A Complete, Type-free "Second-order" Logic and Its Philosophical Foundations

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by

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Part I

In this paper I will motivate and develop a type-free logic within the general ontological framework of (nonextensional) properties, relations, and propositions. Though syntactically second-order, the logic is complete with respect to its semantics. This, along with its soundness and consistency, will be proved in Part II. As I'll point out, many of the central formal ideas I will draw upon have appeared in one guise or another elsewhere; the interest of the present work lies in the formally elegant and philosophically fruitful system that results when these ideas are modified and combined as we've done here. The purpose of this rather lengthy introduction is to present the major ideas of the system informally and to discuss to a limited extent its philosophical significance, especially in relation to Russell's paradox.

§1.1 PROBLEMS OF TYPE THEORY

Since Frege, the (arguably) dominant conception of properties, relations, and propositions (or PRPs, following Bealer [1982]) has been typed. The first rigorous formal development of this view by Russell was motivated largely by his discovery of the famous paradox that bears his is name, though he argued (post hoc) at great length for its intuitive acceptability (see, e.g., Whitehead and Russell [1962], ch. 2). More recently the view (albeit in extensionalist garb) has enjoyed a renewed respectability, and a renewed applicability, in linguistics and the philosophy of language prompted by Montague's rich and detailed investigations into the semantics of natural language (Montague [1974]).

Recent years, however, have seen a growing dissatisfaction with the typed conception of PRPs. There are a number of difficulties. First,
many intuitively natural PRPs either find no place within the typed con-
ception or else receive at best a tortured reconstruction. For example,
because a property in a typed framework can be meaningfully ascribed only
to objects of lower type, there can be neither self-exemplifying proper-
ties such as the property of being a property, nor universal properties
which are true of absolutely all objects, such as the property of being
self-identical. Properties of the latter are reconstructed type-theoret-
ically in terms of an ascending sequence of properties of higher and
higher type, each true of all the objects in the preceding type; proper-
ties of the former sort, by the very nature of the typed conception, have
no reconstruction at all. For similar reasons there can be no relations
which lie in their own fields, nor can there be any fully general propo-
sitions about all objects.

Intuitively, our ordinary patterns of reasoning do not seem to
warrant the implicit restrictions imposed on the "range of significance"
(Russell [1908], p. 161) of a property or relation that result in an a
priori dismissal of PRPs like those above. Nor do there appear to be any
implicit restrictions on the ranges of quantifiers. Indeed, many intu-
ively valid inferences suggest precisely the opposite. For instance,
the pattern of inference from

(1) Everything has the property of being self-identical
to

(2) Galen has the property of being self-identical

is no different from that in the inference from (1) to

(3) The property of being self-identical has the property
    of being self-identical.
Yet the only way to countenance the validity of the latter inference (while preserving the intuitive meaning of sentences in question) is, first to include all objects, and in particular, all properties, in the range of the universal quantifier in (1), and second, to allow not just the meaningfulness but the actual possibility of self-exemplification. Moreover, the intuitive soundness of the argument requires the existence of a universal property. In short, despite its similarity in form to the argument from (1) to (2), the argument simply cannot be represented within a typed framework.²

A further difficulty is that even when typed representations of certain valid inferences are possible, they may still impose unjustified alterations of logical form on the inferences, as well as distinctions in meaning unwarranted by ordinary usage. Consider for example the following two arguments.

(A) Annie loves Galen.
    There is something that Annie loves.

(B) Annie desires that she be wise.
    There is something that Annie desires.

Unlike the argument from (1) to (3), both of these arguments can be represented in a typed framework. The representations of the first premise of each argument would, quite rightly, have Annie standing in a relation to another object. However, the object Annie is related to in (A) is, like her, another individual (in the type theoretic sense) whereas the corresponding object in (B) (as most type theoretic accounts would concur) is a proposition. But since a quantifier can’t range over both individu-
als and propositions within a typed framework, (A) and (B) must be repre-
sented as differing in logical form: whereas the conclusion of (A) in-
volves an individual quantification, (B)’s conclusion involves quantifi-
cation of a higher type. The difference in logical form, then, is traced
to a difference in the meanings of the quantifiers in the conclusions
of the arguments. The proponent of the typed conception, that is, must
argue that the existential quantifier ‘There is’ of ordinary language
is systematically ambiguous.

This difficulty with meaning does not only infect quantifiers;
for consider the following argument:

(C) Annie loves wisdom.
     ___________
     There is something that Annie loves.

Since ‘wisdom’ and ‘Galen’ denote objects of different types, we must
once again have different quantifiers in the conclusions of (A) and (C);
but for the same reason we must also have different predicates in the
premises of the two arguments, since in (A) ‘loves’ denotes a relation
between individuals, whereas in (B) ‘loves’ denotes a relation between
an individual and a property. As above, then, the proponent of the typed
conception must argue, appearances to the contrary, that not only quanti-
fiers but many predicates as well in ordinary language are systematically
ambiguous. Intuitively, that just seems wrong (cf. Chierchia [1984], ch.
1); what we appear to have is simply a type theoretic artifact. To the
contrary, a proper representation of the above intuitive and linguistic
data seems to require an ontological framework that is type-free, i.e.,
a framework in which (i) all objects, PRPs no less than particulars like
you, me, the empty set, etc., are individuals and hence fall within the
range of the universal quantifier, and ii) the range of significance of a property or relation is unrestricted—in short, a framework in which everything is a "first-class citizen" (Barwise [1985], p. 3).

§1.2 A TYPE-FREE LANGUAGE AND ITS LOGIC

But aren’t we forgetting the very reason why Russell adopted a typed conception in the first place? Doesn’t the threat of paradox loom perilously large in the sort of framework we’re asking for? True enough, one needs a framework in which self-exemplification is possible in order to generate the paradox, but as I’ll demonstrate below, the root of the paradox lies not in the framework itself, but in an inadequate understanding of its logical structure. To make this clear, let’s turn our attention to developing a language and a logic that capture the framework in question.

We begin, naturally enough, with a sufficient variety of terms to capture the various sorts of objects that populate our chosen ontology, viz., particulars and n-place relations (= propositions, for n=0). For the latter, we introduce n-place predicates, for every n<ω, and for the former, individual terms. Individual terms, however, should be thought of intuitively as potentially taking any objects as their values, not just particulars, since all objects in our untyped framework are individuals. Now, we also want to capture the idea that properties and relations do not have restricted ranges of significance; but what, exactly, does this come to? Just this. Frege, I believe, had it basically right with respect to the distinguishing feature of properties and relations: these entities are in some sense "gappy", or "unsaturated", as he put it. (See Frege [1977], esp. pp. 24, 31, 54-5.) Less picturesquely, any property
or relation has a determinate number of argument places, and these can be "filled" by other objects via the operation of an appropriate logical function (to be described below) to yield PRPs of fewer argument places, as for example filling the second argument place in the relation less than with the number 2 yields the property of being less than 2. (Where Frege went wrong, of course, from the present point of view, was in supposing that the unsaturated nature of properties and relations somehow excluded them from full-fledged objecthood; cp. Russell [1937] §49.) Given this picture, to say that properties and relations have unrestricted ranges of significance is simply to say that any objects whatever can fill their argument places, even those very properties and relations themselves. To capture this idea syntactically we will need more than the standard atomic formulas of first- and second-order logic. Rather, following Cocchiarella [1972] we will allow any term to occupy subject positions in atomic formulas. In our language, then, predicate terms play a dual role of predicates in the traditional sense and as singular terms. So where p is an n-place predicate, and \( t_1 \ldots t_n \) is a sequence of terms of any kind, predicate or individual, \( \forall t_1 \ldots t_n \) is a well-formed formula.

Since each class of n-place relations for some particular n is a determinate class of full-fledged individuals, there is no reason why we shouldn't be able to quantify over just them; indeed there is good reason to think that in fact we do. The following arguments, for example, are intuitively valid, and to all appearances involve quantification over properties and propositions respectively:

(D) Socrates and Plato are both wise.

There is a property that Socrates and Plato share.
(E) Sid believes that there are ghosts.

There is a proposition that Sid believes.

We will take these quantifications at face value and hence include both individual and n-place predicate quantifiers (and so also the appropriate sorts of variables) in our language. We also include the standard logical particles, among which we count the identity sign ‘=’. Complex formulas are built up by the usual recursive clauses. The logic for our language thus far should suffice to represent the arguments and inference patterns above, and the standard principles of identity; given our type free language all we need are straightforward analogues of the usual axiom schemas for the first-order predicate calculus with identity (axiom schemas 1-5 in §2.3 below).

Intuitively, it is quite plausible to think some PRPs are more complex than others, i.e., that some PRPs are in some sense "built up" logically from less complex PRPs, as, for example, the property of being red and square is built up from the property of being red and the property of being square, or as the proposition that Annie is wise is built up from the property of being wise by plugging Annie into its single argument place. The existence of such PRPs is plausibly thought to follow logically from the existence of the less complex objects. An adequate language for the ontological framework of logic thus requires complex as well as simple predicate terms. Here we will make use of the usual λ notation \([\lambda x_1 \ldots x_n \varphi]\) for distinct individual variables \(x_1, \ldots, x_n\) and some formula \(\varphi\). In this notation, then, we can represent the complex property of being red and square by the term \("[\lambda x Rx & Sx]\”, and the proposition that Annie is wise by \("[\lambda W a]\”).
The demand for individual variables $x_1, \ldots, x_n$ here is again a reflection of our type-free framework. The sequence of variables following the $\lambda$ in a complex term corresponds to the argument places in an $n$-place relation. Hence, since any object can fill any argument place of any relation in our framework, only individual variables, i.e., those variables which can take any objects as their values, will do; to allow $n$-place predicate variables would be to countenance properties and relations with argument places that could only meaningfully be filled by $n$-place relations, i.e., properties and relations with restricted ranges of significance—and that is just what we’ve rejected here.$^5$

Since we intend the logic of complex terms to be governed by a form of $\lambda$-conversion (axiom schema 6; see also Theorem 1), not any formula $\varphi$ will do in the construction of complex terms because of the intensional version of Russell’s paradox. The restriction to individual variables blocks the most obvious form of Russell’s ‘‘property’’, viz., $[\lambda F \neg FF]$, where ‘$F$’ is a 1-place predicate variable. But there are other formulations that don’t violate this requirement, e.g., $[\lambda x \exists F (F=x \& \neg Fx)]$. Now we could simply lay down certain ad hoc restrictions, such as that $\varphi$ be homogeneously stratified (following Cocchiarella’s $\lambda HST^*$ [1985a]); but Russell himself suggested a far more appealing strategy.

To identify a principle, such as the vicious circle principle or homogeneous stratification, whose violation leads to inconsistency is not yet to justify its use in constructing one’s logic. For this, according to Russell [1908], we need more: the restrictions we place on our logic must, as he put it, "result naturally and inevitably from our positive doctrines" (p. 155); that is, the restrictions must follow from a broader
view of language, meaning, and reality for which we have independent arguments and motivations. This was to be the role of type theory for Russell (cf. Chihara [1973], ch. 1). We've found reason to think that this is still, at the least, not the correct ontological story, and hence typing restrictions (and so also stratification; see note 18) lose much of their justification. But despite the ills of type theory, Russell's methodology, I think, is still a good one. The construction of a logic is inevitably guided by some general metaphysical conception, some broad picture of reality and its logical structure. Since, presumably, reality is not contradictory, insofar as our logic gets its general structure right, the logic must be consistent. We have used this methodology already in generating our restriction above on the sorts of variables that can be bound by λs; we will continue to do so in our search for the proper restrictions on φ. We will use (NB: not mention) complex terms naively to assist us in our reflections.

Complex PRPs, I said, can be thought of as built up out of less complex PRPs. But what does this notion of being built up amount to? Here I borrow liberally from ideas and methods developed independently by Bealer [1982], McMichael and Zalta [1980], and Zalta [1983], and adapt them to the present framework. The intuitive idea is that the logical structure of PRPs is algebraic, determined by the operations of a variety of families of abstract logical functions. The members of several of these families operate primarily on the argument structures of relations, as I will now describe. First, let [λx Px] (= P, cf. axiom group 7) be the property of being a property, and let [λxyz Qxyz] (= Q) be the relation that holds among three objects a, b, c (in that order) just in
case a kicks b at c (a relation that, I take it, can only hold between an agent capable of kicking, an object capable of being kicked and a location). The first class of functions, the expansion functions, add vacuous argument places to PRPs such as these. Thus, for example, the function \( \text{EXP}_2 \) takes \([\lambda x \ Px] \) into the relation \([\lambda xy \ Px] \), i.e., the relation that obtains between a and b just in case a is a property. The conversion functions switch two argument places in a relation; thus, the function \( \text{CONV}_3 \) takes \([\lambda x y z \ Qxyz] \) into the converse relation \([\lambda zyx \ Qxyz] \) that holds between c, b, and a (in that order) just in case a, b, and c (in that order) stand in the relation \([\lambda xyz \ Qxyz] \), i.e., just in case, at c, b is kicked by a. The reflection functions subsume one argument place in a relation under another; thus, the function \( \text{REF}_2 \) takes \([\lambda x y z \ Qxyz] \) into the 2-place relation \([\lambda x z \ Qxzz] \) that holds between a and c just in case a kicks himself at c.

The remaining functions correspond to more familiar logical operations. \( \text{NEG} \) takes each PRP to its complement, e.g., \( \text{NEG}([\lambda x y \ Px]) = [\lambda x y \sim Px] \). \( \text{CONJ} \) forms the conjunction of two PRPs, e.g., \( \text{CONJ}([\lambda x \ Px], [\lambda yz \ Qyz]) = [\lambda xyz \ Px \& Qyzz] \); and similarly, \( \text{DISJ} \) forms the disjunction of two PRPs. The various generalization functions correspond to the varieties of (universal and existential) quantification—on the one hand, over all individuals, and on the other, over all n-place relations, for some particular n. Now, generalization functions generalize argument places in relations either to all individuals or simply to all n-place relations (for a fixed n), depending on the sort of function in question. So, for example, the existential generalization of the second argument
place of \([\lambda xz Qxxz]\) to individuals generates the property \([\lambda x \exists z Qxxz]\) that an object (viz., a person) \(a\) has just in case there is an object (viz., a location) \(c\) such that \(c\) bears \([\lambda xz Qxxz]\) to \(c\). Again, the universal generalization of the first (and only) argument place in \([\lambda x Px]\) to all 3-place relations, say, yields the (necessarily false) proposition \([\lambda \forall f^3 Pf^3]\) that all 3-place relations are properties. We will use metalinguistic expressions of the form \(\text{\textit{EXIST}_j}\) and \(\text{\textit{UNIV}_j}\) to denote the functions that generalize the \(j^{\text{th}}\) argument places of \(n\)-place relations \((n \geq j)\) to all individuals, and \(\text{\textit{EXIST}_j^{(i)}}\) and \(\text{\textit{UNIV}_j^{(i)}}\) the functions that generalize the \(j^{\text{th}}\) argument places of such relations to \(i\)-place relations \((i \geq 0)\). So in the examples in the previous two sentences, we saw the functions \(\text{\textit{EXIST}_2}\) and \(\text{\textit{UNIV}_1^3}\) respectively. As indicated, the generalization functions are all \textbf{partial}; the domain of \(\text{\textit{UNIV}_1^{(m)}}\) and \(\text{\textit{EXIST}_1^{(m)}}\) is the collection of \(n\)-place relations such that \(0 < i \leq n\).

The next class of functions corresponds to the syntactic operation of simultaneous substitution. Intuitively, the variables in a formula can be thought of as gaps in the formula that can be filled by terms with determinate meanings. (Note that we call formulas with free variables "open".) We've followed Frege's idea that \(n\)-place relations \((n \geq 1)\) too have "gaps". And just as the gaps in a formula can be filled by substituting closed terms for variables, so the gaps in a relation can be filled by the plugging functions. These come in two varieties. Those of the first sort, the plugging \(0\) functions, simply plug sequences of objects into successive argument places. Thus, \(\text{\textit{PLUG}_1^{(0)}}([\lambda xyz Qxyz],\) Buffy, Bill, the zoo) plugs the objects into the three successive argument places of the relation to yield the proposition that Buffy kicks
Bill at the zoo; again, PLUG$^0_{2,1}(\lambda xyz \ Qxyz, \ Bill) \ yields \ the \ relation
[\lambda xz \ Qxbz] \ that \ holds \ between \ a \ and \ c \ just \ in \ case \ a \ kicks \ Bill \ (b) \ at \ c.
(In \ general, \ the \ function \ PLUG^0_{1,j} \ plugs \ j \ objects \ into \ j \ successive \ argument \ places \ of \ a \ given \ n-place \ relation \ \tau \ (n\geq i+j-1) \ beginning \ with \ the \ i^{th} \ argument \ place.)

The properties and relations that fall within the ranges of the members of the plugging$^1$ functions arise most naturally in contexts involving propositional attitudes, for example, the property of desiring that one be wise. Intuitively, an object $o$ has this property just in case $o$ stands in the desiring relation to a proposition in which $o$ is itself a constituent, viz., the proposition that $o$ is wise. Clearly, this property is built up from the desiring relation $D$ and the property $W$ of being wise. Equally clearly, the property is not the result of simply plugging$^0$ the property $W$ into the second argument place of $D$, for that is just the property $[\lambda x \ Dx[\lambda z \ Wz]] (= [\lambda x \ DxW])$ of desiring wisdom, which, even granting it is possible to desire a property simpliciter (as opposed to the proposition that one have that property), is not the property in question. What also needs to happen, figuratively, is for the argument place in $W$ to be in some sense "appropriated" for the new property in such a way that it remains "active", unlike in the property above. This is just what the plugging$^1$ functions do. Thus, the function PLUG$^1_{2,1}([\lambda xy \ Dxy], \ [\lambda z \ Wz])$ plugs $[\lambda z \ Wz]$ into the second argument place of $[\lambda xy \ Dxy]$ and appropriates its $([\lambda z \ Wz]$'s) first (and only) argument place for the resulting relation $[\lambda xz \ Dx[\lambda Wz]]$, in which $o$ and $o^*$ stand just in case $o$ desires that $o^*$ be wise. Our desired property $[\lambda x \ Dx[\lambda Wx]]$ is then obtained by applying REF$^1_2$ to this relation. (In general, once
again, $\text{PLUG}^1_{i,j}$ plugs an $m$-place relation $r$ into the $i^{th}$ argument place of an $n$-place relation $r^*$ and appropriates the first $j$ argument places of $r$ for the resulting relation. It should be clear that plugging functions of either sort are partial.\(^8\)

Now, it is important to note that the preceding paragraphs enable us to clarify the function of complex terms in our language: they are to mirror the operations of the logical functions on the universe of PRPs. More specifically, suppose we are given some initial assignment\(^9\) of objects (of the right sorts) to the noncomplex terms of the language, and let $S$ be the set of all the objects so assigned; then the complex terms correspond to the PRPs in the closure of $S$ under the logical functions above.\(^10\) This provides us with an intuitive criterion for deciding which formulas $\varphi$ are acceptable for the construction of complex terms: $\varphi$ will be acceptable just in case it can be used to construct a complex term $\lbrack \lambda x_1 \ldots x_n \varphi \rbrack$ that represents a finite composition of logical functions applied to the values of the noncomplex terms in $\varphi$.

So let's ask: which formulas, exactly, are acceptable? Atomic formulas $\lbrack pt_1 \ldots t_n \rbrack$ are acceptable, since these can always be used to construct a term that represents, e.g., the proposition $[\lambda pt_1 \ldots t_n]$ which is obtained by plugging the objects $t_1, \ldots, t_n$ into the argument places in the relation $p$. What about formulas of the form $\lbrack t=s \rbrack$?\(^11\) Here things are a little trickier, since to allow '$=$' into the formation of complex terms is to make an existence assumption that I don't believe is warranted on logical grounds alone (cf. below), and so ultimately I will disallow such formulas. Note, however, that doing so in and of itself rules out the second formulation of Russell's property above; I don't
want it to be thought, though, that this is the reason why the property
cannot be generated in the system here. Hence, for the time being I
will allow formulas of the form in question to be considered acceptable.

So suppose now that \( \psi \) and \( \theta \) are acceptable; then it should also be
the case that \( \psi \land \theta \), \( \psi \lor \theta \), and \( \neg \psi \) are as well, since the occurrence of
and of these logical operators in a complex term just represent straight-
forward applications of CONJ, DISJ, or NEG in the construction of the PRP
it denotes. But what about generalizations \( \forall x \psi \), \( \exists F \psi \), \( \exists x \psi \), and
\( \exists F \psi \) (\( F \) an arbitrary predicate variable)? Intuitively, given our view
of the role of complex terms in our language, occurrences of quantifiers
inside of a complex term \( t \) indicate applications of a generalization
function in the construction of the PRP \( t \) denotes. Now, as we've seen,
generalization functions generalize argument places of properties and
relations, and these argument places are represented in complex terms
\( \lbrack \lambda x_1 \ldots x_n \varphi \rbrack \) by the sequence of individual variables \( x_1 \ldots x_n \).
Hence, before we can answer our question about quantifiers, we have to be
clear about the role of such sequences in the construction of complex
terms. And as it happens, we must be careful, for we can't take just any
sequence of variables \( x_1 \ldots x_n \) and any acceptable formula \( \varphi \) and expect
\( \lbrack \lambda x_1 \ldots x_n \varphi \rbrack \) to be a legitimate complex term, i.e., on our present
picture, a term that reflects the operations of the logical functions
on the universe of PRPs. Let me elaborate.

As we've seen, we must allow for complex terms \( t \) of the form
\( \lbrack \lambda x_1 \ldots x_n \varphi \rbrack \) in which one or more of the variables \( x_1, \ldots, x_n \) occurs
free inside of other complex terms that also occur in \( t \). (Say that
such variables are embedded in \(t\).) However, we cannot permit an \(x_i\) to be embedded just anywhere in such a \(t\). To see why, we need two further notions. Say that a term \(t\) occurs in predicate position in an atomic formula \(\text{\textasciitilde}p_{t_1} \ldots t_n\) just in case \(t\) is \(p\). And say that an embedded variable \(y_i\) occurs unhappily in an expression of the form \(\text{\textasciitilde}([\lambda y_1 \ldots y_n \psi])\) just in case \(y_i\) occurs free in some term that occurs in predicate position in some atomic subformula of \(\psi\). Now, note that the need for terms with embedded variables arose out of our reflections on the operations of the plugging\(^1\) functions. (Recall our example \([\lambda x \, Dx[\lambda \lambda x]]\) of desiring that one be wise.) With that in mind, let’s consider the matter at a somewhat more abstract level. Let \(t\) be \(\text{\textasciitilde}([\lambda xy \varphi])\) and let \(s\) be \(\text{\textasciitilde}([\lambda zwu \psi])\), where \(t\) and \(s\) are closed terms with no variables in common, and let \(\mathcal{I}\) and \(\mathcal{S}\) intuitively be the relations they denote. Let us assume that neither \(x\) nor \(y\) occurs unhappily in \(t\), and that neither \(z\), \(w\), nor \(u\) occurs unhappily in \(s\); what we want to show is that no embedded variable in a term \(t'\) that reflects the operation of a plugging\(^1\) function on \(\mathcal{I}\) and \(\mathcal{S}\) occurs unhappily in \(t'\).

A plugging\(^1\) of \(\mathcal{S}\) into \(\mathcal{I}\) is represented by substituting a term derived from \(s\) for \(x\) or \(y\) in \(\varphi\) (which represents the actual filling of one of the “gaps” in \(\mathcal{I}\)) and altering the sequence \(\text{\textasciitilde}([\lambda xy])\) appropriately (which represents the appropriation of one or more of the argument places in \(\mathcal{S}\) for the new relation). Thus, for example, \(\text{PLUG}_{2,2}^{1}(\mathcal{I}, \mathcal{S}) = [\lambda xzw \varphi' y^{\mathcal{S}}[\lambda u \psi]]\). Here, \(\text{\textasciitilde}([\lambda u \psi])\) is the term derived from \(s\), and represents the plugged relation \(\mathcal{S}\) minus its appropriated argument places; those are represented here by \(z\) and \(w\) in the sequence \(\text{\textasciitilde}([\lambda xzw])\), which in turn represents the argument structure of the new relation. By the assumption above, \(\text{\textasciitilde}([\lambda u \psi])\) does not
occur in a term that occurs in predicate position in $\{[\lambda u \, \psi]\}^y$ (since $y$
does not occur unhappily in $t$). And since $y$ is an individual variable,
$\{[\lambda u \, \psi]\}$ cannot itself occur in predicate position in $\{[\lambda u \, \psi]\}^y$. Hence,
neither $z$ nor $w$ can occur unhappily in $\{[\lambda u \, \psi]\}^y$. By assumption, the same
is true of $x$. Hence, none of the $\lambda$-bound variables (and in particular
none of the embedded variables) in $\{[\lambda xzw \, \psi]\}^{y}_{[\lambda u \, \psi]}$ occurs unhappily in
that term.

It should be clear that an analogous result holds for any two
complex terms (under analogous assumptions) and any appropriate plugging
function. Hence, we conclude in general that expressions of the form
$\{[\lambda x_{1} \ldots x_{n} \, \psi]\}$ in which one or more of the $x_{i}$ occurs unhappily do not
reflect the operations of the plugging functions. Accordingly, we dis-
allow such expressions as legitimate terms of our language.

Now we can return to our question of which quantified formulas
are acceptable. So suppose we have a complex term $\{[\lambda x_{1} \ldots x_{n} \, \psi]\}$ con-
structed with what we know at this point to be an acceptable formula $\psi$ in
accordance with the restrictions we arrived at by our reasoning above,
and consider the application of a generalization function to the $i^\text{th}$
argument place of the PRP denoted by that term. Then intuitively, the
resulting PRP will be denoted by $\{[\lambda x_{1} \ldots x_{i-1} x_{i+1} \ldots x_{n} \, Q_{\alpha} \psi_{\alpha}]\}^{x_{i}}$, where
$Q$ is the appropriate sort of quantifier and $\alpha$ is a new variable of the
appropriate sort for the generalization function in question. But since
$x_{i}$ is an individual term, it cannot occur in predicate position in $\psi$; so
neither can $\alpha$ in $\psi_{\alpha}^{x_{i}}$. And furthermore, by our reasoning above, $x_{i}$ can't
occur free in a term occurring in predicate position in $\psi$; so again, nei-
ther can $\alpha$ in $\psi_{\alpha}^{x_{i}}$. This gives us our general criterion of acceptability
for quantified formulas. Let $\phi$ be acceptable; then if $\alpha$ does not occur free (i) in predicate position in (any atomic subformula of) $\phi$, or (ii) in any term occurring in predicate position in $\phi$, then $\forall \alpha \phi$ and $\exists \alpha \phi$ are acceptable as well.

The remaining axioms for our logic are identity axioms governing complex terms, and are of course intended to capture the central logical truths of our ontology of PRPs. Thus, axiom schema 8 expresses that no n-place relation is identical with any m-place relation if $m \neq n$. Axiom schema 9 expresses that the ranges of CONJ, DISJ, NEG, and the generalization functions are all pairwise disjoint, so that, e.g., no conjunction $[\lambda xy P(x) \& Q(y)]$ of two properties is also the negation, say, or universalization (of any sort), of some other PRP. And axiom groups 10–12 express that the logical functions are one-to-one, so that, e.g., if $r$ is the conjunction of $r^*$ and $r^{**}$, it is the conjunction of only those two PRPs and no two others. 13

§1.3 LOGIC AND METAPHYSICS: RUSSELL'S PARADOX

So much for exposition. We turn now to reflect briefly (and incompletely) on the philosophical import of the system. We begin by noting that even if we allow ' = ' in the formation of complex terms, it is clear that our restrictions on the construction of complex terms block the second formulation of the Russell "property" above ($[\lambda x \exists F(x = F \& \neg F(x))]$). For that formulation requires that a quantified predicate variable occur in predicate position inside the formula to the right of the quantifier. More than this, though, since our logic is consistent (see §2.5 below) we can be certain that these restrictions are indeed sufficient for avoiding paradox. Notice, however, that the avoidance of paradox was not part of
the motivation for those restrictions; our concern was solely to generate a language and a logic that accurately reflect our intuitive conception of the logical structure of the universe of PRPs. As Russell had hoped, the absence of paradox turns out to follow naturally and inevitably from our positive view of the ontological framework of logic. The source of Russell's paradox is thus evident: it lies in an inadequate understanding of the logical structure of the universe of PRPs.

Let's spell this out more carefully. Intuitively, the function of complex terms in a system of logic is to denote those complex PRPs whose existence can plausibly be held to follow as a matter of logic given some interpretation of the noncomplex terms of one's language. The paradox can arise only if one takes it to be a principle of logic that any formula (or at least an injudicious number of them) can be transformed into a complex term, i.e. (given an initial interpretation of the noncomplex terms), that any complex formula can be thought to represent the logical construction of a complex PRP from simpler objects. As we've seen, the basic intuition here is sound—many PRPs are plausibly considered to be built up logically from less complex objects. In light of the precise account of the structure of the universe of PRPs afforded by the logical functions, however, we see why the particular principle above is false, and why certain restrictions must be imposed on the formation of complex terms: some formulas simply don't represent the operations of any of the logical functions, hence it can't be assumed on logical grounds alone that there are PRPs corresponding to them. Without the benefit of the picture the functions provide, though, there is no basis for such a judgment, hence no obvious basis for placing any restrictions on the formation of complex terms; and that is where the paradox finds its foothold.
There is thus a parallel between the conception of the universe of PRPs that the logical functions provide and the iterative conception of the universe of sets. To see this more clearly, note that to place no restrictions on acceptable formulas for the construction of complex terms is in effect to adopt the logical version of the naive comprehension principle: it is to assume, roughly, that for every formula there is a corresponding PRP (in fact infinitely many). The iterative conception explained the restrictions Zermelo placed on the corresponding set theoretic principle by showing that the "set of" operation that determines the structure of the universe of sets could never yield every collection determined by every arbitrary formula, the Russell set in particular. In the same way, the conception of the universe the logical functions yield explains the restrictions we’ve placed on the construction of complex terms by showing that the functions which determine the logical structure of the universe of PRPs (beginning with some initial collection of objects) could never yield every PRP determined by every arbitrary formula, and Russell’s property in particular.

There is, however, a point at which the parallel seems to break down. The iterative conception not only explains Zermelo’s restrictions on the naive comprehension principle, it also explains why there cannot be such a thing as the set of all nonselfmembered sets. Once we’re clear about that, Russell’s paradox in its extensional guise disappears. It is not clear, however, that the same thing can be said about the intensional version of the paradox. What we have shown is that, rightly guided by the logical functions, none of our logical intuitions need be altered or suppressed; as there is no genuine Russell paradox in set theory once we are clear about the structure of sets, neither is there any genuine
Russell paradox in logic once we are clear about the logical structure of the universe of PRPs. However, even though we cannot prove the existence of the Russell property by logical means alone, nothing that has been said gives us any clear answer to the question of why there cannot be such a property simpliciter. Is it not conceivable, for example, that there is an exemplification relation, i.e., a relation \( r \) that holds between \( x \) and \( y \) just in case \( y \) is a property and \( x \) has \( y \)? Nothing in our view of the universe clearly entails that there couldn't be such an entity. But if so, then there is also the property \( \text{NEG} (\text{REF}_2 (r)) \)—the Russell property. Hence, there cannot be such a thing as \( r \). However, we have no obvious explanation of this fact. Hence, the paradox remains. But if this is not a paradox of logic, just what is it?

The answer, I think, is straightforward, but it will take a bit of work to give it some cogency. Broadly speaking, logic is the science of general patterns of valid inference. The proper framework of this science, I've argued, is an ontology that includes both particulars and PRPs. However, for the purposes of logic, that is as specific as things should get. For intuitively, what happens in particular to exist in heaven and earth has no bearing whatever on the general forms of inference with which logic has to do. As Russell [1919] himself put it:

...we do not, in this subject [i.e., logic], deal with particular things or particular properties: we deal formally with what can be said about any thing or any property (p. 196).

Accordingly, we might say, it is never a matter of logic in and of itself that any specific thing exist. But what, exactly, does this amount to? We enter murky philosophical waters here; but I think we can provide a reasonably clear answer.
A first attempt might be to claim that no genuinely logical theorems are of the form $\exists x(x=t)^1$, for some closed term t. But this is far too strong, since theorems of this form follow straightforwardly from logically impeccable axioms of quantification and identity in any system whose language includes individual constants; and surely the mere use of constants does not in and of itself push us beyond the domain of logic (pace Russell [1956], pp. 237–8). However, the suggestion is on the right track, for when we look for the specific commitments of a system, the place to begin is with the singular terms of its language. But we have to think about more than this.

Typically, a formal system $S$ has an intended interpretation, a framework it is designed to characterize to some extent. This is true of our purported system of logic here with its framework of particulars and PRPs no less than any other system. When we look for the specific commitments of a system (if any), we have to look both at its singular terms and at how those terms are intended to hook up with specific objects in the intended framework of the system. More exactly, say that a formal system $S$ entails the existence of an object $o$ of its framework just in case, given the intended meaning of $S$'s logical vocabulary, its distinguished constants (if any) and its syntax, there is (or can be defined) a term $t$ of the language of $S$ such that $o$ is the intended interpretation of $t$ and $\vdash_S \exists x(x=t)$. Thus, for example, ZF and Peano arithmetic entail the existence of the von Neumann ordinal $\omega$ and the number 17, respectively. Now, say that $S$ has specific commitments if, in its intended framework, there is an object whose existence $S$ entails. To say, then, that it is never a matter of logic in and of itself that any specific thing exist is simply to say that no system of logic can have specific commitments.
Now it is easy to show that our system here satisfies this condition. Briefly, the idea is this. Our language contains no distinguished noncomplex terms, so these yield no initial commitments. Further, since '=' is not a term in the language, nor do we allow it in the formation of complex terms, its presence leads to no specific commitments either. Finally, since the complex terms reflect the operations of the logical functions, they also generate no specific commitments. To see why, simply note that the logical functions only yield new objects from objects previously given by conjoining, disjoining, or negating them, plugging one into an argument place of another, generalizing an argument place, or by converting, reflecting, or expanding argument structures. Thus, once an object of any sort enters into the logical architecture of a complex PRP, it is, so to speak, there to stay, in at least some form or other. The syntactic upshot is that every complex term contains predicate constants or free predicate variables (or both); since none of these latter terms come with any predetermined intended meaning, it follows that no complex term itself has a unique intended meaning independent of some particular interpretation of the noncomplex terms. Since no defined term that we could add to our language would have any content independent of some interpretation either, it follows that our system is free of specific commitments.

Things are otherwise for systems with stronger term formation rules. This is of course trivially so for Fregean systems that place no restrictions at all on the formation of complex terms, and hence entail the existence of the Russell property \([\lambda x \exists F (x = F \& \sim F x)]\). But the same is true of weaker, apparently consistent systems like Quine's NF (Quine [1937]) and its intensional counterpart \(\lambda HST^*\). The latter theory, for
example, entails the existence of the following PRPs simply in virtue of the intended meaning of its syntax: the property of having a property \((\lambda x \exists F(Fx))\), the relation of being identical or standing in some 2-place relation with \((\lambda xy x = y \vee \exists F(F^2 xy))\), the proposition that some proposition both holds and doesn’t hold \((\lambda \exists F^0 (F^0 \& \neg F^0))\), and so on. The existence of the same or similar PRPs are entailed as well by Bealer’s first-order system once he adds his distinguished exemplification predicate \(\Delta\). But on our Russellian conception of the relation between logic and ontology, insofar as these systems carry specific ontological commitments, they are to that extent not systems of logic. Granted, notions like having a property, standing in a relation, identity, a proposition’s holding or not, etc. are notions that belong in some sense to logic; but that there exist objects corresponding to these notions is not a matter for logic to decide. Systems like the ones in question which do decide the matter have crossed over from the domain of logic into the domain of metaphysics, from the science of valid inference to the science of what (in particular) there is.\(^{19}\)

And thus the answer to our question about the locus of the intensional version of Russell’s paradox: since the paradox essentially boils down to an issue concerning the existence or nonexistence of a specific property, it clearly takes us beyond logic into metaphysics, and in particular to the science that concerns itself with the nature and existence of properties, viz., the theory of PRPs. Just as the problem in the metaphysics of set theory was to determine the conditions under which there is a set corresponding to a given intuitively well-defined predicate, so the problem in the theory of PRPs is to determine the conditions
under which there is a property (or PRP generally) corresponding to such a predicate. But, once again, the latter is no more a question of logic than the former.

§1.4 A NOTE ON MODALITY

A final note. It will be observed that the issue of modality is conspicuously absent. This deserves a word of explanation, given the unabashedly realistic framework I’ve adopted and the intimate connections between that framework and modality. It is my view that the structure of a logic ought to reflect the metaphysical structure of the world. Hence, the treatment of modality in logic should in particular reflect the modal structure of the world. However, this raises a host of difficult philosophical issues. For example: Do all PRPs exist necessarily, or do those that "involve" contingent objects depend for their existence on those objects? Are there such things as possible worlds? If so, what are they? Are there merely possible individuals? The various answers to these questions provide us with a variety of options in logic regarding, e.g., the syntax, semantics, and logic of modalized complex terms, the logic of quantification inside modal contexts, the representation of modality in the semantics, etc. The view of PRPs we’ve assumed here suggests certain options over others. But it does not entail them. Hence, it seems to me that a complete treatment of modality in logic will explore these different options, and provide the syntax, semantics, and logic appropriate to each. This, however, pushes us a good distance beyond the major purpose of the present article, viz., to present a complete, intuitively satisfying system of logic that avoids the infirmities of type theory. Accordingly, the matter will be taken up in another paper. For now, we turn to the formal development of our system and its metatheory.
\section*{Part II}

\section*{2.1 Languages and Their Syntax\index{Language!formal}\index{Syntax}}

Let $\mu$ be any infinite cardinal. By a language $\mathcal{L}$ we will mean a set of cardinality $\mu$ consisting of individual variables $v_1, v_2, \ldots, v_\xi, \ldots$ $(0 < \xi < \mu)$ and $n$-place predicate variables $F^n_1, F^n_2, \ldots, F^n_\xi, \ldots$ $(n < \omega, 0 < \xi < \mu)$, individual constants $c_1, c_2, \ldots, c_\xi, \ldots$ $(0 < \xi < \mu$, for some $\xi < \mu$) and (for each $n < \omega$) $n$-place predicate constants $p^n_1, p^n_2, \ldots, p^n_\xi, \ldots$ $(0 < \xi < \mu$, for some $\xi < \mu$), logical particles $\forall, \&$, $\neg$, $=$, and $\lambda$, square brackets $[ ]$, and parentheses $(, )$. Constants and variables are called simple terms. By an expression of $\mathcal{L}$ we understand any finite sequence of members of $\mathcal{L}$. In order to pick out the terms and formulas from the expressions of a language $\mathcal{L}$ we need to define them as well as a number of auxiliary notions by a simultaneous recursion. Henceforth, the metavariables $i, j, i', \text{ and } j'$, will range only over positive integers, and metavariables ranging over object language terms are to be considered to be indexed only by positive integers.\index{Term}

Let $\alpha$ be any variable.

1) $\text{TRM}_0 = \{ t \mid t \text{ is a simple term of } \mathcal{L} \}$

2) If $s, t \in \text{TRM}_0$, then (i) $\alpha$ occurs free in $s$ iff $\alpha$ is $s$; (ii) $t$ does not occur in predicate position in $s$.

3) $\text{FLA}_0 = \{ \varphi \mid \varphi \text{ is } s = s', \text{ where } s, s' \in \text{TRM}_0, \text{ or } \varphi \text{ is } \text{pt}_1 \ldots \text{pt}_n, \text{ where } p \text{ is an } n\text{-place predicate term and } p, t_i \in \text{TRM}_0, \text{ for all } i \leq n \}$.

4) If $\varphi$ is $s = s' \in \text{FLA}_0$, then (i) $\alpha$ occurs free in $\varphi$ iff $\alpha$ is $s$ or $\alpha$ is $s'$; (ii) $\varphi$ is not acceptable. If $\varphi = \text{pt}_1 \ldots \text{pt}_n \in \text{FLA}_0$, then (i) $\alpha$ occurs free in $\varphi$ iff $\alpha$ is $p$ or $\alpha$ is $t_i$, for some $i \leq n$; (ii) $t \in \text{TRM}_0$ occurs in predicate position in $\varphi$ iff $t$ is $p$; (iii) $\varphi$ is acceptable.
5) \( \text{TRM}_{k+1} = \{ t \mid t \in \text{TRM}_k \text{ or } t = [\lambda x_1 \ldots x_n \varphi], \text{ where (i) } \varphi \in \text{FLA}_k, \)
\( \text{(ii) } \varphi \text{ is acceptable, and (iii) } x_1 \ldots x_n \text{ s any (possibly empty) sequence of pairwise distinct individual variables such that for all } i \leq n, x_i \text{ does not occur free in any term occurring in predicate position in } \varphi \}. \)

6) If \( s = [\lambda x_1 \ldots x_n \varphi] \in \text{TRM}_{k+1} \), then (i) \( s \) is an \( n \)-place predicate term; (ii) \( \alpha \) occurs free in \( s \) iff for all \( i \leq n \), \( \alpha \) is not \( x_i \), and \( \alpha \) occurs free in \( \varphi \); (iii) \( t \in \text{TRM}_{k+1} \) occurs in predicate position in \( s \) iff \( t \) occurs in predicate position in \( \varphi \).

7) \( \text{FLA}_{k+1} = \{ \varphi \mid \varphi \in \text{FLA}_k \text{ or } \varphi = s = s', \text{ where } s, s' \in \text{TRM}_{k+1} \text{, or } \varphi \) is \( pt_1 \ldots t_n \), where \( p \) is an \( n \)-place predicate term and \( p, t_1 \in \text{TRM}_{k+1} \), for all \( i \leq n \), or \( \varphi \) is \((\neg \psi)\) or \((\psi \& \theta)\) or \( \forall \beta \psi \), where \( \psi, \theta \in \text{FLA}_k \), and \( \beta \) is any variable\}.

8) If \( \varphi \in \text{FLA}_{k+1}, t \in \text{TRM}_{k+1} \), then

a) if \( \varphi \) is \( s = s' \), then (i) \( \alpha \) occurs free in \( \varphi \) iff \( \alpha \) occurs free in \( s \) or in \( s' \); (ii) \( \varphi \) is not acceptable.

b) if \( \varphi \) is \( pt_1 \ldots t_n \), then (i) \( \alpha \) occurs free in \( \varphi \) iff \( \alpha \) occurs free in \( p \) or in \( t_i \), for some \( i \leq n \); (ii) \( t \) occurs in predicate position in \( \varphi \) iff \( t \) is \( p \) or \( t \) occurs in predicate position in \( p \) or in \( t_i \), for some \( i \leq n \); (iii) \( \varphi \) is acceptable;

c) if \( \varphi \) is \((\neg \psi)\) \((\psi \& \theta)\), then (i) \( \alpha \) occurs free in \( \varphi \) iff \( \alpha \) occurs free in \( \psi \) (\( \psi \) and \( \theta \)); (ii) \( t \) occurs in predicate position in \( \varphi \) iff \( t \) occurs in predicate position in \( \psi \) (\( \psi \) or \( \theta \)); (iii) \( \varphi \) is acceptable iff \( \psi \) is (\( \psi \) and \( \theta \) are);

d) if \( \varphi \) is \( \forall \beta \psi \), then (i) \( \alpha \) occurs free in \( \varphi \) iff \( \alpha \) is not \( \beta \) and \( \alpha \) occurs free in \( \psi \); (ii) \( t \) occurs in predicate position in \( \varphi \) iff \( t \)
occurs in predicate position in \( \psi \); (iii) \( \varphi \) is acceptable iff \( \psi \) is acceptable and \( \beta \) is any variable that does not occur free in any term occurring in predicate position in \( \psi \).

Henceforth, parentheses will be eliminated where there is no danger of ambiguity. Let \( \text{TRM}_L = U_{k<\omega} \text{TRM}_k \), and \( \text{FLA}_L = U_{k<\omega} \text{FLA}_k \). Let the rank of a term \( t \in \text{TRM}_L \) (formula \( \varphi \in \text{FLA}_L \)) be the least \( n \) such that \( t \in \text{TRM}_n \) (\( \varphi \in \text{FLA}_n \)).

If \( t, t' \) are both individual terms or both \( n \)-place predicate terms for some \( n < \omega \), then we say that \( t \) and \( t' \) are of the same type. If \( \varphi \) is of the form \( pt_1 ... t_n \), then we say that \( \varphi \) is atomic, and an occurrence of a term \( t \) in \( \varphi \) is a primary subject position occurrence in \( \varphi \) iff the result of replacing that occurrence of \( t \) in \( \varphi \) with a new variable \( \alpha \) (not occurring in \( \varphi \)) is a formula of the form \( pt_1 ... t_{i-1} \alpha t_{i+1} ... t_n \). We say that \( t \) occurs in primary subject position in \( \varphi \) iff some occurrence of \( t \) in \( \varphi \) is a primary subject position occurrence in \( \varphi \).

Let \( \text{TF}_L \) be \( \text{TRM}_L \cup \text{FLA}_L \). For \( \rho \in \text{TF}_L \), \( \alpha_1, ..., \alpha_n \) any pairwise distinct variables and \( t_1, ..., t_n \) any terms, \( \rho^{t_1 ... t_n}_{\alpha_1 ... \alpha_n} \) is the result of (simultaneously) replacing every free occurrence of \( \alpha_i \) in \( \rho \) by \( t_i \) \((i \leq n)\).

For constants \( \kappa \), \( \rho^\kappa_t \) is the result of replacing every occurrence of \( \kappa \) in \( \rho \) by \( t \). Note that in both cases the result needn't be well-formed, i.e., an element of \( \text{TF}_L \). This, as well as familiar problems of substitution into contexts involving bound variables (i.e., variables that occur in a formula \( \varphi \), but are not free in \( \varphi \)), necessitates the following definition. Let \( \alpha \) and \( \beta \) be any variables, and \( t \) any term.

1) If \( s \in \text{TRM}_0 \), then \( t \) is free for \( \alpha \) in \( s \).

2) If \( \varphi \in \text{PT}_1 ... t_n \in \text{FLA}_0 \), then \( t \) is free for \( \alpha \) in \( \varphi \) iff (i) \( \alpha \) is \( p \) and \( t \) is an \( n \)-place predicate term, or (ii) \( \alpha \) is not \( p \).
3) If s is \([\lambda x_1 \ldots x_n \varphi]_{\mathsf{ETRM}_{k+1}}\), then t is free for \(\alpha\) in s iff (i) \(\alpha\) does not occur free in s, or (ii) for all \(i \leq n\), \(x_i\) does not occur free in \(t\), and \(t\) is free for \(\alpha\) in \(\varphi\).

4) If \(\varphi \in \mathsf{FLA}_{k+1}\), then
   a) if \(\varphi\) is \(pt_1 \ldots t_n\), then \(t\) is free for \(\alpha\) in \(\varphi\) iff (i) \(\alpha\) is p and \(t\) is an n-place predicate term and for all \(i \leq n\), \(t\) is free for \(\alpha\) in \(t_i\), or ii) \(\alpha\) is not p and for all \(i \leq n\), \(t\) is free for \(\alpha\) in \(p\) and in \(t_i\);  
b) if \(\varphi\) is \(\neg \psi\), then \(t\) is free for \(\alpha\) in \(\varphi\) iff \(t\) is free for \(\alpha\) in \(\psi\);  
c) if \(\varphi\) is \(\psi \& \theta\), then \(t\) is free for \(\alpha\) in \(\varphi\) iff \(t\) is free for \(\alpha\) in \(\psi\) and \(\theta\);  
d) if \(\varphi\) is \(\forall \beta \psi\), then \(t\) is free for \(\alpha\) in \(\varphi\) iff i) \(\alpha\) does not occur free in \(\psi\), or ii) \(\beta\) does not occur free in \(t\) and \(t\) is free for \(\alpha\) in \(\psi\).

The notion of an alphabetic variant will also be important below. We define it as follows.

1) If \(\rho \in \mathsf{ETRM}_0 \cup \mathsf{FLA}_0\), then \(\rho'\) is an alphabetic variant of \(\rho\) iff \(\rho'\) is \(\rho\).

2) If \(t \in \mathsf{ETRM}_{k+1}\), then \(t'\) is an alphabetic variant of \(t\) iff
   a) \(t \in \mathsf{ETRM}_k\) and \(t'\) is an alphabetic variant of \(t\), or  
b) \(t\) is \([\lambda x_1 \ldots x_n \varphi]\) and (i) \(\varphi'\) is an alphabetic variant of \(\varphi\),  
   (ii) \(y_1, \ldots, y_n\) are free for \(x_1, \ldots, x_n\) (respectively) in \(\varphi'\),  
   (iii) for all \(i \leq n\), if \(x_i\) is not \(y_i\), then \(y_i\) does not occur free in \(\varphi'\), and (iv) \(t'\) is \([\lambda y_1 \ldots y_n \varphi' y_1 \ldots y_n]\).
3) If $\varphi \in \text{FLA}_{k+1}$, then $\varphi'$ is an alphabetic variant of $\varphi$ iff
   a) $\varphi$ is $\text{pt}_1 \ldots \text{pt}_n$ and $\varphi' = \text{pt}'_1 \ldots \text{pt}'_n$, where $\text{pt}'$ is an alphabetic
   variants of $\text{pt}$ and and $\text{pt}'_i$ is an alphabetic variant of $\text{pt}_i$, for
   all $i \leq n$; or
   b) $\varphi$ is $\neg \psi$, $\psi'$ is an alphabetic variant of $\psi$, and $\varphi'$ is $\neg \psi'$; or
   c) $\varphi$ is $\psi \& \theta$, $\psi'$ is an alphabetic variant of $\psi$ and $\theta'$ is an alpha-
   betic variant of $\theta$ and $\varphi'$ is $\psi' \& \theta'$; or
   d) $\varphi$ is $\forall \alpha \psi$, and (i) $\psi'$ is an alphabetic variant of $\psi$, (ii) $\beta$ is a
   variable of the same type as $\alpha$ which is free for $\alpha$ in $\psi'$,
   (iii) if $\alpha$ is not $\beta$, $\psi'$ contains no free occurrences of $\beta$, and
   (iv) $\varphi'$ is $\forall \beta \psi' \alpha$.

Finally, we note that the logical particles $\exists$, $\supset$, and $\equiv$ will be
defined in the usual way.

§2.2 SEMANTICS

An interpretation $\mathcal{I}$ for a language $L$ is a 5-tuple $(\mathfrak{B}, \mathfrak{P}, \mathfrak{E}, \mathfrak{T}, \mathfrak{J})$. $\mathfrak{B}$ is an infinite set, and $\mathfrak{P} = \{D_{-1}, D_0, D_1, \ldots\}$ is partition on $\mathfrak{B}$ such
that only $D_{-1}$ is possibly empty. $\mathfrak{E}$ is an extension function on $\bigcup_{n \geq 0} D_n$
such that if $a \in D_n$, $\mathfrak{E}(a) \in \{T, F\}$, and if $a \in D_n$, $n \geq 0$, $\mathfrak{E}(a) = \exists \mathfrak{P}$. $\mathfrak{J} =
\bigcup_{i < \omega} \{\text{EXP}_i\}_{i < \omega}$, $\{\text{CONV}_j\}_{i < j < \omega}$, $\{\text{REF}_j\}_{i < j < \omega}$, $\{\text{NEG}\}$, $\{\text{CONJ}\}$, $\{\text{UNIV}_i\}_{i < \omega}$,
$\{\text{UNIV}_i\}_{i, m < \omega}$, $\{\text{PLUG}_0\}_{i, j < \omega}$, $\{\text{PLUG}_1\}_{i, j < \omega}$ is a set of one-to-one
partial functions whose domains and ranges are as follows $(m, n < \omega)$:

1) $\text{EXP}_i : D_n \rightarrow D_{n+1}$, $i \leq n+1$.
2) $\text{CONV}_j : D_n \rightarrow D_n$, $i < j \leq n$.
3) $\text{NEG} : D_n \rightarrow D_n$, $n \geq 0$.
4) $\text{CONJ} : D_n \times D_m \rightarrow D_{n+m}$, $m, n \geq 0$. 
5) \( \text{UNIV}_i : D_n \rightarrow D_{n-1}, \ i \leq n \).
6) \( \text{UNIV}^m_i : D_n \rightarrow D_{n-1}, \ i \leq n \).
7) \( \text{REF}_j^i : D_n \rightarrow D_n, \ i < j \leq n \).
8) \( \text{PLUG}^0_{i,j} : D_n \times D_m \rightarrow D_{n-j}, \ i \leq n, j \leq (n+1)-i \).
9) \( \text{PLUG}^1_{i,j} : D_n \times D_m \rightarrow D_{n+j-1}, \ i \leq n, j \leq m \).

We stipulate that the ranges of CONJ, NEG, the UNIV, and the UNIV\(^m\) be pairwise disjoint. These functions affect the extensions of the elements of \( U_{n \geq 0} D_n \) in the following manner.

1) If \( a \in D_0 \),
   \[ \text{EXT}(\text{EXP}_i(a)) = \{ b \mid \text{EXT}(a) = T \}; \]
   if \( a \in D_n, \ n > 0 \),
   \[ \text{EXT}(\text{EXP}_i(a)) = \{ \langle a_1, \ldots, a_{i-1}, b, a_i, \ldots, a_n \rangle \mid \]
   \[ \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle \in \text{EXT}(a) \}. \]

2) If \( a \in D_n, \ n > 1 \),
   \[ \text{EXT}(\text{CONJ}_j^i(a)) = \{ \langle a_1, \ldots, a_{i-j-1}, a_j, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle \mid \]
   \[ \langle a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \rangle \in \text{EXT}(a) \}. \]

3) If \( a \in D_n, \ n > 1 \),
   \[ \text{EXT}(\text{REF}_j^i(a)) = \{ \langle a_1, \ldots, a_{i-1}, a_j, \ldots, a_{i-1}, a_j+1, \ldots, a_n \rangle \mid \]
   \[ \langle a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \rangle \in \text{EXT}(a) \text{ and } a_i = a_j \}. \]

4) If \( a \in D_0 \),
   \[ \text{EXT}(\text{NEG}(a)) = \begin{cases} T, & \text{if } \text{EXT}(a) = F \\ F, & \text{otherwise} \end{cases} \]
if $a \in D_n$, $n > 0$,

$\text{EXT}(\text{NEG}(a)) = \{(a_0, \ldots, a_{n-1}) | (a_0, \ldots, a_{n-1}) \notin \text{EXT}(a)\}$.

5) If $a, b \in D_0$,

$\text{EXT}(\text{CONJ}(a, b)) = \begin{cases} T, & \text{if } \text{EXT}(a) = \text{EXT}(b) = T, \\ F, & \text{otherwise}; \end{cases}$

if $a \in D_0$, $b \in D_n$, $n > 0$,

$\text{EXT}(\text{CONJ}(a, b)) = \{(b_1, \ldots, b_n) | \text{EXT}(a) = T \text{ and } \langle b_1, \ldots, b_n \rangle \in \text{EXT}(b)\}$ (similarly if $a \in D_n$, $n > 0$, and $b \in D_0$);

if $a \in D_n$, $b \in D_m$, $n, m > 0$,

$\text{EXT}(\text{CONJ}(a, b)) = \{(a_1, \ldots, a_n, b_1, \ldots, b_m) | \langle a_1, \ldots, a_n \rangle \in \text{EXT}(a) \text{ and } \langle b_1, \ldots, b_m \rangle \in \text{EXT}(b)\}$.

6) If $a \in D_1$,

$\text{EXT}(\text{UNIV}_1^1(a)) = \begin{cases} T, & \text{if for all } b \in D, b \in \text{EXT}(a), \\ F, & \text{otherwise}; \end{cases}$

if $a \in D_n$, $n > 1$,

$\text{EXT}(\text{UNIV}_1^1(a)) = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) | \text{for all } b \in D, \langle a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n \rangle \in \text{EXT}(a)\}$.

7) If $a \in D_1$, $m \geq 0$,

$\text{EXT}(\text{UNIV}_1^m(a)) = \begin{cases} T, & \text{if for all } b \in D_m, b \in \text{EXT}(a), \\ F, & \text{otherwise}; \end{cases}$

if $a \in D_n$, $n > 1$, $m \geq 0$,

$\text{EXT}(\text{UNIV}_1^m(a)) = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) | \text{for all } b \in D_m, \langle a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n \rangle \in \text{EXT}(a)\}$.

8) If $a \in D_n$, $b_1, \ldots, b_j \in D$, and $j = n$,

$\text{EXT}(\text{PLUG}^0_{1,j}(a, b_1, \ldots, b_j)) = \begin{cases} T, & \text{if } \langle b_1, \ldots, b_j \rangle \in \text{EXT}(a) \\ F, & \text{otherwise}. \end{cases}$

if $a \in D_n$, $b_1, \ldots, b_j \in D$, and $j \leq (n+1) - i$,

$\text{EXT}(\text{PLUG}^0_{i,j}(a, b_1, \ldots, b_j)) = \{(a_1, \ldots, a_{i-1}, a_{i+j}, \ldots, a_n) | \langle a_1, \ldots, a_{i-1}, b_1, \ldots, b_j, a_{i+j}, \ldots, a_n \rangle \in \text{EXT}(a)\}$. 
9) If \( a \in D_n \), \( b \in D_m \), \( 0 \leq i \leq n \), \( 0 \leq j \leq m \),
\[
\exists \xi(\text{PLUG}_{i,j}^1(a, b)) = \{(a_1, \ldots, a_{i-1}, b_1, \ldots, b_j, a_{i+1}, \ldots, a_n) \mid (a_1, \ldots, a_{i-1}, \text{PLUG}_{i,j}^0(b, b_1, \ldots, b_j), a_{i+1}, \ldots, a_n) \in \xi(a)\}.
\]

Finally, \( \xi \) is a function on the constants of \( \mathcal{L} \) into \( D \) such that
if \( c_n \) is an individual constant, then \( \xi(c_n) \in D \), and if \( P_m^n \) is an \( n \)-place predicate constant, then \( \xi(P_m^n) \in D_n \).

**Denotations of Complex Terms**

In order to assign truth values to sentences of \( \mathcal{L} \) relative to an
interpretation \( \mathfrak{I} \) we need to assign denotations to its complex terms (i.e.,
the \( \lambda \)-predicates). To do this we need to induce a partition on the terms
of \( \mathcal{L} \). One cell of the partition consists of the simple terms of \( \mathcal{L} \). We
deal with the remaining terms as follows. Let \( A \) be an arbitrary complex
term \( [\lambda x_1 \ldots x_n \varphi] \), for some formula \( \varphi \) and variables \( x_1, \ldots, x_n \).

1) If there is an \( i \leq n \), such that \( x_i \) does not occur free in \( \varphi \) and
\( i \) is the least such number, then \( A \) is an \( i^{th} \)-\text{expansion of}
\[ [\lambda x_1 \ldots x_{i-1} x_{i+1} \ldots x_n \varphi] . \]

2) If \( A \) is not an \( i^{th} \)-\text{expansion}, then let \( z_1 \ldots z_n \) be the permutation
of the sequence \( x_1 \ldots x_n \) which reflects the order in which those
variables occur free in \( \varphi \) (omitting repetitions). If there is an
\( i \leq n \), such that \( x_i \neq z_j \), and \( i \) is the least such number, then where
\( x_j = z_i \), \( A \) is the \( i, j^{th} \)-\text{conversion of} \[ [\lambda x_1 \ldots x_{i-1} x_j x_{i+1} \ldots x_{j-1} x_i x_{j+1} \ldots x_n \varphi] . \]

3) If \( A \) is none of the above, then if there is an \( i \leq n \) such that \( x_i \)
occurs more than once in \( \varphi \), and if \( i \) is the least such number, then
where \( (a) \ k \) is the number of variables in the sequence \( x_1 \ldots x_n \) that
occur free in $\varphi$ between the first and second occurrences of $x_i$, (b) $\varphi'$ is the result of replacing the second occurrence of $x_i$ with the alphabetically earliest variable $y$ not occurring in $A$, and (c) $j = i + k + 1$, $A$ is the $i$th-reflection of $[\lambda x_1 \ldots x_i \ldots x_{i+k} y x_j \ldots x_n \varphi']$.

4) If $A$ is none of the above, then
   a) if $\varphi$ is $\neg \psi$, $A$ is the negation of $[\lambda x_1 \ldots x_n \psi]$;
   b) if $\varphi$ is $\psi \theta$, then where $i$ is the largest integer $\leq n$ such that $x_i$ occurs free in $\psi$, then $A$ is the conjunction of $[\lambda x_1 \ldots x_i \psi]$ and $[\lambda x_{i+1} \ldots x_n \theta]$;
   c) if $\varphi$ is $\forall x \psi$ ($\forall F \psi$, $F$ any $m$-place predicate variable), then where (i) $x_1, \ldots, x_{i-1}$ occur free to the left of $x$'s ($F$'s) leftmost free occurrence in $\psi$, (ii) no other variables in $x_1 \ldots x_n$ occur free to the left of that occurrence of $x$ ($F$, and (iii) $y$ is the alphabetically earliest individual variable not occurring in $\varphi$, $A$ is the $i$th-universalization ($\forall x \forall y \forall x \forall y$) of $[\lambda x_1 \ldots x_{i-1} \forall x_i \ldots x_n \psi] ([\lambda x_1 \ldots x_{i-1} y x_i \ldots x_n \psi^F]$).

5) If $A$ is none of the above, then
   a) if there is a sequence $t_1 \ldots t_j$ of terms occurring successively in primary subject position (p.s.p.) in $\varphi$ such that (i) for some (unique) $i \leq n$, $x_1, \ldots, x_{i-1}$ occur in p.s.p. to the left of $t_1$'s leftmost p.s.p. occurrence in $\varphi$, (ii) no other terms occur in p.s.p. in $\varphi$ to the left of that occurrence of $t_1$, (iii) for all $i' \leq n$, $j' \leq j$, $x_i$, does not occur free in $t_{j'}$, and (iv) $t_1 \ldots t_j$ is the longest such sequence, then where $\varphi = \varphi_1$, and for all $j' < j$, $\varphi_{j'+1}$ is the result of replacing the leftmost p.s.p. occurrence of $t_j$, in $\varphi_j$, with the alphabetically earliest variable $y_j$, not occur-
ring in \( \phi_j \), \( A \) is the \( i^{th} \)-plugging \( \theta_j \) of \( [\lambda x_1 \ldots x_{i-1} y_1 \ldots y_j x_i \ldots x_n \; \phi] \) by \( t_i \ldots t_j \);

b) if \( A \) is not an \( i^{th} \)-plugging \( \theta_j \), then if there is a term \( t \) occurring in p.s.p. in \( \phi \) that is not identical with any \( x_i \), but in which some \( x_i \), occurs free (\( i, i' \leq n \)), and \( t \) is the leftmost such term, then where (i) \( x_i, x_{i+1}, \ldots, x_{i+j-1} \) (and no other variables in \( x_1 \ldots x_n \)) occur free in \( t = [\lambda y_1 \ldots y_m \; \psi] \), and (ii) \( \psi' \) is the result of replacing the leftmost p.s.p. occurrence of \( t \) in \( \phi \) with the alphabetically earliest variable \( y \) not occurring in \( \phi \), \( A \) is the \( i+1^{th} \)-plugging \( \theta_j \) of 

\[ [\lambda x_1 \ldots x_{i-1} y_{i+j} \ldots x_n \; \phi'] \] 

by 

\[ [\lambda x_1 \ldots x_{i+j-1} y_1 \ldots y_m \; \psi] \].

6) If \( A \) is none of the above, then \( A \) is called elementary.

A moment's reflection makes it clear that this is indeed a partition. For consider any term \( t \). Either \( t \) is simple or complex. If it is simple, then we're done. So suppose it's complex, i.e., that \( t = [\lambda x_1 \ldots x_n \; \phi] \), for some formula \( \phi \). The first thing to check is whether for all \( i \leq n \), all of the variables \( x_i \) occur free in \( \phi \). If not, \( t \) is an \( i^{th} \)-expansion, for some unique \( i \). If so, then check whether the variables \( x_1, \ldots, x_n \) occur in that order in \( \phi \) (ignoring repetitions). If not, then \( t \) is an \( i,j \)-conversion, for some unique \( i,j \). If so, check whether for some \( i \leq n \), \( x_i \) occurs free more than once in \( \phi \). If so, \( t \) is an \( i,j \)-reflection, for unique \( i,j \). If not, look at \( \phi \). If it is a negation, \( t \) is a negation. If it is a conjunction, then \( t \) is a conjunction. If \( \phi \) is \( \forall \psi \), for some \( \psi \), then \( t \) is an \( i^{th} \)-universalization \( (m) \), for some unique \( i \) (\( m \leq \omega \)). If \( \phi \) is none of the above, then it is atomic. If some term \( t' \) distinct
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from every \( x_i \) (\( i \leq n \)) occurs in primary subject position in \( \varphi \), then \( t \) is either an \( i^{th} \)-plugging \( j \) or an \( i^{th} \)-plugging \( j \) (but not possibly both) for unique \( i, j \). If not, then all the variables \( x_1, \ldots, x_n \) occur only once in that order in p.s.p. in \( \varphi \), and no other terms occur in p.s.p. in \( \varphi \), i.e., \( t \) is of the form \( [\lambda x_1 \cdots x_n \; p x_1 \cdots x_n] \), for some \( n \)-place predicate \( p \); and that is just what it is for \( t \) to be elementary.

Call each cell in the partition a syntactic category (relative to \( \ell \)). A syntactic category is primary iff it consists of the conjunctions, the negations, or for some \( i \), the \( i^{th} \)-universalizations or the \( i^{th} \)-universalizations \( ^m \), for some \( m < \omega \).

Since we've defined the terms (and formulas) of \( \ell \) by recursion on the natural numbers, we can prove general facts about terms by induction. In such a proof, of course, the inductive step consists in supposing that the hypothesis is true for terms \( t \in \text{TRM}_k \), and then showing that this implies it is true for the terms in \( \text{TRM}_{k+1} \). However, as the partition makes clear, terms in general have a much greater structural complexity than is revealed merely by their rank in the hierarchy of \( \text{TRM}_k \)'s. E.g., the rank of any complex term \( [\lambda x_1 \cdots x_n \; \varphi] \) is a function solely of \( \varphi \)'s rank in the hierarchy of \( \text{FLA}_k \)'s, irrespective of the sequence of variables \( x_1 \cdots x_n \). As a result, proofs and definitions that rely on the recursive definition of terms above can turn out to be rather tedious and inelegant. There is however a more fine-grained measure of a term's complexity available in virtue of the following

**FACT:** Each term \( t \in \text{TRM}_\ell \) has a unique finite decomposition tree induced by the partition whose terminal nodes are all simple terms of \( \ell \).
Proof: This is obvious for $t \in \text{TRM}_0$. For the inductive step we need four subfacts: (i) If $t$ is an $i^{th}$-expansion, then there is a unique term $t'$ of $\text{TRM}_i$ which $t$ is an $i^{th}$-expansion. (ii) If $t$ is an $i, j^{th}$-conversion (negation, $i^{th}$-universalization $(m)$), then there is a unique term $t'$ of which $t$ is the $i, j^{th}$-conversion (negation, $i^{th}$-universalization $(m)$). (iii) If $t$ is a conjunction, then there are unique terms $t'$ and $t''$ such that $t$ is the conjunction of $t'$ and $t''$. (iv) If $t$ is an $i^{th}$-plugging $^1$ $(\text{plugging}_j^1)$, then there are unique terms $t'$, $t''$ $(t_1 \ldots t_j)$ such that $t$ is the $i^{th}$-plugging $^1$ $(\text{plugging}_j^1)$ of $t'$ by $t''$ $(t_1 \ldots t_j)$. These four subfacts give us uniqueness. Finitude is proved (tediously) by showing that a term's decomposition according to the partition can go only finitely far before one encounters a term (or terms, in the cases of pluggings and conjunctions) in $\text{TRM}_k$, whose decomposition tree(s) is (are) finite by the inductive hypothesis.

Given this fact, we can define the complexity of an arbitrary term $t$ of $\mathcal{L}$ to be simply one less than the height of its decomposition tree. We make use of this in defining denotations for all the terms of $\mathcal{L}$. Given an interpretation $\mathcal{I}$, we define an assignment function $h$ to be a function from the variables of $\mathcal{L}$ into $\mathcal{D}$ such that $h(v_i) \in \mathcal{D}$ and $h(F^n_i) \in \mathcal{D}_n$. We define the denotation of a term $t$ of $\mathcal{L}$ relative to $\mathcal{I}$ and $h$ ($d^\mathcal{I}_h(t)$) as follows. For terms of complexity 0,

1) $d^\mathcal{I}_h(t) = \mathcal{I}(t)$, if $t$ is a constant,

2) $d^\mathcal{I}_h(t) = h(t)$, if $t$ is a variable.

Suppose then we’ve defined the denotations of terms of complexity $< n$, and suppose $t$ is of complexity $n$:
3) If \( t = [\lambda x_1 \ldots x_n \ p x_1 \ldots x_n] \) is elementary, then \( d^X_h(t) = d^X_h(p) \).
4) If \( t \) is an \( i \)th expansion of \( t' \), then \( d^X_h(t) = \text{EXP}_i(d^X_h(t')) \).
5) If \( t \) is the \( i,j \)th conversion of \( t' \), then \( d^X_h(t) = \text{CONV}^i_j(d^X_h(t')) \).
6) If \( t \) is the negation of \( t' \), then \( d^X_h(t) = \text{NEG}(d^X_h(t')) \).
7) If \( t \) is the conjunction of \( t' \) and \( t'' \), then \( d^X_h(t) = \text{CONJ}(d^X_h(t'), d^X_h(t'')) \).
8) If \( t \) is the \( i \)th universalization of \( t' \), then \( d^X_h(t) = \text{UNIV}_i(d^X_h(t')) \).
9) If \( t \) is the \( i \)th universalization\(^m\) of \( t' \), then \( d^X_h(t) = \text{UNIV}^m_i(d^X_h(t')) \).
10) If \( t \) is the \( i,j \)th reflection of \( t' \), then \( d^X_h(t) = \text{REF}^i_j(d^X_h(t')) \).
11) If \( t \) is the \( i \)th plugging\(^0\) of \( t' \) by \( t_1 \ldots t_j \), then \( d^X_h(t) = \text{PLUG}^0_{i,j}(d^X_h(t'), d^X_h(t_1), \ldots, d^X_h(t_j)) \).
12) If \( t \) is the \( i \)th plugging\(^1\) of \( t' \) by \( t'' \), then \( d^X_h(t) = \text{PLUG}^1_{i,j}(d^X_h(t'), d^X_h(t'')) \).

**Satisfaction, Truth, and Models**

For any assignment function \( h \), let \( h(a|a) \) be the assignment function \( h' \) which differs from \( h \) only in that \( h'(a) = a \).

1. \( \phi \) is true in \( \mathcal{M} \) iff \( \mathcal{M}, a \models \phi \).
   (i) If \( \phi \) is \( t = s \), then \( \mathcal{M}, a \models \phi \) iff \( d^X_h(t) = d^X_h(s) \).
   (ii) If \( \phi \) is atomic, then \( \mathcal{M}, a \models \phi \) iff \( \text{SAT}(d^X_h([\lambda \phi])) = T \).
2. If \( \phi = \lnot \psi \), then \( \mathcal{M}, a \models \phi \iff \mathcal{M}, a \not\models \psi \).
3. (i) If \( \phi = \forall v_1 \psi \), then \( \mathcal{M}, a \models \phi \) iff for all \( a \in \mathcal{M} \), \( \mathcal{M}, a \not\models \psi \).
   (ii) If \( \phi = \exists v_1^n \psi \), then \( \mathcal{M}, a \models \phi \) iff for all \( a \in \mathcal{M} \), \( \mathcal{M}, a \not\models \psi \).

A formula \( \phi \) is **true** in \( \mathcal{M} \) iff \( \mathcal{M}, a \models \phi \), for all \( a \). If \( \Gamma \) is any set of formulas of \( L \), we say that \( \mathcal{M} \) is a **model** of \( \Gamma \) iff for all \( \phi \in \Gamma \), \( \phi \) is true in \( \mathcal{M} \). We write \( \Gamma \models \phi \) to signify that every model of \( \Gamma \) is a model of \( \phi \), and we say that \( \phi \) is **logically valid** (\( \vdash \phi \)) iff \( \forall \Gamma \), i.e., iff \( \phi \) is true in all interpretations.
§2.3 LOGIC—THE SYSTEM LPRP

Say that a term \( t \) is convertible iff the complexity of \( t \) is 2.25

Then any instance of any of the following is an axiom. Unless otherwise stated, \( '\alpha' \) (subscripted or not) and \( '\beta' \) stand for any variables.

1. Propositional tautologies.
2. \( \forall \alpha \psi \supset \phi_\alpha^\beta \), where either \( \alpha \) is an individual variable and \( t \) is any term, or \( \alpha \) is an \( n \)-place predicate variable and \( t \) is any \( n \)-place predicate term. In either case, \( t \) must be free for \( \alpha \) in \( \psi \).
3. \( \forall \alpha (\psi \supset \psi) \supset (\psi \supset \forall \alpha \psi) \), where \( \alpha \) is not free in \( \psi \).
4. \( \alpha = \alpha \).
5. \( \alpha = \beta \supset (\psi \equiv \phi_\alpha^\beta) \), \( \beta \) any variable free for \( \alpha \) in \( \psi \).
6. \( \forall \alpha_1 \ldots \alpha_n ([\lambda x_1 \ldots x_n \psi] \alpha_1 \ldots \alpha_n \equiv \psi_\alpha_1 \ldots \alpha_n^{x_1 \ldots x_n}) \), where \([\lambda x_1 \ldots x_n \psi] \) is convertible, and for all \( i \leq n \), \( \alpha_i \) is free for \( x_i \) in \( \psi \).
7. \( [\lambda x_1 \ldots x_n px_1 \ldots x_n] = p \), for any \( n \)-place predicate \( p \).
8. \( \psi \equiv \psi \), where \( \psi \) is any \( n \)-place predicate term, \( \beta \) is any \( m \)-place predicate term, and \( m \neq n \).
9. \( \psi \equiv \psi \), where \( \psi \) and \( \beta \) belong to different primary syntactic categories.
10. \( \psi \equiv \psi \), where \( \psi \) and \( \beta \) are (the) \( i \)-th expansions, \( i,j \)-th conversions, \( i,j \)-th reflections, \( i \)-th universalizations, \( m \)-negations) of \( \psi \) and \( \beta \) respectively.
11. \( \psi \equiv \psi \), where \( \psi \) and \( \beta \) are the conjunctions of \( \psi \) and \( \beta \) respectively.
12. \( \psi \equiv \psi \), where \( \psi \) and \( \beta \) are the \( i \)-th pluffings of \( \psi \) and \( \beta \) by \( t \) \( t_1 \ldots t_j \) and \( t' \) \( t'_1 \ldots t'_j \), respectively.
Modus ponens and Generalization (\(\forall \varphi \text{ follows from } \varphi\), for any \(\alpha\)) are the rules of inference. Proofs are understood in the usual way, and we write \(\Gamma \vdash \varphi\) to indicate that there is a proof of \(\varphi\) from \(\Gamma\). \(\Gamma\) is said to be consistent iff there is no formula \(\varphi\) such that \(\Gamma \vdash \varphi\) and \(\Gamma \vdash \lnot \varphi\). We say that \(\varphi\) is a logical theorem (\(\vdash \varphi\)) iff \(\varnothing \vdash \varphi\), i.e., iff \(\varphi\) is provable from the axioms of LPRP alone.

Axiom group 6 appears on the face of it to be uncomfortably narrow and restrictive. Given the following definition we can prove a more comprehensive \(\lambda\)-conversion schema for all complex terms. In this definition, 'B' and 'C' will stand for the longer metalinguistic expressions written out in detail in the definition of the syntactic categories in §2.2, and the metavariables 'y', 'ψ', 'θ', etc. will correspond exactly to their counterparts in that definition.

So once again, let \(A\) be an arbitrary complex term \([\lambda x_1 \ldots x_n \varphi]\).

1) If \(A\) is an \(i\)th-expansion of \(B\), then \(\varphi_A = \varphi_B^i\).

2) If \(A\) is the \(i,j\)th-conversion of \(B\), then \(\varphi_A = \varphi_B^j\).

3) If \(A\) is the \(i,j\)th-reflection of \(B\), then \(\varphi_A = (\varphi_B^j)^y_{x_i}\).

4) (a) If \(A\) is the negation of \(B\), then \(\varphi_A = \lnot \varphi_B\).

(b) If \(A\) is the conjunction of \(B\) and \(C\), then \(\varphi_A = \varphi_B \& \varphi_C\).

(c) If \(A\) is the \(i\)th-universalization \(\langle m \rangle\) of \(B\), then \(\varphi_A = \forall x \varphi_B \langle \forall \langle \psi_y_B^x \rangle \rangle\).

5) (a) If \(A\) is the \(i\)th-plugging \(^0\) of \(B\) by \(t_1 \ldots t_j\), then \(\varphi_A = ((\varphi_j^i)_B)^{y_1 \ldots y_j}_{t_1 \ldots t_j}\).

(b) If \(A\) is the \(i\)th-plugging \(^1\) of \(B\) by \(C\), then \(\varphi_A = (\varphi_B^j)^y_t\), where \(t\) is \([\lambda y_1 \ldots y_m \psi \, x_i \ldots x_{i+j-1} \psi_{y_1 \ldots y_m}]^{26}\).

6) If \(A\) is elementary, then \(\varphi_A = \varphi\).

Now, where \(A\) is any complex term \([\lambda x_1 \ldots x_n \varphi]\), we can prove by a straightforward induction on the complexity of terms the following general \(\lambda\)-conversion schema:
Theorem 1: \( \forall \alpha_1 \ldots \alpha_n (\exists x_1 \ldots x_n) (\varphi_{A_1} \alpha_1 \ldots \alpha_n) \equiv (\varphi_{A_1}) \alpha_1 \ldots \alpha_n \), where for all \( i \leq n \), \( \alpha_i \) is free for \( x_i \) in \( \varphi \).

§2.4 SOUNDNESS AND COMPLETENESS

Lemma 1: For any \( \varphi \) and \( h \), and any \( s, t \in \text{TRM}_{A} \), if \( s \) is free for \( \alpha \) in \( t \), then where \( h' = h(\alpha | d^\varphi_h(s)) \), \( d^\varphi_{h'}(t'_s) = d^\varphi_{h'}(t) \).

Proof: The lemma is proved by a simple induction on the complexity of terms given the following facts: (i) If \( t \) is an \( i^{th} \)-expansion of \( t' \), then \( t'_s \) is an \( i^{th} \)-expansion of \( t'_{s'} \). (ii) If \( t \) is the \( i,j^{th} \)-conversion (negation, \( i^{th} \)-universalization \( \langle m \rangle \), \( i,j^{th} \)-reflection) of \( t' \), then \( t'^s \) is the \( i,j^{th} \)-conversion (negation, \( i^{th} \)-universalization \( \langle m \rangle \), \( i,j^{th} \)-reflection) of \( t'_{s'} \). (iii) If \( t \) is the conjunction of \( t' \) and \( t'' \), then \( t'^s \) is the conjunction of \( t'_{s'} \) and \( t''_{s'} \). (iv) If \( t \) is the \( i^{th} \)-plugging \( j_i \) (\( -\text{plugging}^0_j \)) of \( t' \) by \( t''(t_1 \ldots t_j) \), then \( t'^s \) is the \( i^{th} \)-plugging \( j_i \) (\( -\text{plugging}^0_j \)) of \( t'_{s'} \) by \( t''_{s'}((t_1')_{s'} \ldots (t_j')_{s'}) \). Then, for example, if the lemma holds for terms of complexity \( \leq n \), and \( t \) is of complexity \( n \) and is an \( i^{th} \)-expansion of \( t' \), then we have

\[
\begin{align*}
    d^\varphi_{h'}(t'_s) &= \text{EXP}_i(d^\varphi_{h'}(t'_s)) & \text{by (i) and definition of } d^\varphi_{h'} \\
    &= \text{EXP}_i(d^\varphi_{h'}(t')) & \text{by the induction hypothesis} \\
    &= d^\varphi_{h'}(t) & \text{by definition of } d^\varphi_{h'}.
\end{align*}
\]

The other cases are just as straightforward.

Lemma 2: For any \( \varphi \) and \( h \), and for any \( \varphi \in \text{FLA}_{A} \) and \( t \in \text{TRM}_{A} \), if \( t \) is free for \( \alpha \) in \( \varphi \), then \( \vDash_{\text{A} \varphi}[h] \iff \vDash_{\text{A} \varphi}[h(\alpha | d^\varphi_h(t))] \).

Proof: Suppose so for formulas of rank \( \leq n \), and suppose \( \varphi \)’s rank is \( n \). If \( \varphi \) is atomic or an identity, then the lemma follows directly from Lemma 1. If \( \varphi \) is a conjunction or negation, the lemma is proved as usual. So let \( \varphi \) be \( \forall \beta \psi \). If \( \alpha \) is \( \beta \), then the lemma is immediate. So suppose not.
$\forall x \phi^x_t[h] \text{ iff for all } a \in D_n, \exists \exists \psi^x_t[h(\beta|a)]$

iff for all $a \in D_n$, $\exists \exists \psi[h(\beta|a)\{a|d^x_{h(\beta|a)}(t)\}]$ (ind. hyp.)

iff for all $a \in D_n$, $\exists \exists \psi[h(\beta|a)\{a|d^x_{h(t)}(t)\}]$ (since $\beta$ does not occur free in $t$)

iff for all $a \in D_n$, $\exists \exists \psi[h(a|d^x_{h(t)}(t))\{\beta|a\}]$ (since $\beta$ is not $a$)

iff $\exists \exists \phi[h(a|d^x_{h(t)}(t))]$.

Lemma 3: Every axiom of LPRP is logically valid.

Proof: The lemma is proved for axiom groups 1, 3, and 5 essentially along the same lines as their standard first-order analogues, modulo of course the somewhat lengthier inductions needed here. The validity of (every instance of) axiom group 2 follows quickly from Lemma 2. The validity of axiom group 4 is obvious, and that of axiom group 6 is proved just by checking the various possible cases of convertible terms. The validity of axiom group 7 is an immediate consequence of the definition of the denotation function. The validity of axiom group 8 is implied by the following

Easy Lemma: If $p$ is an $n$-place predicate term, then $d^x_{h}(p)\in D_n$.

The proof is again by induction on complexity. The validity of axiom group 9 follows from the pairwise disjointness of the ranges of CONJ, NEG, and the UNIV_{i}^{(m)}}, and the validity of axiom groups 10–12 follows from the fact that the logical functions $\forall \epsilon \in \mathcal{F}$ are all one-to-one.

Lemma 4: Let $\Gamma$ be any set of sentences of $L$. Then if $\Gamma \vdash \psi$ and $\Gamma \vdash \forall \phi \psi$, then $\Gamma \vdash \forall \phi \psi$ and $\Gamma \vdash \psi$.

Theorem 2 (Soundness): Let $\Gamma$ be a set of sentences of $L$. Then $\Gamma \vdash \psi$ only if $\Gamma \models \psi$.

Proof: An easy induction on length of proof, given the above lemmas.
We turn now to the completeness theorem.

Lemma 5: Let $\rho \in TF_\lambda$. Then if a variable $\beta$ does not occur at all in $\rho$, and $\beta$ is free for $\alpha$ in $\rho$, then $\alpha$ is free for $\beta$ in $\rho^\alpha_\beta$.

Proof: A trivial induction on the construction of terms and formulas, requiring appeal only to the requisite syntactic definitions.

Lemma 6 (Deduction Theorem): If a proof of $\psi$ from $\Gamma \cup \{\varphi\}$ involves no application of Generalization to a variable that occurs free in $\varphi$, then $\Gamma \vdash \varphi \Rightarrow \psi$.

Proof: Exactly as its first-order analogue, using axiom groups 1 and 3.

Lemma 7: If $t$ is an alphabetic variant of $t'$, then $\Gamma \vdash t = t'$.

Proof: By induction on complexity of terms, using axiom groups 4, 7, and 10-12.

Lemma 8: If $\varphi, \varphi' \in FLA_\lambda$ and $\varphi'$ is an alphabetic variant of $\varphi$, then $\Gamma \vdash \varphi \equiv \varphi'$.

Proof: By strong induction on the rank $n$ of $\varphi$. Axiom groups 2-5 and Lemmas 6 and 7 are needed.

Lemma 9: Let $\varphi \in FLA_\lambda$ and let $\kappa$ be a constant that doesn't occur in any formula of $\Gamma$. Then if $\Gamma \vdash \varphi$, there is a variable $\beta$ (not occurring in $\varphi$) of the same type as $\kappa$ such that $\Gamma \vdash \forall \beta \varphi^\kappa_\beta$.

Proof: By strong induction on length of proof, using Lemma 8.

Lemma 10: Let $\alpha$ be a variable of the same type as $\kappa$. If $\Gamma \vdash \varphi^\alpha_\kappa$, and $\kappa$ does not occur at all in $\varphi$ or in any formula in $\Gamma$, then $\Gamma \vdash \forall \alpha \varphi$.

Proof: By Lemmas 5, 6, 8, and 9, and axiom group 2.

Theorem 3 (Completeness): Let $\Gamma$ be a set of sentences of $L$. Then $\Gamma \vdash \varphi$ only if $\Gamma \models \varphi$. 
We use a Henkin construction to show that every consistent set of sentences has a model. So let $\Gamma$ be a consistent set of sentences of $L$, where the cardinality of $L$ is $\mu$. Now let $K_{n-1}$ be a set of $\mu$ new individual constants $\{d_\xi \mid \xi < \mu\}$, and for (finite) $n \geq 0$, let $K_n$ be a set of $\mu$ new $n$-place predicate constants $\{Q_{\xi}^n \mid \xi < \mu\}$, and let $K = \bigcup_{n \geq 1} K_n$. Let $E = \{\pi_\xi \mid \xi < \mu\}$ be an enumeration of all pairs $\pi = (\varphi, \alpha)$ such that $\varphi \in \text{FLA}_{\Sigma \Delta \varphi}$ is a formula with at most one free variable, and $\alpha$ is the variable occurring free in $\varphi$, if there is one, or $\alpha$ is any variable otherwise. Now, for each $\pi_\xi = (\varphi, \alpha)$, let $\sigma_\xi$ be the formula $\neg \forall \varphi \supset \neg \varphi^\alpha$, where $\kappa$ is the alphabetically earliest new constant in $K$ of the same type as $\alpha$ that does not occur in $\varphi$ or in any $\sigma_\zeta$, for $\langle \zeta \xi \rangle$, and let $\Sigma = \{\sigma_\xi \mid \xi < \mu\}$. Note that $\Gamma$ is a consistent set of formulas of $\Sigma \Delta \varphi$.

**Lemma 11:** $\Gamma \cup \Sigma$ is consistent.

**Proof:** A straightforward transfinite induction, using Lemma 10.

**Lemma 12:** Let $\Omega \in \text{FLA}_L$. If $\Omega \varphi$, then $\Omega \cup \neg \varphi$ is consistent.

**Proof:** As usual.

**Lemma 13:** $\Gamma \cup \Sigma$ can be extended to a maximal consistent set $\Delta$ of sentences of $\Sigma \Delta \varphi$.

**Proof:** By the usual Lindenbaum construction. Lemma 12 is needed.

We now construct an interpretation $\mathfrak{I}$ for $\Sigma \Delta \varphi$. For any closed term $t$ of $\Sigma \Delta \varphi$, let $[t] = \{t' \mid t' \in \text{TRM}_{\Sigma \Delta \varphi}$ and $t \equiv t' \in \Delta\}$. We define $\mathfrak{I}$ to be the set of all such equivalence classes, i.e., $\mathfrak{I} = \{[t] \mid t$ is a closed term of $\Sigma \Delta \varphi\}$. $\mathfrak{I}$ will be the obvious partition, viz., for $n \geq 0$, we let $D_n = \{[p] \mid p$ is a closed $n$-place predicate term of $\Sigma \Delta \varphi\}$, and $D_{-1} = \mathfrak{I} - \bigcup_{n \geq 0} D_n$. (Note that $D_{-1}$ might be empty.)
Lemma 14: For every closed term $t \in \text{TRM}_{\Delta \cup K}$ there is a $\kappa \in K$ such that $\kappa$ is of the same type as $t$ and $t = \kappa \in \Delta$.

Proof: Let $\alpha$ be any variable of the same type as $t$. Then by axiom group 2 we get $\Gamma \vdash \forall \alpha \sim (a=t) \supset \sim (t=t)$, so by axiom group 4, $\Gamma \vdash \forall \alpha \sim (a=t)$. By the maximal consistency of $\Delta$, $\forall \alpha \sim (a=t) \in \Delta$, and since $\Gamma \cup \Sigma \subseteq \Delta$, $\forall \alpha \sim (t=a) \supset \sim (t=a)^{\kappa} \in \Delta$, for some $\kappa \in K$ of the same type as $\alpha$ (and so also $t$), and so by the maximal consistency of $\Delta$ again, $\sim (t=\kappa) \in \Delta$, so $t = \kappa \in \Delta$, by axiom group 1.

Thus, by Lemma 14, every equivalence class $a \in D_n$, for every $n \geq 0$, contains at least one $n$-place predicate constant $\kappa \in K$, and if $D_{-1}$ is not empty, then each $a \in D_{-1}$ contains at least one individual constant $\kappa \in K$. In light of this, for $a \in D_n$, $n \geq 0$, let $K(a)$ be the alphabetically earliest element $\kappa \in Kn_a$ such that $\kappa$ is an $n$-place predicate term. If $a \in D_{-1}$, let $K(a)$ be any $\kappa \in Kn_a$. We define the extension function $\text{EXT}$ as follows:

If $a \in D_0$, then $\text{EXT}(a) = \top$, if $K(a) \in \Delta$.

If $a \in D_n$, $n > 0$, then $\text{EXT}(a) = \{(\llbracket t_1 \rrbracket, \ldots, \llbracket t_n \rrbracket) \mid K(a) t_1 \ldots t_n \in \Delta\}$.

By axiom groups 2 and 5, we have that

$$K(a) t_1 \ldots t_n \& t_1 = t'_1 \& \ldots \& t_n = t'_n \supset K(a) t'_1 \ldots t'_n.$$ 

Hence, $\text{EXT}$ is well-defined, i.e., the definition of $\text{EXT}(a)$ for $a \in D_n$ ($n > 0$) is independent of our choice of equivalence class representatives $t_i$.

We now define the logical functions $\text{F}_e \in \mathcal{F}$ as follows.

1) If $a \in D_n$, $n \geq 0$, then

$$\text{EXP}_i^a = \llbracket (\alpha v_1 \ldots v_{i-1} v_{i+1} v_i \ldots v_n K(a) v_1 \ldots v_n) \rrbracket.$$

2) If $a \in D_n$, $n > 1$, then

$$\text{CONV}_j^a = \llbracket (\alpha v_1 \ldots v_{i-1} v_j v_{i+1} \ldots v_{j-1} v_i v_{j+1} \ldots v_n K(a) v_1 \ldots v_n) \rrbracket.$$
3) If \( a \in D_n \), \( n \geq 0 \), then
\[ \text{NEG}(a) = \left[ (\lambda v_1 \cdots v_n \neg K(a)v_1 \cdots v_n) \right]. \]

4) If \( a \in D_n \), \( b \in D_m \), \( m \geq n \geq 0 \), then
\[ \text{CONJ}(a,b) = \left[ (\lambda v_1 \cdots v_n v_{n+1} \cdots v_{n+m} K(a)v_1 \cdots v_n \& K(b)v_{n+1} \cdots v_{n+m}) \right]. \]

5) If \( a \in D_n \), \( n \geq 0 \), then
\[ \text{UNIV}_i^1(a) = \left[ (\lambda v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \forall v_i K(a)v_1 \cdots v_n) \right]. \]

6) If \( a \in D_n \), \( n, m \geq 0 \), then
\[ \text{UNIV}_i^m(a) = \left[ (\lambda v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \forall v_i K(a)v_1 \cdots v_{i-1} F_i^m v_{i+1} \cdots v_n) \right]. \]

7) If \( a \in D_n \), \( n > 1 \), then
\[ \text{REF}_j^i(a) = \left[ (\lambda v_1 \cdots v_{i-1} v_{j-1} v_{j+1} \cdots v_n K(a)v_1 \cdots v_i \cdots v_{j-1} v_i v_{j+1} \cdots v_n) \right]. \]

8) If \( a \in D_n \), \( n > 0 \), and \( b_1, \ldots, b_j \in B \), then
\[ \text{PLUG}_{i,j}^0(a, b_1, \ldots, b_j) = \left[ (\lambda v_1 \cdots v_{i-1} v_{i+j} \cdots v_n \varphi) \right], \]
where \( \varphi \) is \( K(a)v_1 \cdots v_{i-1} K(b_1) \cdots K(b_j)v_i \cdots v_n \).

9) If \( a \in D_n \), \( b \in D_m \), \( n, m \geq 0 \), then
\[ \text{PLUG}_{i,j}^1(a, b) = \left[ (\lambda v_1 \cdots v_{i-1} v_{n+1} \cdots v_{n+j+1} \cdots v_n K(a)v_1 \cdots v_{i-1} v_{i+j+1} \cdots v_n) \right], \]
where \( t \) is \( v_{n+j+1} \cdots v_{n+m} K(b)v_{n+1} \cdots v_{n+j+1} \cdots v_{n+m} \).

Finally, let \( \mathcal{J} \) be defined on the constants \( \kappa \in \mathcal{D} \cup \mathcal{K} \) in the obvious way, viz., \( \mathcal{J}(\kappa) = [\kappa] \).

It is straightforward, though tedious, to show that the above structure \( \mathcal{M} \) in fact satisfies the criteria for being an interpretation for \( \mathcal{L} \). There are only three things we need to check: (i) that the effects of the logical functions as we've defined them on the extensions of the elements of \( U_{n \geq 0} D_n \) are as required; (ii) that the logical functions are one-to-one; and (iii) that the ranges of CONJ, NEG, and the \( \text{UNIV}_i^m \) are pairwise disjoint.
Proving (i) is essentially a matter of spelling out the definitions. The only case that requires a little more care is that of the \textsc{plug}^1_2 functions which we'll demonstrate here. For clarity, I'll deal with a sufficiently general special case. Let \(a \in D_3\) and \(b \in D_4\). We'll prove (i) for \textsc{plug}^1_2 applied to these objects. By definition of \textsc{plug}^1_2,

\[
\text{EXT}(\text{plug}^1_2(a, b)) = \text{EXT}(\lambda v_1v_4v_5v_3\ K(a)v_1[\lambda v_6v_7K(b)v_4v_5v_6v_7]v_3). \]

By definition of \text{EXT} and \(K(\ldots)\), and by axiom 2, where \((\bar{t})\) is short for

\[
\langle [t_1],[t_4],[t_5],[t_3] \rangle, \]

this is just

\[
(\ast) \quad \{ (\bar{t}) \mid [\lambda v_1v_4v_5v_3\ K(a)v_1[\lambda v_6v_7K(b)v_4v_5v_6v_7]v_3]t_1t_4t_5t_3 \in \Delta \}. \]

By axiom group 6, \((\ast)\) is equal to

\[
(\ast\ast) \quad \{ (\bar{t}) \mid K(a)t_1[\lambda v_6v_7K(b)t_4t_5v_6v_7]t_3 \in \Delta \}, \]

which by definition of \text{EXT} again is equal to

\[
(\ast\ast\ast) \quad \{ (\bar{t}) \mid \langle [t_1],[[\lambda v_6v_7K(b)t_4t_5v_6v_7]], [t_3] \rangle \in \text{EXT}(a) \}. \]

Now by definition of \textsc{plug}^0_1,2,

\[
\text{plug}^0_1,2(b, [t_4],[t_5]) = [[[\lambda v_3v_4K(b)K([t_4])K([t_5])v_3v_4]]], \]

and by definition of \(K(\ldots)\), axiom group 12, and Lemma 7,

\[
\text{plug}^0_1,2(b, [t_4],[t_5]) = [\lambda v_3v_4K(b)t_4t_5v_6v_7], \]

so, as required, \((\ast\ast\ast)\) is equal to

\[
\{ (\bar{t}) \mid \langle [t_1],\text{plug}^0_1,2(b, [t_4],[t_5]), [t_3] \rangle \in \text{EXT}(a) \}. \]

We turn now to (ii). I will prove it here just for the case of \(\text{exp}_i\), since the general structure of the proof is essentially identical in all the other cases. So suppose \(\text{exp}_i\) is not one-to-one, for some \(i<\omega\). Then there are distinct \(a,b \in D\) such that \(\text{exp}_i(a) = \text{exp}_i(b)\). By
axiom group 8 and the definition of EXP, a and b must belong to the same \( D_n \). By hypothesis, \( K(a) \neq K(b) \in \Delta \). Also by hypothesis, \( \text{EXP}_i(a) = [(\lambda v_1 \cdots v_{n+1} v_1 \cdots v_n K(a) v_1 \cdots v_n)] = [(\lambda v_1 \cdots v_{n+1} v_i \cdots v_n K(b) v_1 \cdots v_n)] = \text{EXP}_i(b) \), so \( [(\lambda v_1 \cdots v_{n+1} v_1 \cdots v_n K(a) v_1 \cdots v_n)] = [(\lambda v_1 \cdots v_{n+1} v_i \cdots v_n K(b) v_1 \cdots v_n)] \in \Delta \), by definition of \( \{\ldots\} \). But by axiom group 10, it follows that \( [(\lambda v_1 \cdots v_n K(a) v_1 \cdots v_n)] = [(\lambda v_1 \cdots v_n K(b) v_1 \cdots v_n)] \in \Delta \), and by axiom group 7, \( \vdash [(\lambda v_1 \cdots v_n K(a) v_1 \cdots v_n)] = K(a) \) and \( \vdash [(\lambda v_1 \cdots v_n K(b) v_1 \cdots v_n)] = K(b) \), so \( K(a) = K(b) \in \Delta \), contradicting the consistency of \( \Delta \).

Pairwise disjointness of the ranges of CONJ, NEG, and the UNIV\(_i\) can be proved similarly by means of axiom group 9 and the maximal consistency of \( \Delta \). So \( \mathfrak{A} \) is in fact an interpretation for \( \text{LK} \). Now we want to show it is a model of \( \Delta \).

**Lemma 15:** If \( t \) is a closed term of \( \text{LK} \), then for any assignment function \( h \), \( d^\mathfrak{A}_h(t) = [t] \).

**Proof:** For closed terms of complexity 0, i.e., the individual and predicate constants, the lemma follows from the definition of \( \mathcal{J} \), since for constants \( k \in \text{LK} \), \( d^\mathfrak{A}_h(k) = \mathcal{J}(k) = [k] \). So suppose the lemma holds for terms of complexity \( < n \), and let \( t \) be of complexity \( n \).

**Case 1:** If \( t \) is elementary, then the lemma follows by definition of the denotation function.

**Case 2:** If \( t \) is an \( i \)th-expansion of \( t' \), then for any \( h \), \( d^\mathfrak{A}_h(t) = \text{EXP}_i(d^\mathfrak{A}_h(t')) = \text{EXP}_i([t']) \) (by the induction hypothesis) = 

\[
[(\lambda v_1 \cdots v_{i-1} v_{n+1} v_1 \cdots v_n K([t']) v_1 \cdots v_n)]
\]

(by definition of \( \text{EXP}_i \)) = 

\[
[(\lambda v_1 \cdots v_{i-1} v_{n+1} v_i \cdots v_n t' v_1 \cdots v_n)]
\]

(by definition of \( K(\ldots) \)). Now, 

\[
[\lambda v_1 \cdots v_{i-1} v_{n+1} v_i \cdots v_n t' v_1 \cdots v_n]
\]

is an \( i \)th-expansion of 

\[
[\lambda v_1 \cdots v_n t' v_1 \cdots v_n],
\]

and by axiom group 7, \( \vdash [\lambda v_1 \cdots v_n t' v_1 \cdots v_n] = t' \),
hence by axiom group 10, \( h[\lambda v_1 \ldots v_{i-1} v_{i+1} \ldots v_n t' v_1 \ldots v_n] = t \). Hence, 
\( d^n(t) = \llbracket t \rrbracket \), as required.

The remaining cases fall out in essentially the same way, given axiom group 7 and axiom groups 10-12.

**Lemma 16:** For all sentences \( \varphi \in \text{FLA}_{\text{UK}} \), \( \varphi \in \Delta \) iff \( \varphi \) is true in \( \mathbb{K} \).

**Proof:** Suppose so for all sentences of \( \text{UK} \) of rank \( < n \), and let the rank of \( \varphi \) be \( n \). If \( \varphi \) is \( t = s \), then the lemma follows immediately from Lemma 15. So suppose not.

**Case 1:** \( \varphi \) is atomic. Then \( \varphi \in \Delta \) iff \( [\lambda \varphi] \in \Delta \) iff \( K([\llbracket [\lambda \varphi] \rrbracket]) \in \Delta \) iff 
\( \llbracket [\llbracket [\lambda \varphi] \rrbracket] \rrbracket = T \) iff for all \( h \), \( \llbracket [d^n_h([\llbracket [\lambda \varphi] \rrbracket])] \rrbracket = T \) (by Lemma 15) iff for all \( h \), \( \models^h \varphi \) iff \( \varphi \) is true in \( \mathbb{K} \).

**Cases 2 and 3:** \( \varphi \) is \( \neg \psi \) or \( \psi \wedge \theta \); proofs as usual.

**Case 4:** \( \varphi \) is \( \forall \alpha \psi \). For the only-if direction there are two subcases.

**Subcase 1:** \( \alpha \) is an individual variable.

\( \varphi \in \Delta \implies \) for all closed terms \( t \in \text{TRM}_{\text{UK}} \), \( \psi_{\alpha} \in \Delta \)

\( \implies \) for all \( k \in K \), \( \psi_{\alpha} \in \Delta \)

**Hyp** for all \( k \in K \), \( \psi_{\alpha} \) is true in \( \mathbb{K} \)

**Lem 14** for all closed terms \( t \in \text{TRM}_{\text{UK}} \), \( \psi_{\alpha} \) is true in \( \mathbb{K} \)

\( \implies \) for any \( h \), for all closed terms \( t \in \text{TRM}_{\text{UK}} \), \( \models^h \psi_{\alpha} \)

**Lem 2** for any \( h \), for all closed terms \( t \in \text{TRM}_{\text{UK}} \), \( \models^h \psi_{\alpha} \)

\( \implies \) for any \( h \), for all \( a \in \mathcal{D} \), \( \models^h \psi_{\alpha} \)

\( \implies \) \( \varphi \) is true in \( \mathbb{K} \).

**Subcase 2:** \( \alpha \) is an \( n \)-place predicate variable. In the above proof, simply change \( \text{TRM}_{\text{UK}} \) to \( \{ p \mid p \text{ is a closed } n \text{-place predicate term of } \text{UK} \} \), \( K \) to \( K_n \), and \( \mathcal{D} \) to \( D_n \).
For the if direction, for some \( \xi \subset \mu \) we have \( \pi_{\xi} = \langle \psi, \alpha \rangle \). Thus,

\[
\psi \psi_{\Delta} \implies \neg \forall \psi \in \Delta \implies \neg \psi_{\Delta}^{\alpha} \quad \text{(since } \sigma_{\Delta}, \text{ where } \kappa \text{ is the relevant constant in } K) \implies \psi_{\Delta}^{\alpha} \text{ ind } \frac{\text{hyp}}{\text{hyp}} \psi_{\Delta}^{\alpha} \text{ is not true in } \mathfrak{M} \implies \psi \text{ is not true in } \mathfrak{M}.
\]

So \( \mathfrak{M} \) is a model of \( \Delta \). Hence, \( \mathfrak{M} \) is a model of any subset of \( \Delta \), and so in particular of \( \Gamma \), though in the language \( \mathcal{L} \cup K \). Let \( \mathfrak{M}^* \) be the reduct of \( \mathfrak{M} \) to \( \mathcal{L} \); then \( \mathfrak{M}^* \) is an interpretation of \( \mathcal{L} \) which is a model of \( \Gamma \). Thus, since \( \mathcal{L} \) and \( \Gamma \) were chosen arbitrarily, we have

**Corollary 1:** Every consistent set of sentences of a language has a model.

Completeness now follows straightaway.

**Proof of Theorem 3:** Since \( \Gamma \vdash \psi \) (\( \Gamma \vdash \varphi \)) iff \( \Gamma \vdash \varphi^* \) (\( \Gamma \vdash \varphi^* \)), where \( \varphi^* \) is the universal closure of \( \varphi \), we need consider only sentences. So suppose \( \varphi \) is a sentence of \( \mathcal{L} \) and that \( \Gamma \vdash \varphi \) but not \( \Gamma \vdash \varphi \). By Lemma 12, \( \Gamma \cup \{ \neg \varphi \} \) is consistent. But then by Corollary 1, \( \Gamma \cup \{ \neg \varphi \} \) has a model \( \mathfrak{M} \). But then \( \mathfrak{M} \) is a model of \( \Gamma \) in which \( \varphi \) is false, i.e., \( \neg \varphi \), contradiction.

As in standard first-order model theory, and by the same arguments, compactness and the downward Löwenheim-Skolem theorem follow directly from completeness.\(^{29}\) The upward Löwenheim-Skolem theorem follows from these two theorems, again by the familiar first-order arguments (cf., e.g., Chang and Keisler [1977], §2.1).

§2.5 **CONSISTENCY OF LPRP**

We will now construct an interpretation for an arbitrary language in order to establish the consistency of our logic. So let \( \mathcal{L} \) be an arbitrary language, and let \( \mathcal{L}^* \) be \( \mathcal{L} \cup \{ P \} \), where \( P \) is a 2-place predicate not in \( \mathcal{L} \). We will construct our interpretation so that the formula \( P \gamma_1 \gamma_2 \equiv \gamma_1 = \gamma_2 \)
of \( L^* \) will be true. There are two reasons for introducing \( P \). The first is to verify the claim that our restriction on occurrences of \( = \) in complex terms has no bearing on the question of consistency, since introducing a predicate coextensive with \( = \) is, in effect, to lift the restriction. The second is that it allows the construction below to go through for languages that contain no predicate constants.

The interpretation we construct will be based on a simple word algebra. Let \( A^0_{-1} = \{ t \mid t \text{ is an individual constant of } L^* \} \), and for \( n \geq 0 \), let \( A^0_n = \{ t \mid t \text{ is an } n\text{-place predicate constant of } L^* \} \). Let \( A^0 = \cup_{n \geq 1} A^0_n \).

Consider the following sets of functional expressions:

\[
\{ \varepsilon_i \}_{i < \omega}, \{ c^i_j \}_{i, j \in \omega}, \{ g^i_j \}_{i < j \in \omega}, \{ \lambda i \}_{i \in \omega}, \{ \eta^m_i \}_{i, m \in \omega}, \{ \sigma^i_j \}_{i, j \in \omega}, \{ \xi^i_j \}_{i, j \in \omega}. \]

Let \( \text{FE} \) be union of all these sets of functional expressions, and let \( \Sigma \) be the set of all finite strings of elements of \( A^0 \cup \text{FE} \). Suppose we've defined \( A^k \); we define \( A^{k+1}_n \) as follows. We let \( A^{k+1}_{-1} = A^k_{-1} \). For \( n \geq 0 \), and for any \( \sigma \in \Sigma, \sigma \in A^{k+1}_n \) iff:

(i) \( n \geq 0 \) and \( \exists i \leq n \exists \tau \in A^k_{n-1} \) s.t. \( \sigma = \varepsilon_i \tau \), or
(ii) \( \exists i \exists j, i < j \leq n, \exists \tau \in A^k_n \) s.t. \( \sigma = c^i_j \tau \), or
(iii) \( \exists i \exists j, i < j \leq n+1, \exists \tau \in A^k_{n+1} \) s.t. \( \sigma = g^i_j \tau \), or
(iv) \( \exists \tau \in A^k_n \) s.t. \( \sigma = x \tau \), or
(v) \( \exists m, m' > 0 \), s.t. \( m + m' = n \), and \( \exists \tau \in A^k_m \exists \tau \in A^k_{m'} \) s.t. \( \sigma = g \tau \), or
(vi) \( \exists i \leq n+1, \exists \tau \in A^k_i \) s.t. \( \sigma = \eta^m_i \tau \), or
(vii) \( \exists m \exists i \leq n+1, \exists j \exists \tau \in A^k_{n+1} \exists \tau \in A^k_i \exists \tau \in A^k_i \) s.t. \( \sigma = g^i_j \tau \), or
(viii) \( \exists \sigma \in A^k, A^k \) s.t. \( \sigma = x \tau \), or
(ix) \( \exists j \exists \tau \in A^k \exists \tau \in A^k_i \exists \tau \in A^k_i \) s.t. \( \sigma = g^i_j \tau \), or
(x) \( \exists \sigma \in A^k \) s.t. \( \sigma = \xi^i_j \tau \).

Now let \( A^{k+1} = A^k \cup \cup_{n \geq 1} A^{k+1}_n \), and let \( A = \cup_{k \geq 0} A^k \). Let \( \mathcal{F} \) be the obvious partition \( \{ A_{-1}, A_0, A_1, \ldots \} \), where for each \( n \geq 1 \), \( A_n = \cup_{k \geq 0} A^k_n \).

Note that \( A_{-1} \) is possibly empty. The logical functions \( \sigma \in \mathcal{F} \) on our domain \( A \) should be equally apparent, viz., e.g., \( \text{EXP}_i = \{ \langle \sigma, \sigma' \rangle \mid \sigma' = \varepsilon_i \sigma \} \), \( \text{CONJ} = \{ \langle \sigma, \sigma', \sigma'' \rangle \mid \sigma'' = \varepsilon_i \sigma \} \), \( \text{PLUG}^k_{i,j} = \{ \langle \sigma, \sigma', \ldots, \sigma_j, \sigma' \rangle \mid \sigma = \xi^i_j \sigma \sigma \ldots \sigma \} \), and so on. Note that the functions are all one-to-one, and that the ranges
of CONJ, NEG, the UNIV, and the UNIV are pairwise disjoint. The definition of our extension function will simply mirror the conditions laid down in the description of an interpretation. As indicated, we will let $\mathcal{E}(P) = \{(a,a) \mid a \in A\}$. For the remaining elements of $U_n \in A^n$, let $\mathcal{E}(Q^0) = T$, for all 0-place predicate constants $Q^0$ of $L$, and let $\mathcal{E}(Q^n) = n_A$, for all $n$-place predicate constants of $L$. Suppose then we've defined extensions for the elements of $A^k$, and let $a \in A^{k+1}$. If $a = \delta_1 a'$, then

$$\delta_1(a) = \{b \mid \delta_1(a') = T\};$$

if $a' \in A^k_0$, 

$$\delta_1(a) = \{(a_1, \ldots, a_{i-1}, b, a_i, \ldots, a_n) \mid (a_1, \ldots, a_n) \in \delta_1(a')\}.$$ 

And so on for the remaining cases. It is a simple matter to show that if $a \in A^0_0$, then $\delta_1(a) \in \{T, F\}$, and that if $a \in A^0_n$, $n > 0$, $\delta_1(a) \in n_A$. Finally, let our interpretation function $\delta$ be simply $\{(t, t) \mid t$ is a constant of $L^*\}$.

**Theorem 4:** LPRP is consistent.

**Proof:** It is obvious that the structure $\mathcal{X} = \langle A, \mathcal{F}, \mathcal{E}, \mathcal{G}, \delta \rangle$ is an interpretation for $L^*$, and hence that $\mathcal{X} \in L$ is an interpretation for $L$. Thus, by the soundness theorem, all the axioms of LPRP are true in $\mathcal{X}$. So our logic is consistent. Note also that the $L^*$ formula $Pv_1v_2 \equiv v_1 = v_2$ is true in $\mathcal{X}$, so the consistency of LPRP is independent of our restriction on $=.$
1. And thus has a decidedly first-order cast—hence the scare quotes around ‘SECOND-ORDER’ in the title; for more on this, cf. note 29.

2. At least, not within a traditional ontology. For a typed theory which can nonetheless represent these arguments in the context of a Meinongian ontology, see Zalta [1983].

3. A supposition that forced him into his celebrated denial that the concept (or property) horse is not a concept; cf. Frege [1977], pp. 45–6.

4. This may evoke a sense of unease even among those sympathetic to the type-free conception. For isn’t it clear that there are at least some cases where we have to restrict the range of significance of a property or relation? There are two sorts of worries. One is rooted in fear of Russell’s paradox. As I’ve already mentioned, I will show below that this is ungrounded. The second sort of worry is rooted in the idea that certain attributions are meaningless. Is it not senseless, for example, to ask whether the color green has a weight, or, to use Carnap’s well-worn example, whether this stone is thinking of Vienna? Without taking things too far here, let me just note that it seems to me that this worry is based on a confusion of meaninglessness with necessary falsehood. True enough, green is not the sort of thing which could possibly have a weight, nor is a stone the sort of thing which could even think of a shady patch of earth, let alone Vienna. It is in a certain sense absurd to suppose that there could be colors with weight or stones that think, since such things are contrary to the natures of the objects in question; but neither supposition, no matter how silly, is meaningless; indeed, we have to grant their meaningfulness in order to dismiss them as impossible.
5. Cocchiarella [1972] makes essentially this point in connection with the comprehension principle of his early system $T^*$; cf. pp. 165-168. Although my views on logic and ontology diverge significantly from his, I have been strongly influenced by Cocchiarella’s work in philosophical logic. I wish to thank him for making recent work of his that has not yet appeared available to me.

6. I don’t claim that the following functions are the only logical functions, just the only ones relevant to the current project. The question: In virtue of what are these functions logical? is deep and difficult, and will not be addressed here. The issues are directly related to those surrounding the notion of a logical constant, for which cf. Peacocke [1976], McCarthy [1981], Kuhn [1981], and Westerstahl [1985].

7. In the formal theory itself I will omit DISJ and the various existentialization functions, since they are formally redundant, and hence only complicate matters needlessly. I do, however, think they are distinct functions—the property of being red or square doesn’t seem to me be the same property as being non-(non-red and non-square), even though they are (necessarily) equivalent. One might, for example, not have the concept of negation, in which case it seems one could grasp the first property, but not the second.

8. We can picture the difference between plugging\(^0\) and plugging\(^1\) functions in the following way. (I was introduced to this “notation” by Ed Zalta, who in turn attributes it to Edmund Gettier.) Let

\[
\begin{array}{ccccccc}
  & r & \quad & r' & \quad & x & \quad & y & \quad & z & \quad & w & \quad & u \\
\end{array}
\]
represent the 2-place relation \( r \) of identity, and the 3-place relation \( r^* \) that obtains between \( a, b, \) and \( c \) just in case \( a \) loves both \( b \) and \( c \). The variables beneath the gaps indicate the operative gaps in the relations. Now, \( \text{PLUG}^0_{1,1}(r,r^*) \) can be represented like this:

\[
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{r'} \\
\text{y}
\end{array}
\end{array}
\]

i.e., the property of being identical with \( r^* \). The removal of \('x'\) indicates that the gap in \( r \) has been plugged, and the removal of \('z', 'w', \) and \('u'\) indicate that these gaps in \( r^* \) are not gaps in the new PRP; \( r^* \) is plugged into the gap unchanged. \( \text{PLUG}^1_{1,1}(r,r^*) \), however, is represented like this:

\[
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{r'} \\
\text{z} \\
\text{y}
\end{array}
\end{array}
\]

where the presence of \('z'\) indicates that the first argument place of \( r^* \) has been appropriated as an argument place in the new PRP, viz., the 2-place relation that obtains between \( a \) and \( b \) just in case the relation that obtains between any \( w \) and \( u \) just in case \( a \) loves both \( u \) and \( w \) is identical with \( b \). Similarly, \( \text{PLUG}^1_{1,2}(r,r^*) \) and \( \text{PLUG}^1_{1,3}(r,r^*) \) are represented like this:

\[
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{r'} \\
\text{z} \\
\text{w} \\
\text{y}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{r} \\
\text{r'} \\
\text{z} \\
\text{w} \\
\text{u} \\
\text{y}
\end{array}
\end{array}
\]
I recommend as an exercise the construction of an appropriate representation of the aforementioned property of desiring that one be wise.)

9. I mean here an assignment in the intuitive sense, as in a case where we are actually using the language and so assigning real, specific propositions to the 0-place predicates of our language, properties to the 1-place predicate terms, 2-place relations to the 2-place predicate terms, etc.

10. Though not necessarily to all of the PRPs in the closure. Suppose for example that < is the less than relation among the numbers. Our semantics doesn’t decide whether \( \text{CONV}_2^1(\text{CONV}_2^1(<)) = < \). If not, then even if < is the interpretation of some predicate term p, given the way we partition the complex terms there will not be a term that denotes the PRP above. Cf. §2.2 below.

11. Recall that I’ve classified ‘=’ as a logical particle, not a distinguished predicate, so identities must be treated separately.

12. Indeed, the logic and the metatheoretical results in Part II all remain essentially unchanged if we do not impose this restriction and instead simply treat ‘=’ as a (distinguished) predicate. In fact, the metatheory is somewhat simpler since identities are then just classified as atomic formulas.

13. Roughly these three axiom groups (modulo the syntactic differences in our systems) are part of Bealer’s logic for the second of his two conceptions of PRPs, though I have modified axiom schema 9 somewhat—Bealer wants the ranges of all of his logical functions to be pairwise disjoint (Bealer [1982], p. 53). It seems to me this requirement is too strong.
Consider, for example, the proposition \( [\lambda \rightarrow Ws] \) that Sam is not wise. As I see it, it is quite plausible to think that \( [\lambda \rightarrow Ws] = \text{PLUG}_{1,1}^0(\text{NEG}(W),s) = \text{NEG}(\text{PLUG}_{1,1}^0(W,s)) \). But if so, then the functions \( \text{NEG} \) and \( \text{PLUG}_{1,1}^0 \) both have \( [\lambda \rightarrow Ws] \) in their ranges. Pairwise disjointness of ranges only seems to be an appropriate condition to impose on the functions above. Bealer also adds a further axiom schema (A12, p. 65) which is intended to capture the idea that every PRP has a noncircular logical decomposition. Though I'm inclined to think it is true, I do not find it compelling, and hence do not include it among my axioms. Note also that the presence of predicate quantifiers in our logic enables us to dispense with Bealer's additional inference rule \#3 (p. 65). These differences aside, Bealer's influence on my thought should be evident.

14. Indeed, Bealer [1982] postulates the existence of such a PRP explicitly; cf. ch. 4.

15. Or else (much more dubiously, I think) it must have a very different logical behavior than what we intuitively think it should have; cf. again Bealer [1982], 94ff.

16. I don't mean to imply that one can't use a formal system without committing oneself ontologically to its intended framework; one might well make extensive use of a system, and indeed find its use essential and ineliminable, but have quite another story to tell about its ultimate ontological upshot. Hilbert's view of transfinite arithmetic, and of systems with "ideal objects" in their intended interpretations generally, I take it, would be an example of this (Hilbert [1925]; cf. also Hodes [1984]). However, I do want to claim that insofar as one doesn't take the intended interpretation of a system seriously, one needs, like Hilbert, to have some other tale to tell.
17. Indeed, since the only objects whose existence and uniqueness the logic can prove (given some interpretation) are those denoted by the closed terms of the language, any defined term t would simply be such that for some term t' already in the language \( t = t' \) is provable.

18. Both of these systems appeal to stratification to impose restrictions on the formation of complex terms. Since stratification ultimately derives from type theory (cf. Quine [1938]), I am dubious about its use within an untyped framework.

19. Bealer’s system with \( \Delta \) might thus be more appropriately thought of as a metaphysical theory of the predication relation, and Cocchiarella’s as a theory of a broad class of PRPs genera ble a priori from, in effect, a certain comprehension principle. Also, insofar as the reconstructions of logicism in Bealer [1982] (ch. 6) and Cocchiarella [1985b] depend on the strong ontological commitments of the respective systems, it is questionable whether either succeeds in genuinely reducing arithmetic to logic, as opposed to simply smuggling in a sufficiently rich ontology under the guise of logic. Note that I don’t question the value of, and ingenuity behind, these reconstructions, only their logical character.

One is of course perfectly free to use the system presented here to postulate any of the entities that follow from these stronger systems. If one wants to postulate an identity relation, for example, one need only add a new 2-place predicate \( P \) and the (nonlogical) axiom \( Pxy \equiv x = y \); similarly, if one wants the property of having a property, one can add the axiom \( Qx \equiv \exists F (Fx) \). One can also define the numbers Frege-style by postulating a few new relations as follows. First, we add predicates for the relations of identity (‘=’ itself will do, since the ambiguity
is innocuous) and equinumerosity '≈'. The latter will be governed by
the axiom

$$\exists y \equiv \exists f, g (x = f \land y = g \land \exists z (F \supset \exists u (G \cup H) \cup \forall u (G \cup H) \supset \exists z (F \supset H)).$$

Given this we let 0 = _df_ \[\lambda x \to [\lambda y \neq y]]\], 1 = _df_ \[\lambda x \to [\lambda y \neq 0]]\], 2 = _df_ \[\lambda x \to [\lambda y \neq 0 \lor y = 1]]\], and so on. These are essentially Frege's [1884]
definitions, except that, since concepts (i.e., properties—cf. Frege
[1977], pp. 51ff) are also objects in our framework, there is no need to
appeal to their extensions here.

The definition is elegant: For each numeral \(N\), 'The number of the
concept F is N' amounts simply to 'N(F)', and it is a theorem of our logic
with the above axiom that two concepts have the same number only if they
are equinumerous (i.e., for each \(N\), '1-\(N(F) \land N(G) \supset F \equiv G)\'). Full Peano
arithmetic, of course, requires somewhat more machinery. The present
point is that to use this system for philosophical purposes one must wear
one's specific ontological commitments on one's sleeve.

20. For this issue, cf., e.g., Adams [1981], Fine [1977], Plantinga
[1983], and Pollock [1985].

21. Since we will not be using terms in Part II there is no longer any
danger of ambiguity, so henceforth quotation and quasi-quotation will be
suppressed.

22. Thus, an elliptical string of metavariables like \(x_1 \ldots x_n\) or \(t_1 \ldots t_n\)
representing a sequence of object language terms of arbitrary length n
signifies the empty sequence when n=0.

23. I define the complexity of a term to be one less than the height of
its decomposition tree so that the complexity of simple terms will be 0;
formally unnecessary, of course, but aesthetically more pleasing.
24. I could give this definition a more uniform appearance by using the standard Tarskian definition for the atomic clause. But I prefer to leave it as is, since it underscores what I argued at some length in Part I: that logic doesn’t guarantee an entity corresponding to every formula of the language. Atomic formulas are always acceptable, for the reasons noted above, hence we can employ the simple condition in clause 1 and take advantage of the algebraic semantics directly. But we must resort back to the Tarskian definition in clauses 2 and 3 for nonatomic formulas in general since some are not acceptable for the construction of complex terms.

25. More elaborately, call terms of complexity 1, i.e., terms of the form \([\lambda x_1 \ldots x_m p x_1 \ldots x_n]\) where \(p\) is a simple term of \(L\), simple elementary complex (sec) terms. Then convertible terms are non-sec terms such that the terms from which they are immediately built up are sec terms (or, in the case of \(i^{th}\)-plugging \(_j^0\) terms, either sec terms or simple terms).

26. We need this definition because some instances of the standard \(\lambda\)-conversion schema (schema 6 minus the restriction to convertible terms) are not true in all interpretations. Consider the term \([\lambda x P[\lambda \neg Qx]]\). By standard \(\lambda\)-conversion, \(\vdash [\lambda x P[\lambda \neg Qx]]c \equiv P[\lambda \neg Qc]\). In order for the left side of the biconditional to be true in an interpretation \(\mathfrak{I}\), it must be the case by definition that \(\text{PLUG}^0_{1,1}(\neg d^\mathfrak{I}_h(Q), d^\mathfrak{I}_h(c)) \in \text{EXT}(d^\mathfrak{I}_h(P))\) (for any \(h\)). Conversely, in order for the right side to be true, it must be the case that \(\neg \text{PLUG}^0_{1,1}(d^\mathfrak{I}_h(Q), d^\mathfrak{I}_h(c)) \in \text{EXT}(d^\mathfrak{I}_h(P))\). But if the ranges of \(\text{PLUG}^0_{1,1}\) and \(\neg\) in \(\mathfrak{I}\) are disjoint (which is perfectly permissible), these will be distinct objects. Hence, there is no reason in general why \(d^\mathfrak{I}_h(P)\) shouldn’t have just one of the two in its extension. But if this were the case, the equivalence above would be false in \(\mathfrak{I}\). The definition here
takes this into account by taking the formula $P[\lambda \sim Qx]$ into the formula $P[\lambda x \sim Qx]$. As Theorem 1 states, our axioms then yield the valid equivalence $[\lambda x P[\lambda \sim Qx]]c \equiv P[\lambda x \sim Qx]c$.

27. We assume that if $\xi \neq \zeta$, then $d_\xi \neq d_\zeta$; similarly for the new predicate constants.

28. Note that we have to add this qualification since it is possible that there be formulas of the form $d=t_\Delta$, where $d \in K\alpha$ and $d \in K_-1$ (in which case $d$ would be alphabetically earlier than any predicate term in $K$), whereas our construction will require that $K(a)$ always be a predicate term for $a \in D_n$, $n \geq 0$.

29. In logics based on the standard conception of models as extensional relation structures, by a famous theorem of Lindström [1969], these theorems express the distinguishing properties of first-order logic. Since the conception of model here differs significantly from the standard conception, there is no obvious way of classifying our logic as an extension of standard first-order logic, and hence no obvious way applying Lindström's result directly. However, there is probably an analogous result available relative to a notion of first-order logic that comports with the conception of a model here. This is to be expected; for there is a straightforward (though awkward) way of translating formulas of any of our languages $L$ into formulas of a many-sorted first-order language $L_1$ with countably many distinguished $n$-place exemplification predicates, one for each $n \geq 2$, and a corresponding way of transforming the present notion of an interpretation into one germane to $L_1$. As I've argued in Part I, though, there is a substantial philosophical point to the particular language and syntax we have adopted: the function/argument notation for atomic formulas reflects
the unsaturated nature of properties and relations; that predicates can occur both in predicate and subject position in atomic formulas reflects the type-free conception of PRPs; and the syntax of complex terms reflects the operations of the logical functions. So even though there may indeed be a well-defined abstract model theoretic sense in which our logic is first-order, a first-order syntax would obscure the essential features of the ontological framework that motivated the logic in the first place.

Let me note also that the apparent first-order character of our logic seems to me to be a consequence of the untyped conception of PRPs. A second-order theory is typically understood to be a theory whose \( n \)-place predicate quantifiers range over the entire power set of the \( n \)-th cartesian product of the domain of its intended model. Hence, intuitively, and in particular, on this view there are as many properties as there are elements of the power set of the domain. But on the type-free conception, properties are individuals and hence must be represented formally as composing only a subset of the domain \( \mathcal{D} \), which represents the collection of all individuals (for which of course there is no corresponding "power collection" anyway). Hence, the class of their extensions will include only a subset \( s \) of \( \text{Pow}(\mathcal{D}) \) whose cardinality cannot exceed the cardinality of \( \mathcal{D} \). So in our untyped semantics, there cannot (and fortunately, I think, ought not) be a property corresponding to every subset of \( \mathcal{D} \). For a related discussion, see Maddy [1983].

30. See, e.g., Goguen and Burstall [1982]. In their terminology, the set \( S \) of sorts in the algebra's signature will just be \( \mathbb{N}\cup\{-1\} \), and the operators the members of \( \text{FE} \). The carrier of the algebra will be \( \{U_{\geq i}^n A_n \}_{i \in S} \) (with each \( U_{\geq i}^n A_n \) indexed by \( i \)), and the operations just the logical functions defined below.
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