if there exists real valued functions $\mu_i(a)$ for each $a \in A$, $i = 1,\ldots,r$, and non-central F densities $f_{ab,cd}(w)$ with r degrees of freedom and non-centrality parameters D^2_{ab}, D^2_{cd} such that for a,b,c,d in A

$$q_{ab,cd} = \int_0^1 f_{ab,cd}(w)dw \quad , \tag{7}$$

and such that

$$D^2_{ab} = \sum_{i=1}^{r} (\mu_i(a) - \mu_i(b))^2 \quad . \tag{8}$$

The second condition (8) in the above definition can be expressed more elegantly by utilizing the Young-Householder theorem (Young and Householder, 1938). Define an $(n-1) \times (n-1)$ matrix B_e whose elements b_{ab} are

$$b_{ab} = \frac{1}{2}(D^2_{ea} + D^2_{eb} - D^2_{ab}) \quad , \tag{9}$$

where e is some arbitrarily chosen element in A. Then condition (8) will hold if and only if the matrix B is positive semi-definite.

Rather than state an r-dimensional representation theorem we define a derived numerical (or vector) assignment for an r-dimensional Thurstone system as the function g such that for a in A

$$g(a) = (\mu_{1a},\ldots,\mu_{ra}) \quad . \tag{10}$$

5.5 Successive Interval Systems

In the method of successive intervals a subject is presented with a set B of stimuli and asked to sort them into k categories with respect to some attribute. The categories are simply ordered from the lowest (category 1) to the highest (category k). Generally, either a single subject is required to sort the stimuli a large number of times or else many subjects do the sorting, not more than a few times, possibly once, each. The proportion of times $f_{a,i}$ that a given stimulus a is placed in category i is determined from subjects' responses.

The intuitive idea of the theory associated with the method is that each category represents a certain interval on a one-dimensional continuum, and that each stimulus may be represented by a probability distribution on this continuum. The relative frequency with which subjects place stimulus a in category i should then be equal to the probability integral of the distribution of the stimulus a over the interval representing the category i. In the standard treatments the probability distributions of stimuli are assumed to be normal distributions and the scale value of the stimulus is defined as the mean of the distribution. It is important to emphasize that in the formal analysis we give here it is not necessary to assume, and we do not assume, equality of the variances of the various normal distributions. It is also pertinent to point out that the use of normal distributions is not required. As Adams and Messick (1958) show, a much wider class of distributions may be considered without disturbing the structure of the theory in any essential way. For convenience we shall however restrict ourselves to the normality assumption.

To formalize these intuitive ideas we shall use the following notation. Let B be a non-empty set and for a in B let f_a be a probability distribution on the set $\{1,\ldots,k\}$ of integers, that is, $f_{a,i} \geq 0$ for $i = 1,\ldots,k$ and $\sum_{i=1}^{k} f_{a,i} = 1$. Then a system $\mathcal{B} = \langle B, k, f \rangle$ will be called a <u>successive interval system</u>. Using the notation that $N(x)$ is the unit normal cumulative define an Adams-Messick system as follows:

<u>Definition</u> 32. <u>A successive interval system</u> $\mathcal{B} = \langle B, k, f \rangle$ <u>is an</u> Adams-Messick (or A.M.) system <u>if and only if for all</u> a, b <u>in</u> B <u>and</u> $i = 1,\ldots,k$ <u>there exist real numbers</u> $z_{a,i}$, $z_{b,i}$, δ_{ab} <u>and</u> β_{ab} <u>such that</u>

$$\sum_{j=1}^{i} f_{a,j} = N(z_{a,i}) \tag{1}$$

and

$$z_{a,i} = \delta_{ab} z_{bi} + \beta_{ab} \quad . \tag{2}$$

As with the various pair comparison systems already described, several simple representation theorems can be established for A.M. systems, and corresponding to each representation theorem there is a distinct uniqueness theorem. Rather than emphasize again the non-uniqueness aspect of the representation theorem, we shall limit the present discussion to describing just one of the simple (and customary) representing relations.

Let $N_a(x, \mu_a, \sigma_a^2)$ be the normal cumulative distribution having a mean μ_a and variance σ_a^2. Then we define the relation $R(k, f, \mu)$ as follows:

> $R(k, f, \mu)$ if and only if there are normal cumulatives $N_a(x, \mu_a, \sigma_a^2)$, and there are real numbers $t_0, t_1, \ldots, t_k$ with $t_0 = -\infty$, $t_k = \infty$, and $t_{i-1} \le t_i$ such that
>
> $$f_{a,i} = N_a(t_i, \mu_a, \sigma_a^2) - N_a(t_{i-1}, \mu_a, \sigma_a^2) \quad .$$

The intended interpretation of the numbers t_i should be obvious. The number t_{i-1} is the lower endpoint of category i, and t_i its upper endpoint.

The following two theorems are established by Adams and Messick (1958).

<u>Theorem</u> 32 (<u>Representation Theorem</u>). <u>If the successive interval system</u> $\mathscr{B} = < B, k, f >$ <u>is an A.M. system, then there exists a function</u> μ <u>such</u> $< \mathscr{B}, R, \mu >$ <u>is a derived scale</u>.

In other words, the function μ defined by $R(k, f, \mu)$ is indeed a derived numerical assignment for an A.M. system. Adams and Messick also show the converse of Theorem 32, namely, that if $< \mathscr{B}, R, \mu >$ is a derived scale, the successive interval system $\mathscr{B}$ is an A.D. system.

For the uniqueness theorem we have:

<u>Theorem</u> 33 (<u>Uniqueness Theorem</u>). <u>If the successive interval system</u> $\mathscr{B} = < B, k, f >$ <u>is an A.M. system, then the derived scale</u> $< \mathscr{B}, R, \mu >$ <u>is an interval scale in both the narrow and wide senses</u>.

Adams and Messick also note that if the values of μ_a and σ_a are fixed for one stimulus, or alternatively, if one of the normal cumulatives associated with the stimuli is fixed, then the derived numerical assignment μ is uniquely determined.

For completeness we should point out that the numbers $t_0, t_1, \ldots, t_k$ can also be thought of as the values of a derived numerical assignment for the category boundaries. That is, separate representation and uniqueness theorems can be established for the boundaries as well as for the stimuli, but since the theorems for the boundaries are similar to those just described for the stimuli we shall not pursue these details further.

6. THE PROBLEM OF MEANINGFULNESS

Closely related to the two fundamental issues of measurement--the representation and uniqueness problems--is a third problem which we shall term the meaningfulness problem. While this problem is not central to a theory of measurement it is often involved with various aspects of how measurements may be used, and as such it has engendered considerable controversy. It therefore merits some discussion here although necessarily our treatment will be curtailed.

6.1 Definition

To begin with, it will be well to illustrate the basic source of the meaningfulness problem with two of the statements, (4) and (5), given previously in Sec. 2.2. For convenience they are repeated here.

(4) The ratio of the maximum temperature today (t_n) to the maximum temperature yesterday (t_{n-1}) is 1.10.

(5) The ratio of the difference between today's and yesterday's maximum temperature $(t_n$ and $t_{n-1})$ to the difference between today's and tomorrow's maximum temperature $(t_n$ and $t_{n+1})$ will be $.95$.

In Sec. 2.2 statement (4) was dismissed for having no clear empirical meaning and (5), on the other hand, was said to be acceptable. Here we wish to be completely explicit as to the basis of this difference. Note first that both statements are similar in at least one important respect: neither statement specifies which numerical assignment from the set of admissible numerical assignments is to be used to determine the validity or truth of the statement. Are the temperature measurements to be made on the Centigrade, Fahrenheit or possibly the Kelvin scale? This question is not answered by either statement.

The distinguishing feature of (4) is that this ambiguity in the statement is critical. As an example, suppose we found using the Fahrenheit scale $t_n = 110$ and $t_{n-1} = 100$. We would then conclude that (4) was a true statement, the ratio of t_n to t_{n-1} being 1.10. Note, however, that if the temperature measurements had been made on the Centigrade scale we would have come to the opposite conclusion, since then we would have found $t_n = 43.3$ and $t_{n-1} = 37.8$ and a ratio equal to 1.15 rather than 1.10. In the case of (4) the selection of a particular numerical assignment influences what conclusions we come to concerning the truth of the statement.

In contrast consider statement (5). The choice of a specific numerical assignment is not critical. To illustrate this point assume that when we measure temperature on a Fahrenheit scale we find the following readings, $t_{n-1} = 60$, $t_n = 79.0$ and $t_{n+1} = 99.0$. Then the ratio described in (5) is equal to

$$\frac{60 - 79}{79 - 99} = \frac{-19}{-20} = .95$$

which would indicate that (5) was valid. Now if instead the temperature measurements had been made on the Centigrade scale we would have found $t_{n-1} = 15.56$, $t_n = 26.11$, and $t_{n+1} = 37.20$. But since the ratio of these numbers is

$$\frac{15.56 - 26.11}{26.11 - 37.20} = \frac{-10.55}{-11.09} = .95$$

we would also have come to the conclusion that (5) was true. In fact, it is easy to verify that if we restrict ourselves to numerical assignments which are linearly related to each other then we will always arrive at the same conclusion concerning the validity of (5). If the statement is true for one numerical assignment then it will be true for all. And furthermore, if it is untrue for one numerical assignment then it will be untrue for all. For this reason we can say that the selection of a specific numerical assignment to test the statement is not critical.

The absence of units in (4) and (5) is deliberate, because the determination of units and an appreciation of their empirical significance comes after, not before, the investigation of questions of meaningfulness. The character of (4) well illustrates this point. If the Fahrenheit scale is specified in (4), the result is an empirical statement that is unambiguously true or false, but it is an empirical statement of a very special kind. It tells us something about the weather relative to a completely arbitrary choice of units. If the choice of the Fahrenheit scale were not arbitrary, there would be a meteorological experiment that would distinguish between Fahrenheit and Centigrade scales and thereby narrow the class of admissible transformations. The recognized absurdity of such an experiment is direct evidence for the arbitrariness of the scale choice.

From the discussion of (4) and (5) it should be evident what sort of definition of meaningfulness we adopt.

Definition 33. A numerical statement is meaningful if and only if its truth (or falsity) is constant under admissible scale transformations of

<u>any of its numerical assignments</u>, i.e., <u>any of its numerical functions expressing the results of measurement</u>.

Admittedly this definition should be buttressed by an exact definition of 'numerical statement', but this would take us further into technical matters of logic than is desirable in the present context. A detailed discussion of these matters, including construction of a formalized language, is to be found in Suppes [1959].[6] The kind of numerical statements we have in mind will be made clear by the examples to follow. The import of the definition of meaningfulness will be clarified by the discussion of these examples.

It will also be convenient in what follows to introduce the notion of <u>equivalent statements</u>. Two statements are equivalent if and only if they have the same truth value. In these terms we can say that a numerical statement is meaningful in the sense of Definition 33 if admissible transformations of any of its numerical assignments always lead to equivalent numerical statements.

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[6] It can also be well argued that Definition 33 gives a necessary but not sufficient condition of meaningfulness. For example, a possible added condition on the meaningfulness of statements in classical mechanics is that they be invariant under a transformation to a new coordinate system moving with uniform velocity with respect to the old coordinate system (cf. McKinsey and Suppes [1955]). However, this is not a critical matter for what follows in this section.

6.2 Examples

To point out some of the properties and implications of Definition 33, we discuss a number of specific examples. In each case a particular numerical statement is given as well as the transformation properties of all the numerical assignments referred to in the statement. For each example we ask the question: Is it meaningful? To show that it is meaningful admissible transformations are performed on all the numerical assignments and then the transformed numerical statement is shown to be equivalent to the initial one. In most cases when the two statements are in fact equivalent the transformed statement can generally be reduced to the original statement by using various elementary rules of mathematics or logic. To show that a statement is meaningless we must show that the transformed statement is not equivalent or can not be reduced to the original. When this is not obvious, the simplest way of proceeding is to construct a counterexample, an example in which the truth value of the statement is not preserved after a particular admissible transformation is performed.

In the following examples we shall, for simplicity, conform to common usage and denote numerical assignments by x, y and z instead of by $f(a)$ or $g(a)$. Thus instead of writing

$$\text{for all } a \in A\ ,\ f(a) = \varphi(g(a)) \ ,$$

we shall write simply

$$y = \varphi(x) \ .$$

When it is necessary to distinguish between the values of a given numerical assignment for two distinct objects a and b, subscript notation is used. Thus, for example, x_a and x_b might be the masses of two weights a and b. The parameters of the admissible transformations are j, k, l, m, ... and the numerical assignments which result from these transformations of x, y and z are x', y' and z', respectively.

Example 1.

$$x_a + x_b > x_c \quad . \tag{1}$$

First assume that the numerical assignement x is unique up to a similarity transformation. Then if x is transformed to kx, (1) becomes

$$kx_a + kx_b > kx_c \tag{2}$$

which is obviously equivalent to (1). Hence (1), under the assumption that x lies on a ratio scale, is meaningful.

Assume instead that x is specified up to a linear transformation. Then when x transforms to $kx + \ell$, (1) transforms to

$$(kx_a + \ell) + (kx_b + \ell) > (kx_c + \ell) \quad , \tag{3}$$

which does not reduce to (1). For a specific counterexample, let $x_a = 1$, $x_b = 2$, $x_c = 2$ and $k = 1$ and $\ell = -1$. Equation (1) then becomes

$$1 + 2 > 2 \quad . \tag{4}$$

But substituting the transformed values of x_a and x_b into (1) gives

$$(1 - 1) + (2 - 1) > (2 - 1)$$

or

$$0 + 1 > 1 \tag{5}$$

which obviously does not have the same truth value as (4). Hence (1) is meaningless when x lies on an interval scale. Thus we can say, for example, that if mass is on a ratio scale it is meaningful to say that the sum of the masses of two objects exceeds the mass of a third object. On the other hand, if temperature is on an interval scale then it is not meaningful to say that the sum of the maximum temperatures on two days exceeds the maximum temperature on a third day.

These last remarks should not be interpreted to mean that addition can always be performed with ratio scales and never with interval scales. Consider the statement

$$x_a + x_b > (x_c)^2 \, . \tag{6}$$

This statement is meaningless for a ratio scale since

$$(kx_a) + (kx_b) > (kx_c)^2 \tag{7}$$

is clearly not equivalent to (6).

An example of a meaningful numerical statement which involves both addition and multiplication of a numerical assignment having only interval scale properties is the following.

Example 2. Let S_1 and S_2 be two sets having n_1 and n_2 members respectively. Then

$$\frac{1}{n_1}\sum_{a\epsilon S_1} x_a > \frac{1}{n_2}\sum_{b\epsilon S_2} x_b \quad . \tag{8}$$

One interpretation of (8) is that the mean maximum temperature of the days in January exceeds the mean maximum temperature of the days in February. We wish to show that when x is unique up to a linear transformation that

$$\frac{1}{n_1}\sum_{a\epsilon S_1} x'_a > \frac{1}{n_2}\sum_{b\epsilon S_2} x'_b \tag{9}$$

is equivalent to (8) where x' is another admissible numerical assignment. Substituting $x' = kx + \ell$ into (8) gives

$$\frac{1}{n_1}\sum_{a\epsilon S_1} (kx_a + \ell) > \frac{1}{n_2}\sum_{b\epsilon S_2} (kx_b + \ell)$$

or

$$\frac{1}{n_1}[k(\sum x_a) + n_1\ell] > \frac{1}{n_2}[k(\sum x_b) + n_2\ell] \tag{10}$$

and (10) can be reduced to (8). Hence (8) is meaningful under these conditions. In general, it may be observed that whether or not a particular mathematical operation can be used with a particular scale does not have a simple yes or no answer. The admissibility of any mathematical operation depends not only on the scale type of the relevant numerical assignments but on the entire numerical statement of which the operation is a part.

Example 3.

$$x_a x_b > x_c \quad . \tag{11}$$

If x is unique up to a similarity transformation then (11) is meaningless since

$$(kx_a)(kx_b) > (kx_c) \tag{12}$$

does not reduce to (11) for all values of k. Since (11) is meaningless for a ratio scale it follows a fortiori that it is meaningless for an interval scale as well. It does not follow that (11) is meaningless for all scales. As an example let x be unique up to a power transformation (See Sec. 4.5 for an illustration). Then as x is transformed to x^k, (12) becomes

$$x_a^k \, x_b^k > x_c^k$$

and since this is equivalent to (12), statement (12) is meaningful.

In the following examples we shall consider numerical statements involving at least two independent numerical assignments, x and y.

Example 4.

$$y = \alpha x \quad . \tag{13}$$

If the numerical assignments x and y are not completely unique, then it is clear that (13) will in general be meaningless, since most non-identity transformations applied to x and y will transform (13) to a non-equivalent statement. One approach which is frequently used to make (13) meaningful under more general conditions is to permit the constant α to depend upon the parameters k , ℓ , m , n , ... of the admissible transformations (but not on x or y) . This approach will be used here so that in the remaining discussion statement (13) will be understood to mean the following: there exists a real number α such that for all $a \in A$, $y_a = \alpha x_a$. If x is transformed to x' and y to y' then (13) interpreted in this way will be meaningful if there exists an α' , not necessarily equal to α , such that $y'_a = \alpha' x'_a$ for all $a \in A$.

Assume that x and y are specified up to a similarity transformation. Letting x go to kx , y to my , and α to α' (at present unspecified) (13) transforms to

$$my = \alpha' kx \quad . \tag{14}$$

If we let $\alpha' = (m/k)\alpha$ then (14) will be equivalent to (13) for all (non-zero) values of m and k and therefore (13) will be meaningful.

Assume instead that x and y are both specified up to linear transformations. Let x transform to $(kx + \ell)$, y to $(my + n)$, and α to α' . We then have

$$(my + n) = \alpha'(kx + \ell) \quad . \tag{15}$$

Inspection of (15) suggests that no transformation of α to α' which depends solely on the parameters k , ℓ , m , or n will reduce (15) to (13), This conclusion can be established more firmly by constructing a counterexample. Consider the following in which $A = \{a,b\}$. Let $x_a = 1$, $y_a = 2$, $y_b = 2$ and $y_b = 4$. Then if $\alpha = 2$ equation (13) will be true for all a in A . Now let x go to $(kx + \ell)$, y to

$(my + n)$ where $k = \ell = m = n = 1$ and α to α'. Equation (13) then becomes for a in A

$$3 = \alpha' 2 \tag{16}$$

indicating that if the truth value of (13) is to be preserved we must have $\alpha' = 3/2$. However, for $b \in A$ we have

$$5 = \alpha' 3 \tag{17}$$

which can only be true if $\alpha' = 5/3$ and in particular is false if $\alpha' = 3/2$. Both (16) and (17) cannot be true and the truth value of (13) is not preserved in this example. Hence under these conditions (13) is meaningless.

Example 5.

$$x + y = \alpha \quad . \tag{18}$$

In this example we assume first that x and y are unique up to a similarity transformation. One interpretation of (18) would then be that the sum of the weight and height of each person in A is equal to a constant. As usual we let x transform to kx, y to my and α to α'. Then (18) becomes

$$kx + my = \alpha' \quad , \tag{19}$$

but since it is evident that no value of α' will reduce this equation to one equivalent to (18) we may conclude that (18) is meaningless under these assumptions.

Although this result confirms the common sense notion that it does not make sense to add weight and length, there are conditions under which (18) is certainly meaningful. One example is the assumption that x and y lie on difference scales, that is, are unique up to an additive constant. This can be verified by letting x transform to $x + \ell$, y to $y + n$ and α to $(\alpha + \ell + n)$, for then (18) transforms to

$$(x + \ell) + (y + n) = \alpha + \ell + n \tag{20}$$

which is certainly equivalent to (18).

To illustrate the effects of a derived numerical assignment, as well as another set of assumptions for which (18) is meaningful consider the following. Assume that y is a derived numerical assignment and that it depends in part on x. Assume that when x is transformed to kx, y is transferred to $(ky + 2k)$. Therefore if α transforms to α' (18) becomes

$$kx + (ky + 2k) = \alpha' \tag{21}$$

and clearly if $\alpha' = k(\alpha + 2)$ this equation will be equivalent to (18). Equation (18) is therefore meaningful under these conditions. Thus whether or not it is meaningful to add weight and length depends not so much on the physical properties of bodies but on the uniqueness properties of the numerical assignments associated with weight and length.

Another common dictum frequently encountered is that one can take the logarithm only of a numerical assignment which lies on an absolute scale, or as it is customarily said, of a dimensionless number. With this in mind we consider the following example.

Example 6.

$$y = \alpha \log x \quad . \tag{22}$$

Assume first that x and y have ratio scale properties. Then as x transforms to kx , y to my , and α to α' (22) transforms to

$$my = \alpha' \log kx$$

or to

$$my = \alpha' \log x + \alpha' \log k \tag{23}$$

and (23) cannot be made equivalent to (22) by any value of α' . Therefore under these assumptions we again have the common sense result, viz., that (22) is meaningless.

However, assume next that x is unique up to a power transformation and y is unique up to a similarity transformation. Then when x transforms to x^k , y to my and α to $(m/k)\alpha$ equation (22) transforms to

$$my = \left(\frac{m}{k}\right)\alpha \log \left(x^k\right) \quad . \tag{24}$$

Since (24) is clearly equivalent to (22), equation (22) is meaningful under these conditions, common sense not withstanding.

Another example involving the logarithm of a non unique numerical assignment is the following.

Example 7.

$$y = \log x + \alpha \quad . \tag{25}$$

From the previous example it should be evident that (25) will not be meaningful when x and y have ratio scale properties. But if y is on a difference scale and x is on a ratio scale then (25) will be meaningful. This can be verified by letting x transform to kx, y to $y + n$ and α to $\alpha + n - \log k$ since then (25) transforms to

$$(y + n) = \log kx + (\alpha + n - \log k) \tag{26}$$

and (26) is equivalent to (25). The interesting feature of (25) is that the required transformation of α is not a simple or elementary function of the parameters k and n. Although most, if not all, of the "dimensional constants" encountered in practice have simple transformation properties (usually a power function of the parameters), there is no a priori reason why these constants cannot be allowed to have quite arbitrary transformations. Of course, if the transformation properties of the constants are limited, the conditions under which numerical statements will be meaningful will ordinarily be changed.

6.3 Extensions and Comments

It will be noted that the definition of meaningfulness in Definition 28 contains no reference to the physical operations which may or may not have been employed in the measurement procedure. There are at least two other points of view on this issue which the reader should be aware of.

One point of view (e.g. Weitzenhoffer, 1951) asserts that meaningful statements may only employ mathematical operations which correspond to known physical operations. In terms of empirical and relational systems this point of view may be described as requiring that, for each empirical system, the admissible mathematical operations be limited to those contained in the selected homomorphic numerical system (or alternatively in <u>some</u> homomorphic numerical system).

The second point of view (e.g. Guilford, 1954) appears to be less severe. It asserts that all the rules of arithmetic are admissible when physical addition exists (and, presumably, satisfies some set of axioms such as those in Sec. 4.5). Without physical addition the application of arithmetic is limited (to some undefined set of rules or operations). In terms of relational systems, this point of view appears to imply that all statements "satisying" the rules of arithmetic are meaningful when the selected homomorphic numerical system contains the arithmetical operation of addition (or, perhaps, when at least one homomorphic numerical system exists which contains the addition operation). When this is not the case, fewer statements are meaningful.

In contrast to these two positions, Definition 28 implies that the meaningfulness of numerical statements is determined solely by the uniqueness properties of their numerical assignments, not by the nature of the operations in the empirical or numerical systems.

One of our basic assumptions throughout this section has been that a knowledge of the representation and uniqueness theorems is a prerequisite to answering the meaningfulness question. It can be argued though that these assumptions are too strong, since in some cases the uniqueness properties of the numerical assignments are not precisely known. For example, although we have a representation theorem for semi-orders

(Sec. 4.2) we do not have a uniqueness theorem. In these cases, we often have an approximation of a standard ordinal, interval or ratio scale. Without stating general definitions the intuitive idea may be illustrated by consideration of an example.

Suppose we have an empirical quaternary system $\mathfrak{A} = \langle A, D \rangle$ for which there exists a numerical assignment f such that $abDcd$ if and only if

$$f(a) - f(b) \leq f(c) - f(d) \quad . \tag{1}$$

We also suppose that the set A is finite. In addition, let $\mathfrak{N}$ be the full numerical relational system defined by (1). Let $\langle \mathfrak{A}, \mathfrak{N}, f \rangle$ be a scale and $\langle \mathfrak{A}, \mathfrak{N}, g \rangle$ be any scale such that for two elements a and b in A, $f(a) \neq f(b)$ and $f(a) = g(a)$ and $f(b) = g(b)$, i.e., f and g assign the same number to at least two elements of A which are not equivalent. We then say that $\langle \mathfrak{A}, \mathfrak{N}, f \rangle$ is an ϵ-approximation of an interval scale if

$$\max_{a,g} |f(a) - g(a)| \leq \epsilon \quad ,$$

where the maximum is taken over all elements of A and numerical assignments g satisfying the condition just stated. It is clearly not necessary that there be an ϵ such that the scale $\langle \mathfrak{A}, \mathfrak{N}, f \rangle$ be an ϵ-approximation of an interval scale. Consider, for instance, the three element set $A = \{a,b,c\}$, with $f(a) = 1$, $f(b) = 1.5$, and $f(c) = 3$. Then if $g(a) = 1$ and $g(b) = 1.5$, we may assign any number for the value $g(c)$ provided $g(c) > 2$, and thus there is no finite $\max|f(a) - g(a)|$. On the other hand, if A has twenty or more elements, say, and no intervals are too large or too small in relation to the rest, $< \mathcal{O}\!\mathcal{C}, \mathcal{N}, f >$ will be an ϵ-approximation of an interval scale for ϵ reasonably small in relation to the scale of f.

The relation between ϵ-approximations of standard scales and issues of meaningfulness is apparent. A statement or hypothesis that is meaningful for interval or ratio scales has a simple and direct analogue that is meaningful for ϵ-approximations of these scales. For example, consider the standard proportionality hypothesis for two ratio scales, i.e., that there exists a positive α such that for every a in A

$$f(a) = \alpha\, h(a) \quad . \tag{2}$$

This equation is meaningful, as we have seen (Sec. 6.2, Example 4), when f and h are determined up to a similarity transformation. If f and h, or more exactly $< \mathcal{O}\!\mathcal{C}, \mathcal{N}, f >$ and $< \mathcal{O}\!\mathcal{C}', \mathcal{N}', h >$, are ϵ- and δ-approximations of ratio scales, respectively, then (2) is no longer meaningful, but the appropriate analogue of (2), namely

$$|f(a) - \alpha\, h(a)| \leq \epsilon + \alpha\, \delta$$

is meaningful.

Problems of meaningfulness and the issue of the applicability of certain statistics to data that are not known to constitute an interval or ratio scale have been closely tied in the psychological literature. Unfortunately, we do not have the space to analyze this literature.

We believe that the solution lies not in developing alternative definitions of meaningfulness but rather in clarifying the exact status of the measurements made. One way is to make explicit the empirical relational system underlying the empirical procedures of measurement. A second is along the lines we have just suggested in sketching the theory of ϵ-approximations. A third possibility for clarification is to give a more explicit statement of the theory or hypotheses to which the measurements are relevant.

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