Symmetries in turbulent boundary layer flows

By M. Oberlack

1. Motivation and objectives

The motivation for the present analysis was the finding in Oberlack (1995) that the logarithmic mean profile is a self-similar solution of the two-point correlation equation. The latter can be achieved by introducing the similarity variable $\tilde{r}_i = \frac{r_i}{x_2}$ in the correlation equation. As a result the coordinate x_2 disappears in the two-point correlation equation which finally only depends on $\tilde{r}_i$. This simple scaling may appear trivial. However, it is worth noticing that in the two-point correlation equation non-local terms like $(\bar{u}_k(\boldsymbol{x}+\boldsymbol{r}) - \bar{u}_k(\boldsymbol{x}))\frac{\partial R_{ij}}{\partial r_k}$ appear which makes guessing of other similarity solutions a non-trivial task.

The objective is the development of a new theory which enables the algorithmic computation of *all* self-similar mean velocity profiles. The theory is based on Lie-group analysis and unifies a large set of self-similar solutions for the mean velocity of stationary parallel turbulent shear flows. The results include the logarithmic law of the wall, an algebraic law, the viscous sublayer, the linear region in the middle of a Couette flow and in the middle of a rotating channel flow, and a new exponential mean velocity profile not previously reported. Experimental results taken in the outer parts of a high Reynolds number flat-plate boundary layer, strongly support the exponential profile. From experimental as well as from DNS data of a turbulent channel flow the algebraic scaling law could be confirmed in both the center region and in the near wall region. In the case of the logarithmic law of the wall, the scaling with the wall distance arises as a result of the analysis and has not been assumed in the derivation. The crucial part of the derivation of all the different mean velocity profiles is to consider the invariance of the equation for the velocity fluctuations at the same time as the invariance of the equation for the velocity product equations. The latter is the dyad product of the velocity fluctuations with the equation for the velocity fluctuations. It has been proven that all the invariant solutions are also consistent with similarity of all velocity moment equations up to any arbitrary order.

2. Governing equations

The bases for the following analysis are the incompressible Navier-Stokes equations in a rotating frame. Using the standard Reynolds decomposition, $U_i = \bar{u}_i + u_i$ and $P = \bar{p} + p$, where the overbar denotes a time or ensemble average, the Reynolds averaged Navier-Stokes equations for a parallel flow are

$$K + \nu \frac{\partial^2 \bar{u}_1}{\partial x_2^2} - \frac{\partial \overline{u_1 u_2}}{\partial x_2} = 0 \tag{1}$$

$$-\frac{\partial \bar{p}}{\partial x_2} - \frac{\partial \overline{u_2 u_2}}{\partial x_2} - 2\,\Omega\,\bar{u}_1 = 0 \tag{2}$$

$$\frac{\partial \overline{u_3 u_2}}{\partial x_2} = 0 \tag{3}$$

and the fluctuation equations are

$$\frac{\partial u_i}{\partial t} + \bar{u}_1 \frac{\partial u_i}{\partial x_1} + \delta_{i1} u_2 \frac{d\bar{u}_1}{dx_2} - \frac{d\overline{u_i u_2}}{dx_2} + \frac{\partial u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k^2} + 2\,\Omega\, e_{i3l}\, u_l = 0. \tag{4}$$

The corresponding continuity equation for u_i is

$$\mathcal{C} = \frac{\partial u_k}{\partial x_k} = 0 \tag{5}$$

In (1)-(4) and subsequently the density has been absorbed with the pressure. In the case of a pressure driven flow in the x_1 direction the mean pressure $\bar{p}$ has been replaced by $-x_1 K + \bar{p}(x_2)$, where K is a constant. The only axis of rotation is normal to the mean shear in x_3-direction, and hence we take $\Omega = \Omega_3$.

Equations (1)-(3) can be rewritten and unified with the equation for the fluctuation (4) by solving (1) and (2) for the gradient of the Reynolds stresses and using the result in (4),

$$\begin{aligned}\mathcal{N}_i(\boldsymbol{x}) = \frac{\partial u_i}{\partial t} + \bar{u}_1 \frac{\partial u_i}{\partial x_1} + \delta_{i1} u_2 \frac{d\bar{u}_1}{dx_2} - \delta_{i1}\left(K + \nu \frac{\partial^2 \bar{u}_1}{\partial x_2^2}\right) \\ + \delta_{i2}\left(\frac{\partial \bar{p}}{\partial x_2} + 2\,\Omega\,\bar{u}_1\right) + \frac{\partial u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k^2} + 2\,\Omega\, e_{i3l}\, u_l = 0.\end{aligned} \tag{6}$$

The present analysis is restricted to stationary parallel shear flows

$$\frac{\partial \bar{u}_1}{\partial x_1} = \frac{\partial \bar{u}_1}{\partial x_3} = \frac{\partial \bar{u}_1}{\partial t} = \frac{\partial \bar{p}}{\partial x_1} = \frac{\partial \bar{p}}{\partial x_3} = \frac{\partial \bar{p}}{\partial t} = 0, \tag{7}$$

and hence $\bar{u}_1$ and $\bar{p}$ are only functions of the remaining spatial coordinate x_2.

From a wide variety of different experiments it is well known that high Reynolds number turbulent flows are Reynolds number invariant. Cantwell (1981) has already investigated this from a group theoretical point of view. Using this, we impose an additional restriction on the viscosity dependence in the mean quantities. In the limit of large Reynolds numbers, the leading order $\bar{u}_1$ and $\bar{p}$ are assumed to be independent of viscosity and hence

$$\frac{\partial \bar{u}_1}{\partial \nu} = \frac{\partial \bar{p}}{\partial \nu} = 0. \tag{8}$$

The latter assumption does not restrict the number or the functional form of the self-similar solutions to be computed later. It only limits the appearing constants in the self-similar solutions to be independent of viscosity. An explicit Reynolds number dependence in the scaling laws will be investigated in a future approach since the functional dependence can not be captured with the present analysis.

The system of Eqs. (6) describes the fluctuation and mean of an arbitrary parallel turbulent shear flow. The set of equations is underdetermined. In the classical approach of finding turbulent scaling laws the latter difficulty has motivated the introduction of second moment equations. However, in the next section the above set of equations will be analyzed with regard to its symmetry properties, without any further introduction of higher correlation equations which contain more unclosed terms.

In order to do that, an equation is introduced, which can be directly derived from Eq. (6) without introducing further unclosed terms. It is the velocity product equation, which is the dyad product of $\mathcal{N}_i$ and u_j

$$\mathcal{N}_i u_j + \mathcal{N}_j u_i = 0. \tag{9}$$

The set of Eqs. (5)-(9) to be analyzed result to three mayor differences between the present and the classical similarity approach using the Reynolds stress transport equations. Firstly, in the present approach only the Reynolds stresses appear as unclosed terms in the equations and no higher order correlations as the pressure-strain correlation, the dissipation or the triple correlation need to be considered. Hence, in the present approach only a finite number of variables are present in the system to be analyzed.

Secondly, it is easy to see that any scaling law valid for the mean and the fluctuation velocities obtained from the velocity product Eqs. (9) still holds for the Reynolds stress equations. This fact is crucial for the present approach to obtain scaling laws which are consistent with averaged quantities. The averaging procedure applied to the velocity product equations does not affect the scaling properties of the equation.

Thirdly, is has been proven by Oberlack (1996a) that any scaling law for the velocity fluctuation and the second order velocity product Eqs. (9) is also a scaling law for all nth order velocity product equations. The nth order velocity product equations are defined as the nth order dyadic product of the velocity fluctuations with the equation for the velocity fluctuations. Since the averaging procedure does not change the scaling properties of the nth order velocity product equations, it is also consistent with all nth order correlation equations. In the classical approach using correlation functions, it may not be possible to show that all higher order velocity correlations are consistent with the scaling in the Reynolds stress equations. The Reynolds stress equations is the first in a row of an infinite number of correlation equations which need to be considered in principle in the classical approach.

3. Lie point symmetries in turbulent parallel shear flows

A set of differential equations is said to admit a symmetry if a transformation to a new set of variables exists which leaves the equations unchanged. If the symmetries are computed all self-similar solutions can be obtained as will be pointed out below.

The set of variables considered in the subsequent transformation consist of

$$\boldsymbol{y} = \{x_1,\ x_2,\ x_3,\ t,\ \nu,\ u_1,\ u_2,\ u_3,\ p,\ \bar{u}_1,\ \bar{p}\}. \tag{10}$$

The purpose of the symmetry analysis is to find all those transformations

$$\tilde{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y};\varepsilon) \tag{11}$$

which, under consideration of (7) and (8), satisfy

$$\mathcal{C}(\boldsymbol{y}) = \mathcal{C}(\tilde{\boldsymbol{y}}),\ \ \mathcal{N}_i(\boldsymbol{y}) = \mathcal{N}_i(\tilde{\boldsymbol{y}}) \tag{12}$$

and the extended system also including (9)

$$(\mathcal{N}_i u_j + \mathcal{N}_j u_i)(\boldsymbol{y}) = (\mathcal{N}_i u_j + \mathcal{N}_j u_i)(\tilde{\boldsymbol{y}}). \tag{13}$$

Lie gave an infinitesimal form of the transformation (11)

$$\tilde{\boldsymbol{y}} = \boldsymbol{y} + \varepsilon\boldsymbol{\zeta} + \mathcal{O}(\varepsilon^2) \text{ with } \boldsymbol{\zeta} = \left.\frac{\partial \boldsymbol{f}}{\partial\varepsilon}\right|_{\varepsilon=0} \tag{14}$$

where, instead of $\boldsymbol{f}$, all the infinitesimal generators $\boldsymbol{\zeta}$ need to be calculated, each element depending on $\boldsymbol{y}$.

It can be proven that the infinitesimal transformation (14) is *fully equivalent* to the global transformation (11) (see Bluman (1989)). The direct approach finding $\boldsymbol{f}$ only from (12) and (13) using the global transformation (11) would have been almost impossible.

The calculation of $\boldsymbol{\zeta}$ is fully algorithmic and results in more than a hundred linear overdetermined PDE's for $\boldsymbol{\zeta}$. Its derivation has been aided by SYMMGRP.MAX, a software package for MACSYMA (1993) written by Champagne (1991). The solution has been calculated by hand. The complete set of solutions is given in Oberlack (1996a).

For the present approach, the global transformation (12) is not needed since only the self-similar solutions for the mean flow will be investigated. The equation for the self-similar solutions is the invariant surface condition (ISC). In the present case of parallel flow, the ISC for the mean flow is given by

$$\frac{dx_2}{\xi_{x_2}} = \frac{d\bar{u}_1}{\eta_{\bar{u}_1}} \tag{15}$$

where

$$\xi_{x_2} = a_1 x_2 + a_3 \text{ and } \eta_{\bar{u}_1} = [a_1 - a_4]\bar{u}_1 + a_2 \tag{16}$$

are the infinitesimal generators.

Four different solutions for different combinations of parameter $a_1 - a_4$ have to be distinguished. Each case has a specific meaning for the corresponding turbulent flow in terms of an external time, length, or velocity scale which may break some of the scaling symmetries as has been point out by Jiménez (1996).

A non-zero angular rotation rate will be considered only in the subsection (3.2). In this case the set of transformations to be obtained later contain a reduced number of parameters. The rotation rate will be considered as a branching parameter for the two different cases of $\Omega = 0$ and $\Omega \neq 0$.

3.1 Turbulent shear flows with zero system rotation

Algebraic mean velocity profile: ($a_1 \neq a_4 \neq 0$ and $a_2 \neq 0$)

The present case is the most general of all. No scaling symmetry is broken. As a result the mean velocity $\bar{u}_1$ has the following form

$$\bar{u}_1 = C_1 \left(x_2 + \frac{a_3}{a_1} \right)^{1-\frac{a_4}{a_1}} - \frac{\dfrac{a_2}{a_1}}{1 - \dfrac{a_4}{a_1}}. \tag{17}$$

In the domain where the algebraic mean velocity profile is valid there can be no external length and velocity scale acting directly on the flow since non-zero and unequal parameters a_1 and a_4 are needed for its derivation. It will be pointed out in section (4) that the case of an algebraic scaling law applies both in the vicinity of the wall and in the center region of a channel flow.

Barenblatt (1993) developed an algebraic scaling law based on the idea of incomplete similarity with respect to the local Reynolds number. The proposed scaling law involves a special Reynolds number dependence of the power exponent and the multiplicative factor. It emerges that the familiar logarithmic law is closely related to the envelope of a family of power-type curves. George (1993) proposed an asymptotic invariance principle for zero pressure-gradient turbulent boundary layer flows. They found that the profiles in an overlap region between the inner and outer regions are power laws. Using the limit of infinite Reynolds number, the usual logarithmic law of the wall is recovered in the inner region.

Logarithmic mean velocity profile: ($a_1 = a_4 \neq 0$ and $a_2 \neq 0$)

For the present combination of parameters we can see in the infinitesimals (16) that no scaling symmetry with respect to the velocity $\bar{u}_1$ exists and hence an external velocity scale is symmetry breaking. The mean velocity $\bar{u}_1$ can be integrated to

$$\bar{u}_1 = \frac{a_2}{a_1} \log \left(x_2 + \frac{a_3}{a_1} \right) + C_2. \tag{18}$$

In case of the classical logarithmic law of the wall it is the friction velocity u_τ, which breaks the scaling symmetry for the velocities. The present case coincides with the usual derivation of the logarithmic law of the wall as first given by von Kármán (1930) where the friction velocity u_τ is the only velocity scale in the near wall region. So far a logarithmic scaling law has only been found in the vicinity of the wall. The wall breaks the translational symmetry with respect to x_2 and hence a_3 has to be zero.

Exponential mean velocity profile: ($a_1 = 0$ and $a_4 \neq a_2 \neq 0$)

Since a_1 is zero in the present case there is an external length scale which breaks the symmetry in (16) with respect to the spatial coordinates. As a result the spatial coordinate is an invariants with only a constant added to the infinitesimal in (16) resulting from the frame invariance in the x_2 direction.

The mean velocity $\bar{u}_1$ turns out to have the following form

$$\bar{u}_1 = \frac{a_2}{a_4} + \exp\left(-\frac{a_4}{a_3}x_2\right)C_3. \tag{19}$$

It will be shown in section (4) that (19) applies to the flat plate high Reynolds number boundary layer flow. It appears that the boundary layer thickness is the external length scale which is symmetry breaking.

Linear mean velocity profile: ($a_1 = a_4 = 0$ and $b_1 \neq a_3 \neq 0$)

In the present case there is an external velocity and length scale symmetry breaking. Only the linear mean velocity profile is a self-similar solution

$$\bar{u}_1 = \frac{a_2}{a_3}x_2 + C_4. \tag{20}$$

The latter profiles may apply in the viscous sublayer where ν/u_τ and u_τ are the symmetry breaking length and velocity scales respectively. Another example is the center region of a turbulent Couette flow where the symmetry is broken due to the moving wall velocity and channel height (see Bech (1995) and Robertson (1970)).

3.2 Turbulent shear flows with non-zero system rotation

Here we consider the symmetries of the Eqs. (6)-(9) with $\Omega \neq 0$. The infinitesimal generators to be obtained are very similar to those in non-rotating case but with one important difference: $a_4 = 0$ and hence the scaling symmetry with respect to the time has been lost.

The rotation rate Ω scales with x_2 and only the linear profile is a self-similar solution

$$\bar{u}_1 = C_5\Omega x_2 + C_6. \tag{21}$$

The present case is distinguished from the previous linear mean velocity profiles since a scaling of the spatial coordinates still holds ($a_1 \neq 0$). The present linear law applies in the center region of a rotating turbulent channel flow where the time scale is the inverse of the rotation rate Ω.

4. Experimental and numerical verification of the scaling laws

Some of the mean velocity profiles derived in the previous section have been already obtained by means of other methods and verified in several experiments and DNS data. The best known result is von Kármán's (1930) logarithmic law of the wall which has been verified in a large number of experiments since its derivation.

Another well known mean velocity profile, derived in the previous section, is the linear mean velocity which can be found in the viscous sublayer of the universal law of the wall, and it is valid up to about $y^+ = 3$. May be less well known is the linear mean velocity profile which covers a broad region in the center of a turbulent Couette flow. This has been shown by the experimental study of Robertson (1970) and in the DNS of Bech (1995) to name only two. In both of the latter two cases there is a length and a velocity scale dominating the flow and hence break two scaling symmetries. In the viscous sublayer the length and the velocity scale are ν/u_τ and u_τ and in the turbulent Couette flow it is b and u_w, the channel width and the wall velocity respectively. As a consequence, no scaling symmetry exists as has been already pointed out in the previous section and only the linear mean velocity profile is a self-similar solution.

A third linear mean velocity profile, which from a similarity point of view is distinct from the previous two cases, can be found in the center region of a rotating channel flow. Here the external time scale Ω^{-1} acts on the flow and hence it is symmetry breaking which results in $a_4 = 0$. However, in contrary to the previous case a scaling symmetry with respect to the spatial coordinates still exists. The linear mean velocity as given by Eq. (21) is well documented in the experimental data of Johnston (1972) and in the DNS results of Kristoffersen (1993). In both investigations they found the value C_5 to be approximately 2.

In order to avoid the duplication of well documented invariant solutions, we will focus on basically two cases. The first one is the verification of the exponential law, which has never been reported in the literature. This has been found to match a broad region in the outer part of a turbulent boundary layer flow. The second one is the algebraic law which fits about 80% of the core region of the turbulent channel flow. In addition the algebraic scaling law has also been identified in the vicinity of the wall in low Reynolds DNS data of a turbulent channel flow.

Zero pressure-gradient turbulent boundary layer flow

There is a considerable amount of data available for canonical boundary layer flows but the Reynolds number is usually low and some of the data contain too much scatter. For the present purpose the data need to be very smooth.

Three sets of experimental data have been chosen for comparison with the exponential velocity profile. These data are at medium to high Reynolds numbers, and we believe that they have been taken very carefully. The data of DeGraaff (1996) are very smooth and cover the Reynolds number range $Re_\theta = 1500 - 20000$, where $\theta = \int_0^\infty (1 - \bar{u}/\bar{u}_\infty)\bar{u}/\bar{u}_\infty dy$ is the momentum thickness and $\bar{u}_\infty$ is the free stream velocity. The second set of data are from Fernholz (1995) with the highest Reynolds number of $Re_\theta = 60000$. The third data set of Saddoughi (1994) reaches the unchallenged Reynolds number of $Re_\theta = 370000$.

Figure 1 shows DeGraaff's data for the mean velocity profiles taken at six different Reynolds numbers, in the usual wall variables in semi-log scaling. The extension of the viscous subregion and the logarithmic region are visible, with extension depending on the Reynolds number. In outer scaling the log-region extends approximately to $y/\Delta = 0.025$ where $\Delta = \int_0^\infty (\bar{u}_\infty - \bar{u})/u_\tau dy$ is the Rotta-Clauser length scale and

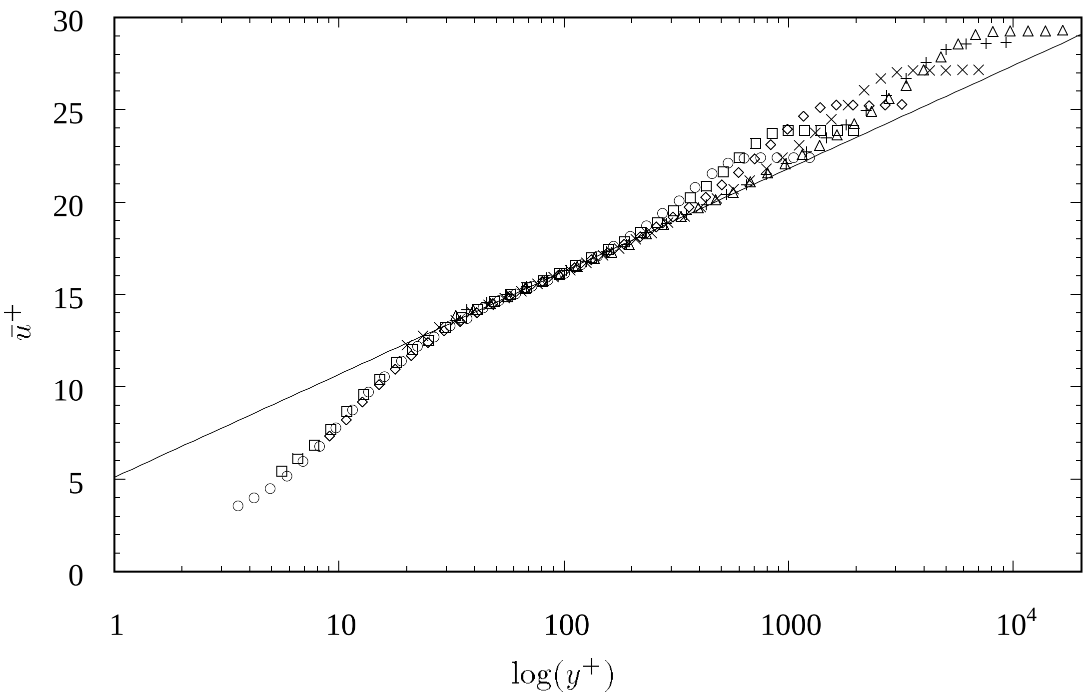

FIGURE 1. Mean velocity of the zero-pressure gradient turbulent boundary layer in log-linear scaling from DeGraaff (1996): ○, $Re_\theta = 1500$; □, $Re_\theta = 2300$; ◇, $Re_\theta = 3800$; ×, $Re_\theta = 8600$; +, $Re_\theta = 15000$; △, $Re_\theta = 20000$; ——— , $2.41 \ln(y^+) + 5.1$.

u_τ is the friction velocity.

As has been pointed out above, it appears that the exponential law (19) matches the outer part of a high Reynolds number flat plate boundary layer flow. In order to match the theory and the data, the mean velocity profile in Eq. (19) will be re-written in outer scaling

$$\frac{\bar{u}_\infty - \bar{u}}{u_\tau} = \alpha \exp\left(-\beta \frac{y}{\Delta}\right) \tag{22}$$

where α and β are universal constants.

In Fig. 2 the turbulent boundary layer data are plotted as $\log\left[\frac{\bar{u}_\infty - \bar{u}}{u_\tau}\right]$ vs. $\frac{y}{\Delta}$. If the data match the scaling law given by (22) a straight line is visible. In the scaling of Fig. 2 the log region is valid up to $y/\Delta \approx 0.025$ and does not follow the exponential (22). For all Reynolds number cases, there is no Reynolds number dependence within the measurement accuracy, and all the data appear to converge to a straight line in the region $y/\Delta \approx 0.025 - 0.15$. The data of Saddoughi (1994) show an extended region for the exponential law up to about $y/\Delta \approx 0.23$. With increasing Reynolds number the applicability of the exponential law appears to increase. For the medium Reynolds number cases, the applicability is approximately five times longer than the logarithmic law and for the high Reynolds number case it is about eight times longer.

The outer part of the boundary layer does not match the exponential (22) and

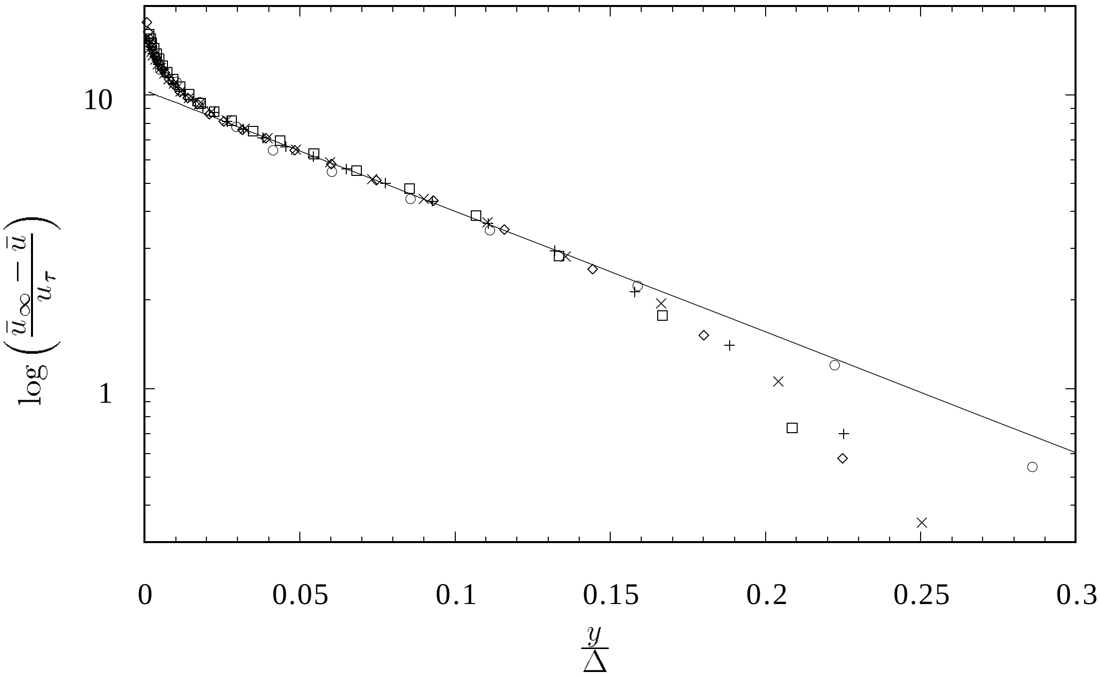

FIGURE 2. Mean velocity of the zero-pressure gradient turbulent boundary layer in lin-log scaling of the defect law: $\circ$, $Re_\theta = 370000$ (Saddoughi (1994)); $\square$, and $\diamond$, $Re_\theta = 60000$ (Fernholz (1995)); $+$, $Re_\theta = 15000$ and $\times$, $Re_\theta = 20000$ (DeGraaff 1996)); ——— , $10.34 \exp(-9.46 y/\Delta)$.

it appears that a weak Reynolds number dependence exists. This seems to be in contradiction to Coles (1962) who found the wake parameter to be constant for $Re_\theta > 5000$. However, several explanations can be given for this behavior. It is common to have a few percent of error in experimental data. Since the data are plotted in log coordinates, and the free stream velocity is subtracted, a few percent error in the free stream velocity has a large impact on the lower part of the curve. This is almost invisible in the upper part. In fact from $y/\Delta \approx 0.3$ the data for the medium Reynolds number flows exhibit no clear trend. This is due to the error accumulation coming from the difference of two almost equally large numerical values.

$y/\Delta \approx 0.3$ corresponds approximately to the boundary layer edge. It is also possible that the outer-region large-scale intermittency plays a dominant role for the scaling of the mean velocity.

If the exponential velocity profile (22) were be valid over the entire boundary layer, an integration of (22) from zero to infinity would give $\alpha = \beta$. A least square fit of the presented data leads to approximately the latter equivalence with $\alpha = 10.34$ and $\beta = 9.46$.

Even though the exponential (22) in Fig. 2 shows an excellent agreement with the experimental data, one may object that, unlike the channel flow, boundary layer flows are not strictly fully parallel flows. However, since the stream line curvature is usually very small, locally the flow can be considered as parallel. The dependence on the streamwise position is hidden in the Rotta-Clauser length Δ and hence does not appear in the experimental results explicitly. Recently Oberlack (1996c) has derived the exponential mean velocity by a group analysis of the two-point correlation equation for a two-dimensional mean flow. It corresponds to a linear growth rate of the boundary layer thickness.

The two dimensional turbulent channel flow

Most data for the turbulent channel flow exhibits too much scatter and cannot be used for the present purpose. A fair comparison between data and algebraic law can only be made in double log plots. Here the experimental data of Niederschulte (1996), Wei (1989) and the low Reynolds number DNS data of Kim (1987) will be used for the investigation of the algebraic scaling law.

Beside the classical wall based scaling laws, here we found another algebraic regime which scales on a wall normal coordinate with its origin in the center of the channel. The validity of an algebraic scaling law based on the center-line appears to be more clear than for the near wall region. The reason for that can be found in the infinitesimal generators (16). Since for the algebraic scaling law both constants a_1 and a_4 have to be non-zero and different from each other, the region where the algebraic scaling law applies has the highest degree of symmetry. The center region seems to be more suitable for that because in the near wall region u_τ is symmetry breaking which results to $a_1 = a_4$ and eventually leads to the log law.

Regarding the algebraic law in the center of the channel we find the appropriate outer scaling for the channel is similar to the turbulent boundary layer flow

$$\frac{\bar{u}_c - \bar{u}}{u_\tau} = \varphi \left(\frac{y}{b}\right)^\gamma, \tag{23}$$

where φ and γ are constants, y originates on the channel center line, $\bar{u}_c$ is the center line velocity and b is the channel half width.

In Fig. 3 the data of Wei (1989) and Niederschulte (1996) have been plotted in double log scaling for the Reynolds number range $Re_c = 18000 - 40000$, where Re_c is based on the center line velocity and channel half width. Even though the data exhibit some scatter, there is some obvious indication that the center region up to about $y/b = 0.8$ closely follows an algebraic scaling law given by (23). The unknown constants in (23) have been fitted to $\varphi = 5.83$ and $\gamma = 1.69$ using Niederschulte's data. We believe Niederschulte's experiment has been done very carefully and the algebraic scaling law has a large extension towards the center line.

An even more profound indication regarding the algebraic law can be obtained from the DNS data of Kim (1987). In Fig. 4 the data are plotted in double log scaling and an almost perfectly straight line is visible for both $Re_c = 3300$ and 7900 from the centerline up to about $y/b = 0.75$. The scaling extends slightly further out for the $Re_c = 7900$ case. Since both Reynolds numbers in the DNS are low, a weak Reynolds number dependence of both φ and γ exists.

At this point it may be instructive to refer to a recent result of Oberlack (1996b) who analyzed circular parallel turbulent shear flows with respect to the self-similarity using the present theory. For this case he also found the existence of an algebraic

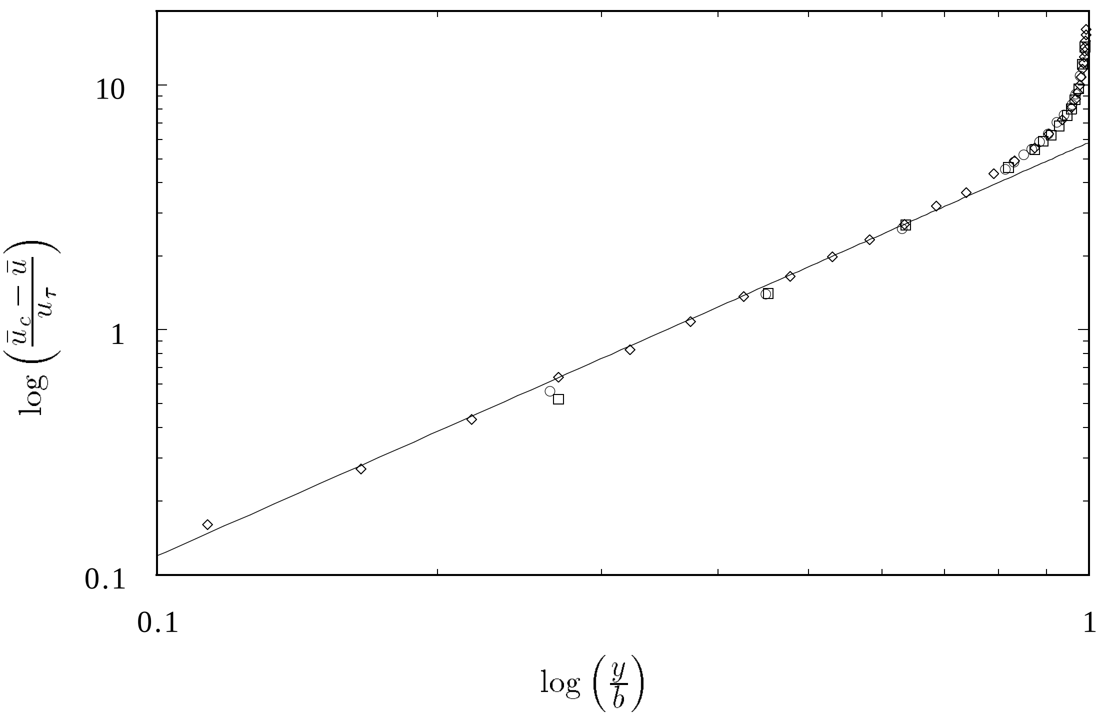

FIGURE 3. Mean velocity of the turbulent channel flow in double-log defect law scaling: ○, $Re_c = 40000$; □, $Re_c = 23000$ Wei (1989); ◇, $Re_c = 18000$ Niederschulte (1996); ——— , $5.83\,(y/b)^{1.69}$.

scaling law. Oberlack analyzed the high Reynolds number data of Zagarola (1996) and here also he found an almost perfect fit, covering 80% of the center of the pipe.

It has been mentioned earlier that in appendix of Oberlack (1996a) the two-point correlation equations have been analyzed with respect to its self-similarity of a parallel shear flow. The resulting equation for the mean flow derived there is fully equivalent to the Eqs. (15)/(16). Further more scaling laws for the two-point correlations could be obtained.

Hunt (1987) have analyzed the two-point correlations with respect to self-similarity using the data of Kim (1987). They investigated the near wall region assuming the logarithmic law to hold. The surprising result here is that the self-similarity of R_{22} has a much longer extension towards the centerline as could be expected from the fairly short log region in the mean flow. The result could be clarified using the fact that the near wall region does not follow a log, but rather an algebraic scaling law. Figure 5 shows the mean velocity of the channel flow data in double log coordinates. Up to about $y^+ = 3$ the linear law of the viscous sublayer is valid. In the range $50 < y^+ < 250$ for $Re_c = 7900$ an almost perfectly straight line is visible and a least square fit of an algebraic law in this range results in a much higher correlation coefficient than a least square fit of a logarithmic function. Since the algebraic law extends much further than a logarithmic law, we can also expect the self-similarity of the two-point correlation R_{22} to hold much further. The only difference for R_{22} regarding the two different scaling laws is that in case of the algebraic scaling law

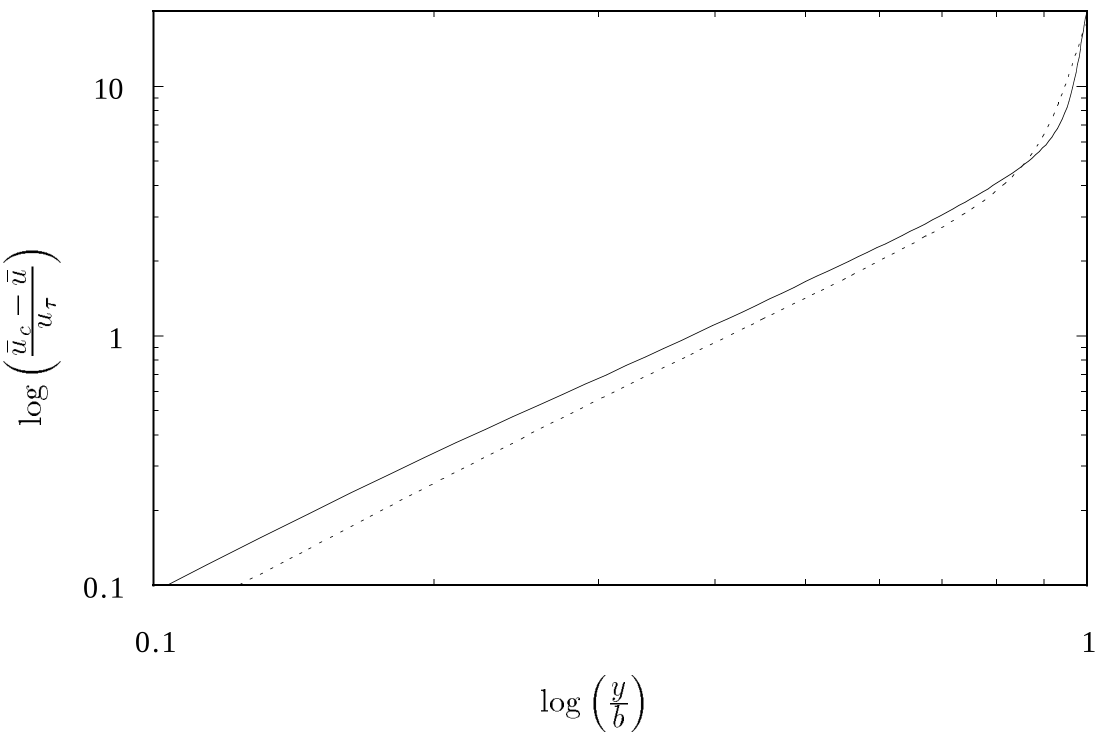

FIGURE 4. Mean velocity of the turbulent channel flow in double-log defect law scaling from Kim, Moin & Moser (1987): ——— , $Re_c = 7900$; – – –, $Re_c = 3300$.

R_{22} also scales with the wall distance, while for the log law this is not the case.

5. Future plans

In the near future the theory presented herein will be applied to turbulent flows with higher dimensions up to 3D time dependent ones. If possible, all self-similar flows will be empirically validated using experimental and DNS data.

An important application of the present theory is in turbulence modeling. Common statistical turbulence models may not be consistent with all the symmetries calculated in the present theory and hence can not capture the associated scaling laws. As an example, consider the standard k-ε model which, interesting enough, formally admits all the symmetries of the unaveraged Euler equations (see Pukhnachev (1972)). This is somewhat misleading since it has been shown in the previous sections that turbulence has different symmetry properties than the unaveraged Navier-Stokes equations.

The standard k-ε model captures some non-trivial scaling laws like the exponential law. However, it can be shown in the case of a turbulent channel flow that the symmetry groups of the k-ε are not consistent with the present finding. As a result, k-ε misses the correct exponent for the algebraic mean velocity profile in the center of the channel.

The present theory can be used as an guide to develop new or improve existing turbulence models. It is proposed that turbulence models should have all of the symmetry properties computed in the present analysis. This is a necessary condition

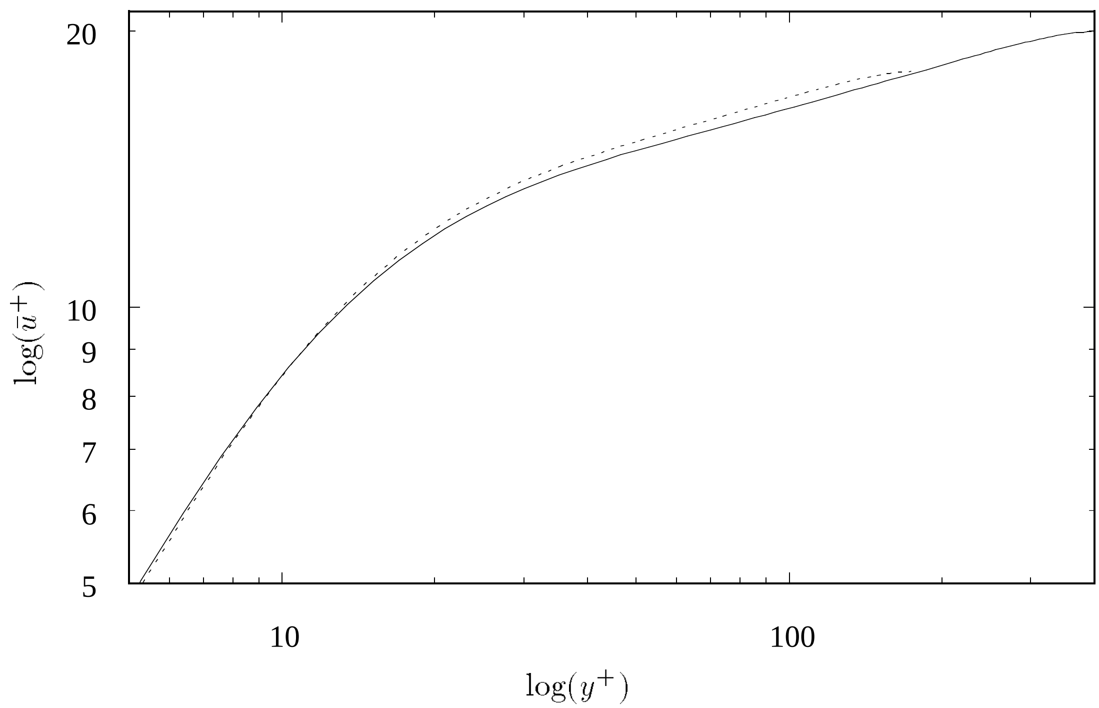

FIGURE 5. Mean velocity of the turbulent channel flow in double log scaling from Kim, Moin & Moser (1987): ——— , $Re_c = 7900$; - - - - , $Re_c = 3300$.

in order to capture the turbulent scaling laws and the associated turbulent flows. The presented symmetry properties in turbulent flows can be considered as a new realizability concept. A more general theory on symmetries in turbulence models is now under investigation and will be published elsewhere.

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