On the reflectivity of sponge zones in compressible flow simulations

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1. Motivation and objectives

External flow simulations are typically computed on truncated domains. In these computations the exact conditions at the truncation boundary are usually unknown, and the external boundaries are often treated artificially. The aim of these artificial treatments is to allow for out-going flow features to leave the computational domain without reflecting any signature inside. These methods involve a variety of techniques (see Colonius 2004) including characteristic-based decomposition (e.g., Giles 1990; Poinset & Lele 1992; Tam & Dong 1996), flow dissipation (e.g., Israeli & Orszag 1981; Freund 1997), grid-stretching/slow-down operators (e.g., Colonius et al. 1993; Karni 1996), supergrid modeling (Colonius & Ran 2002), and perfectly matched layers (Hu et al. 2008; Hu 2008). While some of these treatments require sophisticated procedures, they still fail to accomplish perfect nonreflectivity.

One simple approach to treat the external boundaries is to use the sponge terms (Israeli & Orszag 1981; Bodony 2006). With this method the compressible Navier-Stokes equations are artificially modified as follows:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) &= \sigma (\rho_{\text{ref}} - \rho) , \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j + p \delta_{ij} - \tau_{ij}) &= \sigma [(\rho u_i)_{\text{ref}} - \rho u_i] , \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E + p) u_j + q_j - u_k \tau_{kj}] &= \sigma (E_{\text{ref}} - E) ,
\end{align*}
\]

where, \( \rho, u_i, p, E, \tau_{ij}, \) and \( q_j \) are the density, velocity, pressure, total energy, viscous stress tensor and heat flux, respectively. The unphysical terms on the right-hand side are active only near external boundaries where they damp the flow variables to a known reference solution. The damping coefficient, \( \sigma \), which is an inverse time scale, is the same for all conserved variables due to stability considerations (Bodony 2006). One can also show that this form maintains Galilean invariance.

The sponge treatment has been widely adopted owing to its simplicity, robustness, non-stiff nature, and flexibility to handle complex geometries and unstructured grids. Examples of computations with sponge treatment include compressible mixing layers (Bogey et al. 2000; Barone & Lele 2005), jets (Bodony & Lele 2008; Shoeybi et al. 2010), cavity flows (Gloerfelt et al. 2003; Larsson et al. 2004), airfoils (Bodony 2009), ramps (Adams 2006), and bluff bodies (Mani et al. 2009).

Each sponge region can be characterized by its length, \( l_{\text{sp}} \), and strength, which is an integral measure of \( \sigma \) (defined below). As we shall see, larger sponges perform better than small sponges with the same strength in the sense that they damp flow features more quietly. However, larger sponges demand larger computational domains and are more expensive. This trade-off sets an optimal sponge design for each simulation.
Existing mathematical analyses of numerical sponge layers establish adequate confidence regarding their usefulness and concerns such as well-posedness, proof of convergence, and stability (see Israeli & Orszag 1981; Bodony 2006, for example). However, they do not provide sponge design guidelines for general purpose CFD applications. Neither does there exist an adequate understanding of sponge failure mechanisms and its margins. Many previous simulations relied on a combination of trial-and-error and sponge sensitivity analyses to achieve satisfactory sponge settings. Because of this, cost-optimal sponges have rarely been achieved in practice.

In this report we elucidate the basic “physics” of sponge/flow interactions beyond the one-dimensional model and provide a quantitative understanding of its different reflectivity mechanisms. By comparing basic power law profiles we identify the optimal sponge profile, which we will investigate over a wide range of conditions. We provide results aiming to determine the correct sponge length and strength for an arbitrarily set performance requirement. Our results cover basic concerns with sponges such as reflectivity of sound/sponge interactions and vortex/sponge interactions and are presented in terms of dimensionless parameters which can be estimated a priori. We then present an example of nonlinear Euler calculation of vortex/sponge interactions and show that a sponge designed by our guidelines achieves a performance comparable with that by a perfectly matched layer for the same computational cost over moderate sponge lengths.

Many researchers combined sponge terms with grid-stretching and viscous (or higher-order) dissipation (e.g., Bogey et al. 2000, 2003). Equivalently, coordinate transformation may be used instead of stretching (e.g., Appelo & Colonius 2009). Grid-stretching allows for larger sponge without additional computational cost, but needs to be accompanied by high-order dissipation to prevent the reflection of unsupported features. Even though we do not include viscous dissipation in our analysis, the presented results can still be useful, since the sponge requirements are often dictated by the low wavenumber flow content, which is least affected by the high-order dissipation terms.

2. Basic physics of sponge/flow interactions

We present our analysis using the linearized Euler equations which enables capture of the leading order mechanisms of sponge reflectivity. The linearization is justified since sponges are typically employed in the “far field” where the amplitude of incident fluctuations is expected to be small.

\begin{align}
\frac{\partial u}{\partial t} + M \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= -\sigma(x)u, \\
\frac{\partial v}{\partial t} + M \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} &= -\sigma(x)v, \\
\frac{\partial p}{\partial t} + M \frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\sigma(x)p,
\end{align}

(2.1)

where \(u, v\), are perturbation velocities in the \(x\)- and \(y\)-directions, respectively, and \(p\) is perturbation in pressure. These perturbations are relative to a uniform freestream with Mach number \(M\) and are nondimensionalized using free stream speed of sound, density, and an arbitrary length scale. As shown in Figure 1a, the \(x-y\) coordinate system is such that the sponge isocontours are aligned with the \(y\)-direction. The background flow is considered to have only an \(x\)-component since the \(y\)-component can be eliminated by

\[\text{Figure 1a: Coordinate system with sponge isocontours aligned with \(y\)-direction.}\]
Figure 1. Schematics of an oblique incident wave approaching a sponge zone (a). An oblique sound wave moving towards a sponge near an inflow boundary with $M = -0.3$ (b), and the resulting reflected vorticity wave and sound wave (c) obtained from a nonlinear Euler calculation.

Transforming to a moving frame of reference. Flow features approach the sponge from $x < 0$ and interact with it through $0 \leq x \leq l_{sp}$.

This sponge treatment maintains decoupling of entropy fluctuations from sound and vorticity modes. One can verify that sponge performs “ideally” on the entropy modes; i.e., entropy fluctuations attenuate exponentially through the sponge as they advect, regardless of their scale and alignment. However, the sponge term can cause coupling between incoming and outgoing acoustic modes as well as coupling between acoustics and vorticity modes. As a demonstrative example, Figure 1b and 1c show reflection of a sound wave as well as a vorticity wave due to interactions of an incident sound wave with a sponge. Before presenting the full analysis of Eq. (2.1), we present analysis of two simplified cases to gain insight into the basic physics of sponge/flow interactions.

2.1. One-dimensional limit

In this case Eq. (2.1) reduces to the following system:

$$\frac{\partial R}{\partial t} + (M+1) \frac{\partial R}{\partial x} = -\sigma(x)R,$$

$$\frac{\partial v}{\partial t} + M \frac{\partial v}{\partial x} = -\sigma(x)v,$$

$$\frac{\partial L}{\partial t} + (M-1) \frac{\partial L}{\partial x} = -\sigma(x)L,$$

(2.2)

where $R = u + p$ and $L = u - p$. One can see that all characteristics remain uncoupled. This system can be physically interpreted as advection of characteristics with their physical speed, but their amplitude is damped exponentially as they experience the sponge zone. If the boundary condition after the sponge at $x = l_{sp}$ remains decoupled (such as the widely used 1D characteristic-based boundary condition), the sponge plus boundary condition would be perfectly nonreflective. Otherwise, there will be reflections, but the sponge will exponentially suppress the effects as the flow features pass through it. In this case three general scenarios or their combination can happen:

1. sound-sound reflectivity: an incident sound wave can reflect in the form of a sound wave,
2. sound-vorticity reflectivity: an incident sound wave can reflect in the form of a vorticity wave for an inflow boundary with \( M < 0 \), and

3. vorticity-sound reflectivity: an incident vorticity wave can reflect in the form of sound wave for an outflow boundary with \( M > 0 \).†

For the sound-sound reflectivity, one can show that the ratio of the reflected to incident sound amplitude is

\[
\eta \equiv \frac{|L\text{reflected}|}{|L\text{incident}|} = \eta_{BC} \exp \left( -\frac{2}{1-M^2} \int_{0}^{l_{sp}} \sigma(x)dx \right),
\]

where \( \eta_{BC} \) is the (sound-sound) reflection coefficient of the boundary condition at \( x = l_{sp} \). Similar expressions can be obtained for other mechanisms.‡ The exponential term in Eq. (2.3) is the sole damping contribution of the sponge. This term can be interpreted as a two-passage experience of the sponge with velocities \( 1 + M \) and \( 1 - M \), and residence times inversely proportional to the velocities. In this one-dimensional setting sponge performs “ideally” in the sense that it does not cause its own reflection and achieves its destined (or targeted) exponential damping. We define sponge target-damping as

\[
\eta_{target} \equiv 20 \log \left( \exp \left( -\frac{2}{1-M^2} \int_{0}^{l_{sp}} \sigma(x)dx \right) \right) = -20 \frac{2 \log e}{1-M^2} \int_{0}^{l_{sp}} \sigma(x)dx.
\]

In practical situations where a sponge encounters misaligned features, depending on its degree of failure and the end boundary condition \( (\eta_{BC}) \), the overall damping, \( \eta \), maybe better or worse than \( \eta_{target} \). One can see that the integral, \( \int \sigma dx \), is a direct measure of sponge strength. In this paper we refer to \( -\eta_{target} \) (i.e., the same integral but properly normalized) as the sponge strength. As an example, a sponge with 40dB strength would damp the amplitude of an incident sound wave by a factor of 100 under the one-dimensional condition.

### 2.2. Oblique waves with no convection

To gain insight into the performance of sponges when encountering oblique waves, we first analyze a simple case in which \( M = 0 \). Eq. (2.1) can be transformed to an ODE after taking Fourier transforms in the homogeneous directions.

\[
\frac{d}{dx} \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} = (i\omega - \sigma) \begin{bmatrix} 0 & 1 + \frac{k_{\parallel}^2}{(i\omega - \sigma)^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix},
\]

where \( \omega = 2\pi f \) is the angular frequency and \( k_{\parallel} = 2\pi \sin \theta / \lambda \) is the transverse wavenumber with \( \theta \) being the angle of the wavefront normal relative to the \( x \)-direction (defined outside of the sponge zone). As mentioned above, a simple choice for the boundary condition at \( x = l_{sp} \) is the 1D characteristics-based (Thompson) condition:

\[
\hat{p}(l_{sp}) - \hat{u}(l_{sp}) = 0.
\]

† We note that in the most general form the boundary condition at \( x = l_{sp} \) could also provide entropy coupling. However, this will not be the case for the boundary conditions that we study in this report.

‡ Only the factor of \( 2/(1-M^2) \) will be replaced by the sum of inverse incident and reflected characteristic velocities.
We first consider the case of constant \( \sigma \). Under this condition Eq. (2.5) can be diagonalized to the following system:

\[
\frac{d}{dx} \begin{bmatrix} R \\ L \end{bmatrix} = (i\omega - \sigma) \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix}, \quad 0 < x < l_{sp},
\]

where \( c = \sqrt{1 + k_y^2/(i\omega - \sigma)^2} \). \( R \) and \( L \) are the right-going and the left-going characteristics, respectively. The eigenvectors associated with these characteristics can be obtained from the following transformation:

\[
\begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} c & -c \\ 1 & 1 \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix}.
\]

The key observation is that for the case where \( k_y \neq 0 \), both eigenvectors are dependent on \( \sigma \); hence they would change across any interface with a jump in \( \sigma \). As a right-going incident characteristic enters the sponge it will not remain a pure right-going characteristic any more. This would cause reflection of a portion of an incident wave off the sponge to maintain continuity (for \( \hat{u} \) and \( \hat{p} \)) at the interface. To see this in the analysis we present the solution to the ODE system of Eq. (2.7):

\[
\begin{bmatrix} \hat{u}(l_{sp}) \\ \hat{p}(l_{sp}) \end{bmatrix} = \begin{bmatrix} c & -c \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \exp[-c(\sigma - i\omega)l_{sp}] & 0 \\ 0 & \exp[c(\sigma - i\omega)l_{sp}] \end{bmatrix} \begin{bmatrix} \frac{1}{2c} & \frac{1}{2c} \\ -\frac{1}{2c} & \frac{1}{2c} \end{bmatrix} \begin{bmatrix} \hat{u}(0) \\ \hat{p}(0) \end{bmatrix}.
\]

With the boundary condition \( \hat{u}(l_{sp}) = \hat{p}(l_{sp}) \), one obtains

\[
\begin{aligned}
\frac{\hat{u}(0)}{\hat{p}(0)} &= \frac{\cosh [c(\sigma - i\omega)l_{sp}] + c \sinh [c(\sigma - i\omega)l_{sp}]}{\cosh [c(\sigma - i\omega)l_{sp}] + \frac{1}{i} \sinh [c(\sigma - i\omega)l_{sp}]}.
\end{aligned}
\]

Equation (2.10) is the “apparent” effective boundary condition at \( x=0 \) resulting from the system of the sponge plus the end boundary. To obtain the reflectivity of this overall condition, one can use the inverse of Eq. (2.8) to translate Eq. (2.10) in terms of the ratio of reflected to incident characteristics (at \( x = 0^- \)).

\[
\eta = \left| \frac{\mathcal{L}(0^-)}{R(0^-)} \right| = \left| \frac{c(0^-) - \frac{\hat{u}(0)}{\hat{p}(0)}}{c(0^-) + \frac{\hat{u}(0)}{\hat{p}(0)}} \right|,
\]

where \( c(0^-) = \sqrt{1 - k_y^2/\omega^2} = \cos \theta \). Substitution of Eq. (2.10) into Eq. (2.11) results in a closed-form expression for reflectivity in terms of incidence angle (\( \theta \)), and the products \( l_{sp}f \) and \( l_{sp}\sigma \) (nondimensionalized by speed of sound, \( a \), and independent of the reference length).

### 2.3. Physical interpretations

Before extending our analysis to a general condition, we present the physical interpretation of results for the case of \( M=0 \) and \( \sigma=\text{const} \). The trends that we discuss here are applicable to the general condition and provide useful insight into the design of sponge zones. We discuss our result in the limit of small incident angle, \( \theta \approx k_y/\omega \ll 1 \). Substitution of Eq. (2.10) into Eq. (2.11) yields

\[
\eta = \left| \frac{\theta^2}{4} \left( 1 - \frac{\omega^2}{(\omega + i\sigma)^2} \right) + \frac{\theta^2}{4} \frac{\omega^2}{(\omega + i\sigma)^2} \exp(-2il_{sp}\sigma + 2il_{sp}\omega) \right|.
\]
Figure 2. Contours of density field computations for flow over cylinder at Re=10,000 and M=0.4. The outer boundary and sponge isocontours are circles, but in (a) they are centered around the cylinder and in (b) they are aligned with the dominant radiating acoustic fronts. The visible spurious sound contamination is eliminated by sponge alignment and profile optimization.

The first term in Eq. (2.12) represents reflection by the sponge interface at \( x = 0 \), and the second term represents reflection off the boundary at \( x = l_{sp} \), which is damped by the sponge similar to the 1D case. While a stronger sponge suppresses the reflection effects of the end boundary, higher \( \sigma \) is not always desired as it causes reflection from the sponge itself. Following is a summary of important observed trends which are also applicable to our extended analysis presented in the next sections.

**Scaling with the angle of incidence:** For small \( \theta \) the reflection coefficient scales as \( \theta^2 \). Therefore, significant improvement can be achieved by simply aligning the sponge isocontours with the outgoing wavefronts. Figure 2 shows an example of this practice.

**Large wavelength limit:** One can verify that in the limit of small \( \omega \) the reflection coefficient is

\[
\lim_{\omega \to 0} \eta_{\omega} = \frac{\theta^2}{4},
\]

(2.13)

which is the same as the reflection coefficient without the sponge. In other words, sponges are ineffective for large \( \sigma \)'s compared to the incident frequency.

**Short wavelength limit:** In the large \( \omega \) limit, the reflection coefficient becomes

\[
\lim_{\omega \to \infty} \eta_{\omega} = \frac{\theta^2}{4} \exp(-2l_{sp}\sigma),
\]

(2.14)

which is identical to the ideal 1D reflectivity described by Eq. (2.3), with \( \eta_{BC} = \theta^2/4 \) and \( M = 0 \). Achieving this ideal limit (and simultaneously avoiding the ineffective limit above) requires that \( \omega \gg \sigma \). Noting \( \omega = 2\pi/\lambda \), this leads to the following requirement

\[
\frac{l_{sp}}{\lambda} \gg \eta_{\text{target}},
\]

(2.15)

This conclusion is generally applicable to non-constant sponge profiles, but its severity can be significantly relaxed by profile optimization (see below).

Figure 3a shows reflectivity of a sponge with constant \( \sigma \) and \( \eta_{\text{target}}=-20\text{dB} \) as a function of dimensionless sponge length (using frequency, \( f \), and speed of sound, \( a \)). One can see
that for the sponge length smaller than 30% of the wavelength, the sponge is ineffective and the reflection coefficient is the same as that without sponge ($\eta \approx 0.25\theta^2$). For sponges larger than 10 wavelengths, an almost ideal performance is achievable.

The oscillations in the plot of Figure 3a is due to interference between two reflections at $x = 0$ and $x = l_{sp}$ represented by two terms in Eq. (2.12). Figure 3b shows the upper envelope of this curve (i.e., the worst interference scenario) and compares it with sponges at higher strength. Achieving ideal performance at higher target-dampings requires much larger sponges. For example, a sponge with $\eta_{\text{target}} = -60\text{dB}$ needs to be as long as $10^4$ wavelengths to achieve ideal performance. This indicates that sponges with constant profile are practically ineffective.

2.4. Effect of sponge profile

The stringent requirement on sponge length by Eq. (2.15) can be significantly relaxed if one resorts to non-uniform sponges. Extension of our analysis to variable $\sigma$ is straightforward. One only needs to discretize the sponge profile and assume constant $\sigma$ in each interval with length $\Delta x$ (as done in basic integration methods). The ODE system of Eq. (2.5) can be diagonalized for each interval, and solved analytically resulting in a matrix relation between the two end state-vectors for each interval similar to Eq. (2.9). Multiplying the series of these matrices for all intervals yields the global solution to the system.

Figure 4 compares the performance of different sponge profiles and compares them with that of a constant profile. We consider power-law profiles of the form $\sigma(x) \sim x^n$.
with \( n = 0, 1, 2, \) and \( 3 \). The rate of convergence to ideal performance is much faster for higher-order sponges than for the constant sponge. This is mainly due to distribution of interfacial reflection over the entire length of the sponge when \( \sigma \) is increased gradually. This treatment reduces the overall reflection in two ways: first, in deeper regions (higher \( x \)) the incident signal is already weakened by the sponge and thus the interfacial reflected energy will be smaller, and second, there is a possibility of phase cancellation by reflection from a continuous interface.†

Higher-order sponges, however, have a worse low-frequency performance. As shown in the figure, the quadratic sponge \((\sigma \sim x^2)\) has the best overall performance for \( \eta_{\text{target}} \) in the range \(-20\) to \(-60\)dB with trends suggesting cubic sponges for target-dampings beyond \(60\)dB. We consider the quadratic sponge as our method of choice and in the following sections present a general analysis of its performance.

3. General analysis

Extension of this analysis to the general case with nonzero \( M \) (non-constant \( \sigma \), and general \( \theta \)) is straightforward. Equation (2.1) can be Fourier transformed in homogeneous directions and rearranged to

\[
\frac{d}{dx} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} M & 0 & 1 \\ 0 & M & 0 \\ 1 & 0 & M \end{bmatrix}^{-1} \begin{bmatrix} i\omega - \sigma(x) & 0 & 0 \\ 0 & i\omega - \sigma(x) & -ik_y \\ 0 & -ik_y & i\omega - \sigma(x) \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix}. \tag{3.1}
\]

The sponge profile can be discretized into constant-\( \sigma \) intervals, and system (3.1) can be diagonalized and analytically solved in each interval to achieve

\[
\begin{bmatrix} \hat{u}(x + \Delta x) \\ \hat{v}(x + \Delta x) \\ \hat{p}(x + \Delta x) \end{bmatrix} = S(x) \exp \left( \Lambda(x) \Delta x \right) S^{-1}(x) \begin{bmatrix} \hat{u}(x) \\ \hat{v}(x) \\ \hat{p}(x) \end{bmatrix}, \tag{3.2}
\]

where \( S \) is the local matrix of eigenvectors and \( \Lambda \) is the diagonal matrix of local eigenvalues. Multiplying the matrices for all intervals yields a matrix relation between the state-vector at \( x = 0 \) and that at \( x = l_{sp} \). Again, a reasonable choice for the boundary condition at \( x = l_{sp} \) is the 1D characteristic-based condition:

\[
\hat{u}(l_{sp}) - \hat{p}(l_{sp}) = 0, \\
\hat{v}(l_{sp}) = 0 \quad \text{if} \quad M < 0. \tag{3.3}
\]

The second condition ensures that the incoming 1D vorticity wave will be zero. For \( M > 0 \) this condition would be replaced by another condition set at \( x = 0 \). For analysis of an incident sound, the incident vorticity would be set to zero and vice versa.

We note that in the general case with \( M \neq 0 \), the nondimensional sponge length, \( l_{sp} f / a \), is not equal to the ratio of the sponge length to the wavelength. In the physical domain, \( x \in (-\infty, 0) \), the relations between the spatial wave number and frequency are \( \omega = k(1 + M \cos \theta) \), and \( \omega = kM \cos \phi \) for the sound wave and the vorticity wave, respectively, with \( \theta \) and \( \phi \) being the angle of the wavefront-normal, measured relative to the \( x \)-direction.

We performed numerical calculations of reflectivity for quadratic sponges in a range beyond \( 60\)dB. We consider the quadratic sponge as our method of choice and in the following sections present a general analysis of its performance.

† A perturbation analysis of reflectivity of a general sponge profile for small \( \theta \) reveals that \( \eta_{l_{sp}, f / a} \) is the Fourier transform of a function involving the sponge profile. Smoother sponges result in a faster drop of the Fourier transform at high frequencies.
of inflow and outflow Mach numbers ($M = -0.8, -0.5, -0.2, 0.2, 0.5, 0.8$) and for sponge strengths of 20dB, 40dB, and 60dB. We used 200 uniform intervals to discretize the sponge profile into a piecewise-constant function and verified satisfactory convergence of the results. The analysis is done for a range of nondimensional frequencies from 0 to 10 with stepping, $l_{sp} \Delta f / a$, equal to 0.04.

3.1. Reflectivity of the sound-sound mechanism

Figures 5 and 6 show the sound-sound reflectivity of the quadratic sponge for inflow and outflow boundaries, respectively. In the limit of small $l_{sp}$ the sponge is ineffective and the results are the same as those without the sponge. But the sponge effect picks up at $l_{sp} f / a$ of order 0.5 to 2, depending on the Mach number and target-damping.

The explored range of incidence angles are from 0 to $\cos^{-1}(-M)$; the waves outside of
this range are irrelevant since they do not have positive group velocity in the $x$-direction. As $\theta$ approaches $\cos^{-1}(-M)$ the waves do not effectively encounter the sponge since they move almost tangentially.

The solid line in each plot indicates the isocontour of $\eta = \eta_{\text{target}}$, and one can see that a wide range of frequencies and incident angles are damped by ratios better than the target level. The results of these plots can guide the design of sponges for CFD applications. One first needs to estimate the minimum relevant frequency, $f_{\text{min}}$, in the system to be simulated (for example, the shedding frequency for flow over a cylinder). The desired simulation accuracy would set the parameter $\eta_{\text{target}}$ which then for the known $M$ narrows the search to a specific sub-plot in Figures 5 or 6. The sponge should be designed to be long enough so that $l_{sp}/f_{\text{min}}$ lies on the right side of the plotted isocontour for the desired ranges of the incidence angles. Based on the results of Figures 5 and 6, a summary of recommended guidelines is presented in Section 4.
3.2. Reflectivity of the sound-vorticity mechanism

For an inflow condition with $M < 0$, it is possible that a portion of the energy of an incident sound wave reflects back into the physical domain in terms of a vorticity wave. Figure 7 shows the results of the reflectivity analysis of the sound-vorticity mechanism. Again, the contours in the zero incident frequency limit indicate reflectivity without the sponge. As frequency increases the sponge effect picks up very rapidly and the reflectivity of the sound-vorticity mechanism is much smaller than that of the sound-sound mechanism over the entire range of parameters. Therefore, this mechanism is typically of lesser concern when designing sponges.
3.3. Reflectivity of the vorticity-sound mechanism

For an outflow condition with $M > 0$ it is possible that a portion of the energy of an incident vorticity wave reflects back into the physical domain in terms of a sound wave. Figure 8 shows the results of the reflectivity analysis of the vorticity-sound mechanism.

We note that only the waves with $|\sin \phi| < M$ create propagating acoustic reflection; the $\phi$’s outside of this range would cause reflection in the form of evanescent waves and are excluded from our analysis. Again, similar to what was observed in Figures 3 to 7, sponges perform better at higher frequencies and lower incidence angle.

In physical scenarios, vorticity is less likely to be encountered in the form of “waves”. Instead compact vortices are likely to be found in the domain. The analysis presented...
in Figure 8 needs to be further processed to obtain estimates of reflectivity due to vortex/sponge interactions and hence provide insights for sponge design when compact vortices must leave the domain.

3.4. Vortex/sponge interactions

Consider a single vortex convecting towards an outflow boundary with a sponge. The vortex can be characterized by its diameter and by the shape of its velocity profile; the amplitude does not affect the reflection coefficient in the linear analysis. While the profile shape affects the quantitative details of the sponge/vortex system, we expect that the trends to be independent of the profile and the vortex diameter to be the most important parameter characterizing the behavior. We assume the following velocity profile for the vortex:

$$v_\theta(r) = v_{\text{max}} \frac{r}{b} \exp \left[ \frac{1}{2} \left( 1 - \frac{r^2}{b^2} \right) \right],$$

(3.4)

where \(v_{\text{max}}\) is the maximum circumferential velocity, and \(b\) is the measure of radius of the vortex with \(v_\theta(b) = v_{\text{max}}\). This velocity field can be decomposed into Fourier vorticity modes. With the analysis that was presented in the previous section, one could compute the amplitude of the reflected sound wave for each mode. Combining these reflected modes together provides a quantitative understanding of the reflected sound field due to an incident vortex.

To make the analysis useful from a practical point of view we assign a simple measure of reflectivity to each vortex/sponge interaction event. The natural choice would be to report the ratio of the energy of the vortex which has been reflected as sound.†

Figure 9 shows results of the reflectivity analysis of vortex/sponge interaction over a range of incident Mach numbers. Each plot shows the energy portion of an incident vortex reflected into the physical domain as a function of dimensionless sponge-length. For sponge lengths larger than 0.5–2 vortex diameters (depending on the outflow Mach number) the reflectivity drops rapidly when increasing the sponge size. A power law fit reveals that reflectivity drops as \(l^{-n}\) with \(3 < n < 4\). In the very large \(l_{\text{sp}}\) limit the reflectivity asymptotes to a constant number since the sponge reaches its “ideal” limit similar to what previously observed in Figures 3 and 4 for the sound-sound mechanism. Stronger

† Note that the ratio of the reflected to incident energy flux for each mode involves the amplitude ratio as well as the ratio of group velocities in the \(x\)-direction. Also the reflected acoustic wave involves both pressure and velocity modes which contribute to energy equally.
sponges provide more damping in the asymptotic limit, but have poorer performance in the small to moderate $l_{sp}$ ranges.

4. Sponge design guidelines

The following presents a summary of our findings:

1. Sponge performs ideally for entropy modes, but creates coupling between sound and vorticity modes.

2. For target-dampings in the range 20-60dB, the quadratic sponge provides reasonably “optimal” performance (assuming the use of Thompson boundary condition at the sponge end). For target-dampings beyond 60dB, cubic and higher-order polynomials need to be used for the sponge profile.

3. Sponges perform best when the incident features are aligned with the sponge. In the small angle limit, reflectivity is proportional to the square of the incidence angle.

4. For a given problem and fixed sponge length, there exists an optimal sponge strength leading to the lowest reflectivity; a strong sponge causes its own reflection, whereas a weak sponge does not prevent reflection from the numerical boundary condition.

5. Conversely, for a given problem and a fixed target-damping (translating into fixed desired accuracy), there is a minimum sponge length requirement. This sponge length needs to be the larger of the two lengths imposed due to incident sound and vorticity (discussed below).

6. For effective damping of incident sound waves, minimum dimensionless sponge lengths from $\sim0.5$ to $\sim2$ are recommended. The length is normalized by minimum relevant incident frequency and speed of sound. For higher target-dampings, longer sponges are recommended. This recommendation covers a wide range of incidence angles, with details presented in Figures 5 to 7.

7. For effective damping of an incident vortices, dimensionless sponge lengths above 2 are recommended. This length is normalized by the maximum relevant vortex diameter. Reflectivity of the sponge/vortex mechanism improves rapidly as the length increases faster than the third power of the sponge length. More details are presented in Figure 9.

8. In low-Mach-number CAA applications reflection of a small portion of an out-going vorticity mode can create significant acoustic errors. Therefore, strong sponges with lengths of $\sim10$ vortex diameters or more are recommended. This higher (vortex-based) length demand is also consistent with sound-sound requirements as acoustic wavelengths become much larger than vortex size in the low-Mach limit. However, the sound-sound mechanism is of lesser concern since the sound-sound mechanism can generally be tackled by improving alignment.
Sponge reflectivity

Figure 10. Maximum reflection error for $v$ velocity along the line $x = 0.9$ near the outflow boundary of the vortex/sponge problem. The cost for sponge is $l_{sp}/D_{vort}$ and for PML is $14l_{sp}/5D_{vort}$ due to additional transport equations. The PML data are from Hu et al. (2008).

5. Comparison with perfectly matched layers

In this section we present results from computation of nonlinear Euler equations applied to a single isentropic vortex interacting with a sponge near an outflow boundary. We compare the performance of sponge with that of a perfectly matched layer (PML) using the data published by Hu et al. (2008). In our computation we replicate the same scenario as that considered by Hu et al. (2008) in all physical aspects including domain size, vortex size and profile (same as in Eq. 3.4), and flow parameters.

We considered a vortex with radius $b = 0.2$ in a physical domain bounded by $-1 < x < 1$ and $-1 < y < 1$. The vortex has a maximum rotational Mach number of 0.25 and convects in a uniform flow with $M = 0.5$. We used a sixth-order compact finite difference scheme on a staggered mesh (Lele 1992) to discretize the nonlinear Euler equations in space. The mesh size was selected to be $1/30$ in each direction. The fourth-order Runge-Kutta scheme with acoustic CFL number of 0.2 was used for time stepping. Similar to the approach of Hu et al., a computation on a larger physical domain (by a factor of 2 in the $y$- and a factor of 4 in the $x$-direction) was used as a reference solution to compute the errors. Sponges with different lengths were used outside of the physical domain to damp the vortex. Guided by the results in Figure 9b, we selected sponge strength of 20dB for our tests.

Figure 10 compares the reflectivity performance of the two methods. We note that, for the same length, a PML calculation is generally more expensive than a sponge treatment. This is because PML requires computation of auxiliary transport equations in addition to the primary quantities. For a single boundary (i.e., only in the $x$-direction), PML requires solving for 9 additional quantities and thus is more expensive by a factor of 14/5. Considering this factor, one can see from Figure 10 that the sponge has comparable performance to PML for moderate sponge lengths ($\sim 5$ to $\sim 10$ vortex diameters). The trends suggest that as the length becomes higher (leading to higher accuracy), the sponge becomes more effective than the PML with the same cost.

6. Discussion and summary

In our analysis we have ignored discretization errors. This is a reasonable assumption, as long as the sponge profile is well resolved and the mesh size does not vary rapidly. The presence of highly stretched meshes, which may be used for cost saving in the sponge region, can cause new sources of reflection. As practiced by many researchers, one remedy
is to combine dissipative numerics with grid-stretching in the sponge zone to avoid high wave number reflections, while at the same time saving computational cost.

In summary, we identified major reflectivity mechanisms for sponge/flow interactions and presented a quantitative analysis of these mechanisms using linearized Euler equations. We showed that sponges perform ideally (exponential damping) for both one-dimensional systems and high-frequency features. The choice of sponge profile was shown to have a significant influence on its performance. To achieve damping in the range 20-60dB the quadratic sponge was found to be a practical choice. The system of a sponge plus 1D characteristics-based end boundary condition was extensively analyzed for different reflectivity mechanisms over a wide range of parameters. These sponge performance studies for different inflow/outflow conditions and target-dampings can be used to estimate sponge requirements for practical CFD purposes as guided in Section 4. Our computation of vortex/sponge interactions indicates that accuracies comparable to PML can be achieved with sponges at the same cost. Simple implementation of sponge treatment and its flexibility in handling complex geometries and unstructured meshes makes it an attractive method to treat the external boundaries in general CFD applications.

Acknowledgments

This work was supported by the United States Department of Energy under the Predictive Science Academic Alliance Program (PSAAP) at Stanford University. The author is grateful for the comments by Dr. G. Lodato and Dr. J. Nichols on a draft of the manuscript.

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