# Wall-modeling in large eddy simulation: length scales, grid resolution and accuracy

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## 1. Motivation and objectives

The promise of large eddy simulation (LES) is that it constitutes a more-or-less optimal compromise between predictive accuracy and computational cost. The energetic, dynamically important and flow-dependent motions are solved directly, leaving only motions with small energy and supposedly universal behavior to be modeled; this leads to predictive accuracy. Moreover, an increase in Reynolds number $Re$ changes the spectrum only at the smallest scales, and hence the computational cost of LES (which is due to solving the large scales) is independent of $Re$.

This favorable picture of LES is true in many situations, but changes completely when LES is applied to turbulent boundary layers. Boundary layers (BLs) are multi-scale phenomena where the energetic and dynamically important motions in the inner layer (the innermost 10-20% of the BL) become progressively smaller as the Reynolds number ($Re$) is increased. For the case of computing the flow over an airfoil, Chapman (1979) estimated the required number of grid points as $N_{\text{total}} \sim Re_c^{1.8}$, where $Re_c$ is the chord Reynolds number. This is close to the cost of direct numerical simulation (DNS), and effectively prevents LES from being used on realistic wall-bounded flows at realistic (high) Reynolds numbers for the foreseeable future.

The solution to this "near-wall problem" has been clear for a long time: the inner layer must be modeled rather than resolved (cf. Deardorff 1970; Schumann 1975). When directly resolving only the outer layer, Chapman (1979) estimated a drastically lower $N_{\text{total}} \sim Re_c^{0.4}$ for his airfoil example. There have been many proposed methods for modeling of the inner layer in LES (cf. the reviews by Piomelli & Balaras 2002; Spalart 2009). In the present study it is convenient to divide existing methods into two general approaches: 1) methods that model the wall shear stress $\tau_w$ directly, and 2) methods that switch to a RANS-like description in the inner layer. The second category includes hybrid LES/RANS and detached eddy simulation (DES).

The most basic and rudimentary requirement on a wall-model for LES is that it accurately predicts the skin friction in an equilibrium BL. A survey of the literature on both wall-stress models and hybrid LES/RANS (or DES) shows that most models do not do this. Nikitin *et al.* (2000) showed clearly how DES is affected by an artificial buffer layer that results in 10-15% underprediction of the wall shear stress $\tau_w$ (i.e., the skin friction). Later studies have shown how artificial forcing can remove this error, but at the price of introducing additional modeling parameters that must be tuned (cf. Piomelli *et al.* 2003; Larsson *et al.* 2007). While DES and hybrid LES/RANS generally underpredict $\tau_w$, wall-stress models may over- or under-predict depending on details of the wall-model, the LES subgrid-model, and the numerics (e.g. Cabot & Moin 1999; Nicoud *et al.* 2001; Pantano *et al.* 2008). The reason for this behavior is that the LES is necessarily underresolved in the first few grid points, and hence numerical and subgrid modeling errors can not be avoided in those first few points. This argument was given by Cabot & Moin (1999), and was given as a major reason by Nicoud *et al.* (2001) for developing a wall model based on control theory. In fact, the idea that numerical and modeling errors are unavoidable in the first few grid points, and, crucially, the idea that these errors necessarily lead to errors in the computed flow field (e.g., the computed skin friction), is now broadly agreed upon in the wall-modeling community (cf. Pantano *et al.* 2008).

The purpose of the present Brief is to show that while numerical and subgrid-modeling errors in the first few grid points are unavoidable, it is possible to ensure that they have negligible impact on the computed flow (specifically, the predicted skin friction). In a certain sense, the proposed method is similar to the use of explicit filtering in LES, in that it, by removing the numerical errors, exposes the errors due to the wall-modeling. We note that the present study applies only to models in the wall-stress category, and not to hybrid LES/RANS or DES methods.

We begin by briefly introducing the governing equations and the wall model. This is then followed by a discussion about turbulence length scales and grid resolution; these concepts then directly lead to the proposed method.

## 2. LES and wall-model equations

The arguments and ideas presented in this Brief apply to wall-stress models, i.e., where a model is used to estimate the instantaneous wall-stress $\tau_w$ that is then applied as a boundary condition to the LES equations. The basic reasoning holds for a wide range of wall-stress models, including equilibrium models (e.g., the famous log-law) and more elaborate approaches that solve the thin boundary layer equations on an auxliary grid near the wall (cf. Cabot & Moin 1999; Wang & Moin 2002; Kawai & Larsson 2010). To simplify the presentation, we consider only a simple equilibrium model here. Given our interest in high-speed flows, the wall-model and the arguments leading to the proposed method are presented for compressible flow, but everything extends trivially to the incompressible case. The equilibrium wall-model used in this study is given by

$$\frac{\partial}{\partial y}\left[(\mu+\mu_t)\,\frac{\partial u}{\partial y}\right]=0\,, \tag{2.1$a$}$$

$$\frac{\partial}{\partial y}\left[(\mu+\mu_t)\,u\frac{\partial u}{\partial y}+c_p\left(\frac{\mu}{Pr}+\frac{\mu_t}{Pr_t}\right)\frac{\partial T}{\partial y}\right]=0\,, \tag{2.1$b$}$$

which was derived from the conservation equations for streamwise momentum and total energy with use of the standard approximations in equilibrium BL flow (cf. Pope 2000).

The eddy-viscosity is taken from a mixing-length model as

$$\mu_t=\kappa\rho y\sqrt{\frac{\tau_w}{\rho}}\left[1-\exp\left(\frac{y^+}{A^+}\right)\right]^2\,, \tag{2.2}$$

where we note that $\sqrt{\tau_w/\rho}$ is the velocity scale in a BL with varying mean density. The modeling parameters are taken as $\kappa=0.41$, $Pr_t=0.9$, and $A^+=17$. A "+" superscript implies normalization by viscous wall quantities, as per usual.

The wall-model given by (2.1) and (2.2) is completed by boundary conditions on $u$ and $T$, taken here as adiabatic no-slip at the wall and equal to the instantaneous LES solution at a grid point above the wall. The wall-model is a system of 2 coupled ODEs that is solved numerically in the present implementation (the computational cost of doing this is minimal). In the incompressible limit without heat transfer, the wall-model can be analytically integrated to $(\nu + \nu_t)\partial_y u = u_\tau^2$, which yields the famous log-law in the inviscid region.

It is important to note the "input-output" character of the wall-model. The wall-model takes information from the LES in the form of instantaneous data at some grid point above the wall; this becomes the upper boundary condition for the wall-model, given in this study by (2.1) and (2.2). The wall-model then returns the wall shear stress and heat flux to the LES, which becomes the boundary condition for the LES at the wall at $y = 0$.

## 3. Length scales and grid resolution

Having introduced the wall-model, we next discuss the numerical and subgrid-modeling errors in the first few grid points. Consider LES of a boundary layer of thickness $\delta$ at high $Re$. A wall-model models the processes between the wall ($y = 0$) and a location $y_m$ (which is typically taken as the first grid point off the wall). The location $y_m$ must be fixed in outer units in order to lead to a computational cost that is independent of $Re$. Moreover, since the classic picture of a BL (cf. Pope 2000) suggests that the inner layer $y/\delta \lesssim 0.1 - 0.2$ is independent of the free-stream (e.g., the pressure gradient), it is clear that $y_m/\delta \approx 0.1$ is a good choice (since it implies that a supposedly "universal" inner layer is modeled). We also assume that the first grid point is located in the (inviscid) log-layer, i.e., that $y_1^+ \gtrsim 50$. This assumption simplifies the analysis, and is reasonable for wall-modeled LES at high $Re$. Using only conservation of momentum, it is easy to derive the average shear-stress balance in the log-layer (i.e., $50 \lesssim y^+ \lesssim 0.2\delta^+$) as

$$\overline{\mu}_{\rm sgs}\partial_y\widetilde{u} - \overline{\rho}\widetilde{u''v''} = \tau_w \, , \tag{3.1}$$

which shows that the mean velocity gradient is a result of a balance between the total stress $\tau_w$, the resolved stress $-\overline{\rho}\widetilde{u''v''}$, and the average subgrid eddy-viscosity $\overline{\mu}_{\rm sgs}$.

The size of the energetic and stress-carrying motions in the log-layer is proportional to the wall-distance $y$ (cf. Pope 2000). It is reasonable to assume different sizes in the different directions, and hence one can take the length scale of the stress-carrying motions as $L_i = C_i y$, where $C_i$ is a (different) constant for each direction $i$. To resolve motions of size $L_i$ accurately, a grid with grid-spacing $\Delta x_i \lesssim L_i/\alpha$ is needed, where the value of $\alpha$ (i.e., the number of grid points per wavelength) depends on the particular numerical method used for the LES. Thus the stress-carrying motions are well resolved only if

$$L_i = C_i y \gtrsim \alpha \Delta x_i \quad , \quad i = 1, 2, 3 \, . \tag{3.2}$$

Consider a uniform grid with $y_j = j\,\Delta y$. The standard approach is to estimate the wall-stress (and wall heat flux) by solving the wall-model (2.1)-(2.2) between the wall and the first grid point at $y_1$. The criterion (3.2) in the $i = 2$ direction implies that the stress-carrying motions at $y_1$ in the LES are accurate only if $C_2\Delta y \gtrsim \alpha\Delta y$, i.e., if $C_2 \gtrsim \alpha$. This is highly unlikely: the numerical Nyquist criterion gives $\alpha \geq 2$, and the kinematic damping by the wall makes $C_2 \lesssim 2$ a very reasonable upper bound. Hence the LES is underresolved at the first grid point $y_1$. Since the wall-model takes as input the instantaneous solution from the LES at the first grid point, it is clear that the wall-model is fed underresolved information. Therefore even a perfect wall-model would not be able to accurately predict the skin friction (when the wall-model is applied between the first grid point $y_1$ and the wall).

The reasoning up to this point is simply a repeat of the arguments given by Cabot & Moin (1999), Nicoud *et al.* (2001) and others to argue that wall-models are unavoidably

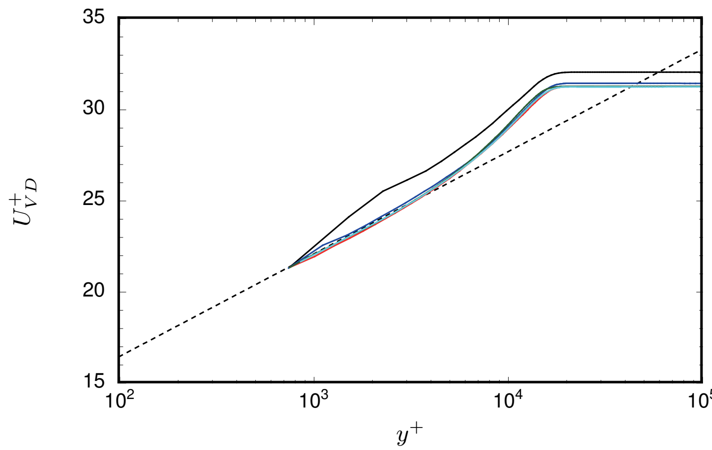



FIGURE 1. Mean velocity (van Driest-transformed) compared to the log-law $\ln(y^+)/0.41 + 5.2$ (dashed line). Fixed $y_m/\delta = 0.05$, varying $m$ and near-wall grid-spacing $\Delta y$ with (increasing $m$ corresponding to lower curves): $m = 1$, $\Delta y/\delta = 0.05$ (black); $m = 2$, $\Delta y/\delta = 0.025$ (blue); $m = 3$, $\Delta y/\delta = 0.0167$ (red); $m = 4$, $\Delta y/\delta = 0.0125$ (green); $m = 5$, $\Delta y/\delta = 0.01$ (cyan).

affected by numerical and subgrid modeling errors, but from here on we depart in a different direction. All previous studies (to our knowledge) always implicitly used $y_m = y_1$, i.e., used information from the first grid point to feed into the wall-model. The crucial point in the proposed method is to realize that there is nothing requiring the wall-model to be applied between the first grid point $y_1$ and the wall: the wall-model equations are valid for *any* interval from the wall and up (provided the upper point is within the inner portion of the BL). Let us allow $y_m \neq y_1$ or equivalently $m \geq 1$; the criterion (3.2) then yields that the LES is well-resolved at height $y_m$ only if

$$\frac{y_m}{\Delta x_i} \gtrsim \frac{\alpha}{C_i} \quad , \quad i = 1, 2, 3 \, .$$

In the wall-parallel and wall-normal directions this becomes

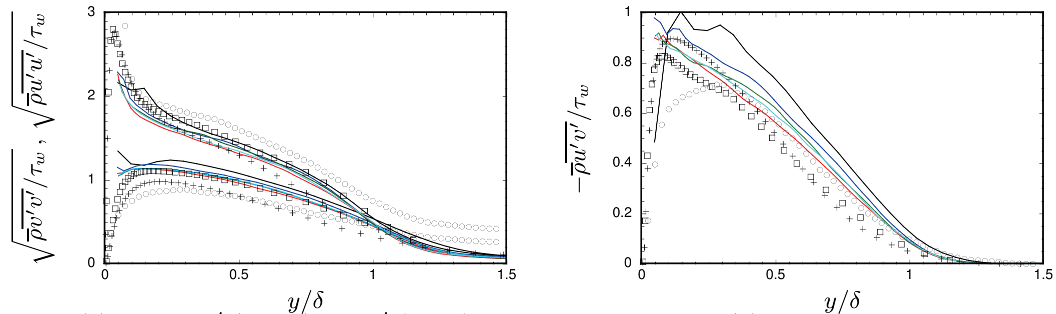
$$\frac{y_m}{\Delta x} \gtrsim \frac{\alpha}{C_1} \quad , \quad \frac{y_m}{\Delta z} \gtrsim \frac{\alpha}{C_3} \; , \tag{3.3$a$}$$

$$\frac{y_m}{\Delta y} = m \gtrsim \frac{\alpha}{C_2} \; , \tag{3.3$b$}$$

respectively. The value of $\alpha$ is related to the numerical method while the values of $C_i$ are dictated by flow physics, specifically the structure of the energetic motions in the log-layer.

Note that the criterion (3.3$b$) in the wall-normal direction is closely related (in spirit) to the technique of explicitly filtered LES (cf. Bose *et al.* 2010): for a given $y_m = m\,\Delta y$, it says that the wall-normal grid-spacing $\Delta y$ must be refined (while keeping $y_m$ fixed) until a point where numerical and subgrid modeling errors at $y_m$ are sufficiently small. In other words, the grid must be refined while still applying the wall-model at the same height above the wall.

We emphasize the overall reasoning here: that the wall-model can only be expected to function properly if fed accurate information from the LES, which in turn implies that the LES must be well resolved at the point $y_m$ where information is fed into the wall-model. Note that the wall (the bottom boundary in both LES and the wall-model




(a) Rms of $u'$ (upper) and $v'$ (lower). (b) Shear stress.

FIGURE 2. Resolved normal stresses (left) and shear stress (right). Fixed $y_m/\delta = 0.05$; varying $m$ and near-wall grid-spacing $\Delta y$ with identical line styles as in Fig. 1. Compared to experiments (circles, Souverein *et al.* 2010), DNS at Mach 2.28 and $Re_\theta = 2300$ (squares, Pirozzoli & Bernardini 2010), and incompressible DNS at $Re_\theta = 900$ (plusses, Wu & Moin 2009).

equations) is still located at $y = 0$. We next turn to numerical experiments to verify these arguments.

## 4. Results

The numerical experiments were performed using the FDL3DI finite-difference code with 6th order accurate compact schemes and a gentle dealiasing filter. Details of this method can be found in Kawai & Larsson (2010). The dynamic Smagorinsky model was used to model the the subgrid stresses (Moin *et al.* 1991). We note that a subset of these numerical tests were also performed in a different code with very different numerical characteristics (non-dissipative through use of split-form derivatives), with the same convergence trends but at different critical values of the parameters $y_m/\Delta x_i$. Thus the conclusions reached in this Brief are dependent on the numerics only insofar as exact numerical values of the parameters are concerned, but not in the larger qualitative sense.

The supersonic flat plate boundary layer experiment by Souverein *et al.* (2010) is considered in this study. The free-stream Mach number is 1.69 and the Reynolds number is $Re_\theta = 50,000$ (based on momentum thickness) and $Re_\delta = 620,000$ (based on BL thickness). We emphasize that this is a *much* higher Reynolds number than what traditional wall-resolved LES is capable of computing. The computational domain is $15 \times 15 \times 3$ in terms of the thickness of the BL at the inlet $\delta_0$ in the streamwise, wall-normal, and spanwise directions, respectively. The boundary layer thickness $\delta$ at the station where statistics are compared is $\delta \approx 1.2\delta_0$. A recycling/rescaling procedure is used to produce realistic turbulence at the inlet. The grid resolution in the wall-parallel directions is held constant at $\Delta x = \Delta z = 0.042\delta$ for all simulations presented here.

The first test uses a fixed location $y_m = 0.05\delta$ at which information is fed to the wall-model, but with varying near-wall grid-spacing $\Delta y$ and hence varying values of $m$. The computed mean velocity from this test is shown in Fig. 1. There is clearly a convergence process as $m$ increases and $\Delta y$ decreases. Further evidence is given in Fig. 2 which shows the resolved turbulence fluctuations for this test.

It is clear that the result for $m = 1$ (using the wall-model from the first grid point in the traditional manner) yields an overprediction of $U_{VD}^+$. This is trivially connected to the wall shear stress $\tau_w$ through $U_{VD}^+ = U_{VD}/\sqrt{\tau_w/\rho_w}$, and further to the skin friction

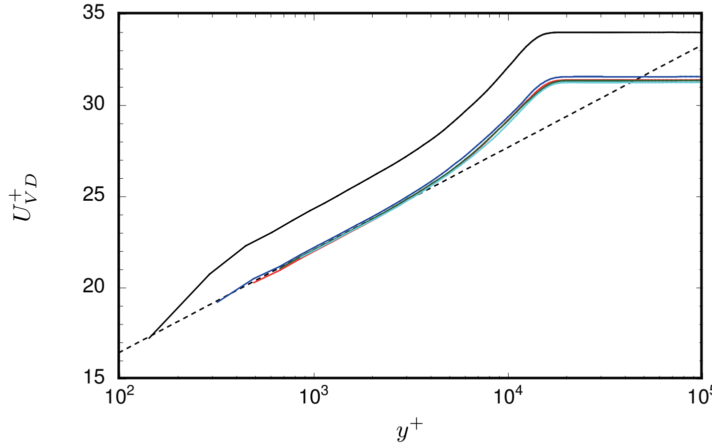



FIGURE 3. Mean velocity (van Driest-transformed) compared to the log-law $\ln(y^+)/0.41 + 5.2$ (dashed line). Fixed grid with $\Delta y/\delta = 0.01$, varying $m$ and $y_m$ with (increasing $m$ corresponding to lower curves): $m = 1$, $y_m/\delta = 0.01$ (black); $m = 2$, $y_m/\delta = 0.02$ (blue); $m = 3$, $y_m/\delta = 0.03$ (red); $m = 4$, $y_m/\delta = 0.04$ (green); $m = 5$, $y_m/\delta = 0.05$ (cyan).

coefficient $C_f = 2\tau_w/(\rho_\infty U_\infty^2)$. An overprediction of $U^+_{VD}$ is therefore directly related to an underprediction of $\tau_w$ and $C_f$.

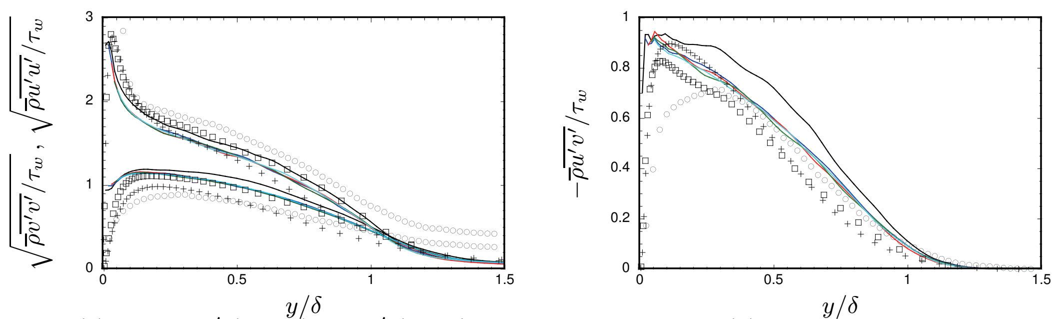
The results (both mean velocity and turbulence statistics) with $m = 2$ are substantially improved, while $m \geq 3$ appears essentially converged with minor differences. This test confirms the criterion on the wall-normal grid-spacing (3.3*b*) and the reasoning leading up to it (since the quantities in the wall-parallel criterion are held constant).

The next test is to use a fixed grid but to vary $m$. One purpose of this test is to vary $y_m/\Delta x$ and $y_m/\Delta z$, thereby testing criterion (3.3*a*). A second purpose, which is actually of greater practical importance, is that this test is closer to the decision one would face in reality: given a fixed grid (related to what is computationally affordable), can the accuracy be improved by increasing $m$? The results of this test are shown in Figs. 3 and 4. Again there is clearly convergence, with much improved results for $m = 2$ and grid-converged results for $m \geq 4$.

The tests so far have shown how the computed statistics, especially the skin friction, are improved by abandoning the established practice of applying the wall-model at the first grid point off the wall. We next show that the method yields physically realistic turbulence near the wall. Fig. 5 shows a snapshot of the instantaneous streamwise velocity very near the wall; note the absence of unphysically smooth regions as commonly seen in hybrid LES/RANS and DES-type methods (cf. Piomelli *et al.* 2003; Larsson *et al.* 2007).

## 5. Summary

This Brief addresses one of the basic problems encountered when modeling the wall shear stress in large eddy simulation on grids that do not resolve the viscous layer: the inevitable presence of numerical and subgrid modeling errors in the first grid point off the wall. In this Brief the wall-model is viewed as taking an input from the LES (instantaneous data at some height $y_m$) and returning an output back to the LES (the instantaneous wall shear stress $\tau_w$ at $y = 0$). In this view, all prior work (to the authors' knowledge) using a wall-stress model have taken the input to the wall-model from the first grid point off the wall, which leads to inevitable errors in the predicted skin friction.




(a) Rms of $u'$ (upper) and $v'$ (lower). (b) Shear stress.

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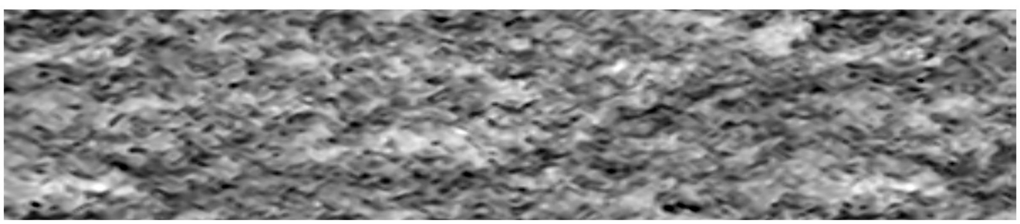

FIGURE 5. Instantaneous snapshot of $u'$, taken from the second test with $m = 5$ and $\Delta y/\delta = 0.01$. Note that $\tau_w$ and $u$ at the wall-model input location are correlated by design for the equilibrium stress model used here. The size of the figure corresponds to the full computational domain, i.e., $15 \times 3$ in terms of the incoming boundary layer thickness.

The main purpose of this Brief is to point out that there is nothing in the wall-modeling approach that requires LES information to be taken from the first grid point. The second purpose is then to show how the error in skin friction can be removed by feeding information to the wall-model from the second or third grid point in the LES, where the LES is well resolved and hence more accurate. This result stems directly from considerations of how turbulence length scales behave in the logarithmic layer. In other words, while the proposed solution is simple, it is based solidly on physical reasoning.

In this Brief we only considered an equilibrium BL and an equilibrium wall-model. However, the basic idea of increasing accuracy by taking the input to the wall-model farther away from the wall can be used much more broadly. For example, Kawai & Larsson (elsewhere in this volume) use this idea in the context of solving the full RANS equations near the wall, with a dynamic procedure to compute the appropriate RANS eddy-viscosity.

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