Subsonic characteristic outflow boundary conditions with transverse terms: a comparative study

By G. Lodato, F. Ham AND H. Pitsch

1. Motivation and objectives

The accuracy of compressible solvers is known to be strongly sensitive to the boundary solution. Several different techniques have been proposed to deal with the mathematical formulation of the boundary problem, and the definition of non-reflecting frontiers for the most general set of flow conditions is particularly challenging. In the present study, we are interested in the class of nonlinear characteristic boundary conditions. Initially developed for hyperbolic systems of Euler equations, the basic idea on which these techniques are grounded is the reformulation of the system of conservation laws in terms of the characteristic waves traveling in a fixed direction (generally, the normal to the boundary is chosen, even though different directions may be used as proposed by Bayliss & Turkel 1982; Roe 1989). Once a distinction is made between waves that leave and enter the computational domain, the boundary problem is then formulated as a set of numerical and physical boundary conditions to be imposed on outgoing and incoming waves, respectively.

The identification of incoming waves allows a direct control over boundary reflection, as the boundary condition can be designed to prevent incoming perturbations or to damp their amplitude while allowing smooth transients (Hedstrom 1979; Thompson 1990; Rudy & Strikwerda 1980). This technique was extended to the Navier-Stokes equations by Poinsot & Lele (1992) by providing the additional boundary conditions necessary for the viscous terms. This procedure, commonly referred to as Navier-Stokes Characteristic Boundary Conditions (NSCBC), makes explicit use of the hypothesis that the flow field at the boundary may be approximated as inviscid and one-dimensional.

A series of modifications was discussed by different authors in order to account for transverse effects. Based on a low Mach number expansion, Prosser (2005) proposed to decouple convective and acoustic effects and to reformulate the non-reflective conditions for the acoustic length scales only. Based on this analysis, transverse gradients were introduced in the computation of unknown incoming wave amplitude variations. Yoo & Im (2007) showed that the inclusion of transverse convective and viscous terms in the computation of incoming waves improves accuracy and convergence rate, while reducing flow distortion. Following the above idea, in the specific context of structured codes on cartesian grids, Lodato *et al.* (2008) developed a systematic procedure to solve the wave coupling induced by the transverse terms at the edges and corners of the computational domain.

Interestingly, as pointed out by Yoo & Im (2007), in the low Mach number limit, the inclusion of transverse convective terms in the NSCBC procedure leads to a formulation which is similar to the second-order subsonic outflow condition developed by Giles (1988, 1990) using a completely different approach, *viz.* deriving a hierarchy of approximate non-reflecting boundary conditions for the linearized Euler equation as initially proposed by Engquist & Majda (1977) for the wave equation.

In the present study, the definition of the *transverse terms* in relation to the characteristic subsonic non-reflecting outflow is addressed, with particular emphasis on well-posedness and reflection coefficients. Based on the mathematical analysis of reflection coefficients, it will be shown that an optimal *mix* of transverse terms can be determined, thus obtaining a formulation similar to the one proposed by Giles (1990).

2. The characteristic formulation and the transverse terms

Let us consider the local formulation of the Euler equations involving the conservative variables' vector $\widetilde{\boldsymbol{U}}$ and the flux vector in the kth direction $\widetilde{\boldsymbol{F}}^k$. Using Einstein summation convention for repeated indices, these may be written as

$$\partial_t \widetilde{\boldsymbol{U}} + \partial_{x_k} \widetilde{\boldsymbol{F}}^k = \boldsymbol{0}, \quad \text{with} \quad \widetilde{\boldsymbol{U}} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho e \end{pmatrix}, \quad \widetilde{\boldsymbol{F}}^k = \begin{pmatrix} \rho u_k \\ \rho u_1 u_k + \delta_{1k} p \\ \rho u_2 u_k + \delta_{2k} p \\ \rho u_3 u_k + \delta_{3k} p \\ (\rho e + p) u_k \end{pmatrix}. \tag{2.1}$$

In the above equations, ρ represents the fluid density, u_k is the velocity component in the kth direction, p is the pressure, δ_{ik} is the Kronecker delta and e is the total energy. In particular, under the hypothesis of constant specific heat capacities, the total energy e is related to the pressure and the kinetic energy by the following relation:

$$\rho e = \frac{1}{2} \rho u_k u_k + \frac{p}{\gamma - 1}. \tag{2.2}$$

Introducing the Jacobians $\partial \widetilde{\boldsymbol{F}}^k / \partial \widetilde{\boldsymbol{U}}$ and the transformation matrix $\boldsymbol{P} = \partial \widetilde{\boldsymbol{U}} / \partial \boldsymbol{U}$ between primitive variables $\boldsymbol{U}$ and conservative variables $\widetilde{\boldsymbol{U}}$, Eq. (2.1) may be reduced in quasi-linear form as (Hirsch 1990; Thompson 1990; Warming *et al.* 1975):

$$\partial_t \boldsymbol{U} + \boldsymbol{A}^k \partial_{x_k} \boldsymbol{U} = \boldsymbol{0}, \tag{2.3}$$

where $\boldsymbol{A}^k = \boldsymbol{P}^{-1} (\partial \widetilde{\boldsymbol{F}}^k / \partial \widetilde{\boldsymbol{U}}) \boldsymbol{P}$. For the particular choice for $\boldsymbol{U}$ given by the density, the three components of velocity and the pressure, $\boldsymbol{P}$ and $\boldsymbol{A}^k$ become

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 \\ \frac{1}{2} q & \rho u_1 & \rho u_2 & \rho u_3 & 1/\varkappa \end{pmatrix}, \tag{2.4}$$

$$\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -u_1/\rho & 1/\rho & 0 & 0 & 0 \\ -u_2/\rho & 0 & 1/\rho & 0 & 0 \\ -u_3/\rho & 0 & 0 & 1/\rho & 0 \\ \frac{1}{2} q \varkappa & -\varkappa u_1 & -\varkappa u_2 & -\varkappa u_3 & \varkappa \end{pmatrix}, \tag{2.5}$$

and

$$\boldsymbol{A}^k = \begin{pmatrix} u_k & \delta_{1k}\rho & \delta_{2k}\rho & \delta_{3k}\rho & 0 \\ 0 & u_k & 0 & 0 & \delta_{1k}/\rho \\ 0 & 0 & u_k & 0 & \delta_{2k}/\rho \\ 0 & 0 & 0 & u_k & \delta_{3k}/\rho \\ 0 & \delta_{1k}\rho a^2 & \delta_{2k}\rho a^2 & \delta_{3k}\rho a^2 & u_k \end{pmatrix}, \tag{2.6}$$

with $q = u_k u_k$, $a = \sqrt{\gamma p/\rho}$ the local speed of sound and $\varkappa = \gamma - 1$.

In order to better classify the different role of the transverse terms, the characteristic problem will be formulated in the most general case of a non-orthogonal rotated reference frame (see, *e.g.*, Parpia 1991; Kim & Lee 2000, 2004). In the transformed reference system, the different contributions to the transverse terms in the boundary conditions will be analyzed. In particular, two main and distinct contributions of the transverse terms will be identified and it will be shown that both have to be treated in different ways. Furthermore, it will be shown that, in order to obtain a consistent formulation, the full content of the transverse components of the material derivative along the bicharacteristics needs to be included in the boundary condition. This leads to a formulation which resembles the approximate second-order unsteady boundary condition proposed by Giles (1990), but starting from a completely different physical interpretation.

If $\boldsymbol{n}$, $\boldsymbol{t}$ and $\boldsymbol{k}$ represent three directions not necessarily orthogonal nor uniform in space, and x_i are the coordinates in the global orthonormal reference frame, then the relations between the coordinates in the two reference frames are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} n_1 & t_1 & k_1 \\ n_2 & t_2 & k_2 \\ n_3 & t_3 & k_3 \end{pmatrix}}_{\boldsymbol{J}} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \underbrace{\begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \tau_1 & \tau_2 & \tau_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix}}_{\boldsymbol{J}^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2.7}$$

where ξ, η, ζ represent the coordinates in the new reference frame. By applying the chain rule, the gradients in the new reference frames are obtained according to the following equation:

$$\partial_{x_k} = \partial_{x_k}\xi\partial_\xi + \partial_{x_k}\eta\partial_\eta + \partial_{x_k}\zeta\partial_\zeta = J^{-1}_{1k}\partial_\xi + J^{-1}_{2k}\partial_\eta + J^{-1}_{3k}\partial_\zeta. \tag{2.8}$$

Eq. (2.3) is then written in the new reference frame as

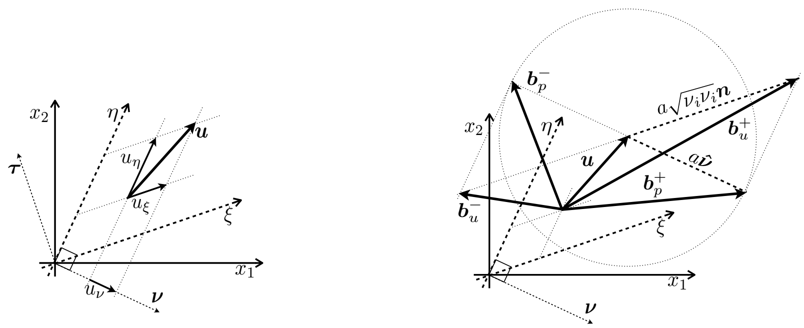
$$\partial_t \boldsymbol{U} + \boldsymbol{\mathcal{A}}^\xi \partial_\xi \boldsymbol{U} + \boldsymbol{\mathcal{A}}^\eta \partial_\eta \boldsymbol{U} + \boldsymbol{\mathcal{A}}^\zeta \partial_\zeta \boldsymbol{U} = \boldsymbol{0}, \tag{2.9}$$

where $\boldsymbol{\mathcal{A}}^\xi = \nu_i \boldsymbol{A}^i$, $\boldsymbol{\mathcal{A}}^\eta = \tau_i \boldsymbol{A}^i$ and $\boldsymbol{\mathcal{A}}^\zeta = \kappa_i \boldsymbol{A}^i$. It should be noted that $\{\boldsymbol{n}, \boldsymbol{t}, \boldsymbol{k}\}$ and $\{\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{\kappa}\}$ represent reciprocal bases, the latter being dual to the former. Hence, the obvious identity $J^{-1}_{ik} J_{kj} = \delta_{ij}$ establishes orthogonality conditions between these two bases, which will be useful in the following developments.

Without losing generality, let assume that the boundary is located at constant ξ and reduce $\boldsymbol{\mathcal{A}}^\xi = \nu_i \boldsymbol{A}^i$ in diagonal form. The relevant eigenvalues may be readily obtained by solving the characteristic equation

$$(u_i\nu_i - \lambda)^3 \left[(u_i\nu_i - \lambda)^2 - \nu_i\nu_i a^2 \right] = 0, \tag{2.10}$$

thus obtaining the eigenvalues $\lambda = u_i\nu_i$ (with multiplicity equal to three), and $\lambda = u_i\nu_i \pm a\sqrt{\nu_i\nu_i}$. In order to obtain the transformation matrix, many different choices are available (Hirsch 1990); nonetheless, a particularly convenient formulation is the one



(a) (b)

FIGURE 1. Projection of a vector $\boldsymbol{u}$ over the axes of the non-orthonormal reference frame (a) and two-dimensional schematic representation of the space-projected bicharacteristics $\boldsymbol{b}_p^{\pm}$ and $\boldsymbol{b}_u^{\pm}$ for the case where the flow is in a subsonic regime (b). The direction of the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ is also indicated showing their orthogonality to the axes $\boldsymbol{t}$ (indicated as η) and $\boldsymbol{n}$ (indicated as ξ), respectively.

proposed by Warming *et al.* (1975), which is here rewritten, slightly modified, as

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} \rho/a & \hat{\nu}_1 & \hat{\nu}_2 & \hat{\nu}_3 & \rho/a \\ -\hat{\nu}_1 & 0 & -\hat{\nu}_3 & \hat{\nu}_2 & \hat{\nu}_1 \\ -\hat{\nu}_2 & \hat{\nu}_3 & 0 & -\hat{\nu}_1 & \hat{\nu}_2 \\ -\hat{\nu}_3 & -\hat{\nu}_2 & \hat{\nu}_1 & 0 & \hat{\nu}_3 \\ \rho a & 0 & 0 & 0 & \rho a \end{pmatrix}, \tag{2.11}$$

$$\boldsymbol{\mathcal{S}}^{-1} = \begin{pmatrix} 0 & -\hat{\nu}_1/2 & -\hat{\nu}_2/2 & -\hat{\nu}_3/2 & 1/(2\rho a) \\ \hat{\nu}_1 & 0 & \hat{\nu}_3 & -\hat{\nu}_2 & -\hat{\nu}_1/a^2 \\ \hat{\nu}_2 & -\hat{\nu}_3 & 0 & \hat{\nu}_1 & -\hat{\nu}_2/a^2 \\ \hat{\nu}_3 & \hat{\nu}_2 & -\hat{\nu}_1 & 0 & -\hat{\nu}_3/a^2 \\ 0 & \hat{\nu}_1/2 & \hat{\nu}_2/2 & \hat{\nu}_3/2 & 1/(2\rho a) \end{pmatrix}, \tag{2.12}$$

where $\hat{\nu}_k = \nu_k/(\nu_i\nu_i)^{1/2}$ is the unit vector obtained by normalizing the vector $\boldsymbol{\nu}$. Note that this vector defines the actual characteristic direction, which is therefore constrained to be always orthogonal to the plane containing $\boldsymbol{t}$ and $\boldsymbol{k}$. If $\boldsymbol{\Lambda}^\xi$ is the diagonal matrix of the eigenvalues of $\boldsymbol{\mathcal{A}}^\xi$, the following identity is readily satisfied:

$$\boldsymbol{\mathcal{A}}^\xi = \boldsymbol{\mathcal{S}}\boldsymbol{\Lambda}^\xi\boldsymbol{\mathcal{S}}^{-1}. \tag{2.13}$$

Combining Eqs. (2.9) and (2.13) and premultiplying by $\boldsymbol{\mathcal{S}}^{-1}$, the following relation is obtained:

$$\boldsymbol{\mathcal{S}}^{-1}\partial_t\boldsymbol{U} + \boldsymbol{\Lambda}^\xi\boldsymbol{\mathcal{S}}^{-1}\partial_\xi\boldsymbol{U} + \boldsymbol{\mathcal{S}}^{-1}\left[\boldsymbol{\mathcal{A}}^\eta\partial_\eta\boldsymbol{U} + \boldsymbol{\mathcal{A}}^\zeta\partial_\zeta\boldsymbol{U}\right] = \boldsymbol{0}, \tag{2.14}$$

which is here synthetically rewritten as

$$\partial_t\boldsymbol{W} + \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}_{\mathrm{t}} - \overline{\boldsymbol{T}} = \boldsymbol{0}. \tag{2.15}$$

The various terms introduced in the above decomposition are defined as follows:

$$\partial_t \boldsymbol{W} = \boldsymbol{\mathcal{S}}^{-1}\partial_t \boldsymbol{U} = \begin{pmatrix} \frac{1}{2}\left[\frac{1}{\rho a}\partial_t p - \partial_t u_\nu\right] \\ \hat{\nu}_1 \partial_t s - \partial_t \omega_1 \\ \hat{\nu}_2 \partial_t s - \partial_t \omega_2 \\ \hat{\nu}_3 \partial_t s - \partial_t \omega_3 \\ \frac{1}{2}\left[\frac{1}{\rho a}\partial_t p + \partial_t u_\nu\right] \end{pmatrix}, \tag{2.16}$$

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{S}}^{-1}\boldsymbol{\mathcal{A}}^\xi \partial_\xi \boldsymbol{U} = \begin{pmatrix} \frac{1}{2}(u_\xi - a\sqrt{\nu_i\nu_i})\left[\frac{1}{\rho a}\partial_\xi p - \partial_\xi u_\nu\right] \\ u_\xi\left[\hat{\nu}_1 \partial_\xi s - \partial_\xi \omega_1\right] \\ u_\xi\left[\hat{\nu}_2 \partial_\xi s - \partial_\xi \omega_2\right] \\ u_\xi\left[\hat{\nu}_3 \partial_\xi s - \partial_\xi \omega_3\right] \\ \frac{1}{2}(u_\xi + a\sqrt{\nu_i\nu_i})\left[\frac{1}{\rho a}\partial_\xi p + \partial_\xi u_\nu\right] \end{pmatrix}, \tag{2.17}$$

$$\boldsymbol{\mathcal{L}}_{\mathrm{t}} = \begin{pmatrix} \frac{1}{2\rho a}(u_\eta - a\hat{\nu}_i\tau_i)\partial_\eta p - \frac{1}{2}u_\eta\partial_\eta u_\nu \\ u_\eta\left[\hat{\nu}_1 \partial_\eta s - \partial_\eta \omega_1\right] \\ u_\eta\left[\hat{\nu}_2 \partial_\eta s - \partial_\eta \omega_2\right] \\ u_\eta\left[\hat{\nu}_3 \partial_\eta s - \partial_\eta \omega_3\right] \\ \frac{1}{2\rho a}(u_\eta + a\hat{\nu}_i\tau_i)\partial_\eta p + \frac{1}{2}u_\eta\partial_\eta u_\nu \end{pmatrix} + \begin{pmatrix} \frac{1}{2\rho a}(u_\zeta - a\hat{\nu}_i\kappa_i)\partial_\zeta p - \frac{1}{2}u_\zeta\partial_\zeta u_\nu \\ u_\zeta\left[\hat{\nu}_1 \partial_\zeta s - \partial_\zeta \omega_1\right] \\ u_\zeta\left[\hat{\nu}_2 \partial_\zeta s - \partial_\zeta \omega_2\right] \\ u_\zeta\left[\hat{\nu}_3 \partial_\zeta s - \partial_\zeta \omega_3\right] \\ \frac{1}{2\rho a}(u_\zeta + a\hat{\nu}_i\kappa_i)\partial_\zeta p + \frac{1}{2}u_\zeta\partial_\zeta u_\nu \end{pmatrix}, \tag{2.18}$$

$$\overline{\boldsymbol{T}} = \begin{pmatrix} -\frac{1}{2}a\left[\tau_i\partial_\eta u_i + \kappa_i\partial_\zeta u_i\right] \\ \frac{1}{\rho}\varepsilon_{1ij}\hat{\nu}_i\tau_j\partial_\eta p + \frac{1}{\rho}\varepsilon_{1ij}\hat{\nu}_i\kappa_j\partial_\zeta p \\ \frac{1}{\rho}\varepsilon_{2ij}\hat{\nu}_i\tau_j\partial_\eta p + \frac{1}{\rho}\varepsilon_{2ij}\hat{\nu}_i\kappa_j\partial_\zeta p \\ \frac{1}{\rho}\varepsilon_{3ij}\hat{\nu}_i\tau_j\partial_\eta p + \frac{1}{\rho}\varepsilon_{3ij}\hat{\nu}_i\kappa_j\partial_\zeta p \\ -\frac{1}{2}a\left[\tau_i\partial_\eta u_i + \kappa_i\partial_\zeta u_i\right] \end{pmatrix}, \tag{2.19}$$

where we have introduced the contravariant components of the velocity vector $\boldsymbol{u}$ in the $\{\boldsymbol{n}, \boldsymbol{t}, \boldsymbol{k}\}$ reference frame (*cf.* Figure 1),

$$u_\xi = \nu_i u_i, \quad u_\eta = \tau_i u_i, \quad u_\zeta = \kappa_i u_i, \tag{2.20}$$

and the arbitrary variations

$$\delta \boldsymbol{W} = \begin{pmatrix} \frac{1}{2}\left[\frac{1}{\rho a}\delta p - \delta u_\nu\right] \\ \hat{\nu}_1 \delta s - \delta \omega_1 \\ \hat{\nu}_2 \delta s - \delta \omega_2 \\ \hat{\nu}_3 \delta s - \delta \omega_3 \\ \frac{1}{2}\left[\frac{1}{\rho a}\delta p + \delta u_\nu\right] \end{pmatrix}, \quad \text{with} \quad \begin{aligned} \delta s &= \delta\rho - \delta p/a^2, \\ \delta u_\nu &= \hat{\nu}_j \delta u_j, \\ \delta \omega_i &= \varepsilon_{ijk}\hat{\nu}_j \delta u_k, \end{aligned} \tag{2.21}$$

where ε_{ijk} is the Levi-Civita symbol.

As it will be detailed below, the $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}_{\mathrm{t}}$ terms account for convective transport, whereas the $\overline{\boldsymbol{T}}$ is a coupling terms between characteristic equations. In order to identify the different nature of the various transverse contributions, we make use of the definition of bicharacteristic, *viz.*, the space-time vector $(\boldsymbol{b}, b_t)$, whose space projection $\boldsymbol{b}$ is not in the direction of $\boldsymbol{\nu}$ and satisfies the condition $\boldsymbol{b}\cdot\boldsymbol{\nu} = \lambda$ (Hirsch 1990). In particular, the four bicharacteristic directions

$$\boldsymbol{b}_p^{\pm} = \boldsymbol{u} \pm a\,\hat{\boldsymbol{\nu}} \quad \text{and} \quad \boldsymbol{b}_u^{\pm} = \boldsymbol{u} \pm a\sqrt{\nu_i\nu_i}\,\boldsymbol{n}, \tag{2.22}$$

schematically represented in Figure 1(b), are of particular relevance in the present case.

Recalling that $\boldsymbol{\nu}$ and $\boldsymbol{n}$ are reciprocal vectors and making use of the orthogonality conditions between dual bases, *i.e.*, $\nu_i n_i = 1$ and $n_i \tau_i = n_i \kappa_i = 0$, it is readily verified that $\boldsymbol{b}_p^{\pm} \cdot \boldsymbol{\nu} = \boldsymbol{b}_u^{\pm} \cdot \boldsymbol{\nu} = u_i \nu_i \pm a\sqrt{\nu_i \nu_i}$ and that the material derivatives along these bicharacteristics are, respectively

$$\frac{\mathrm{d}^{\pm}_{\mathrm{b}p}}{\mathrm{d}t} = \partial_t + (u_k \pm a\hat{\nu}_k)\partial_{x_k} = \partial_t + (u_\xi \pm a\hat{\nu}_i\nu_i)\partial_\xi + (u_\eta \pm a\hat{\nu}_i\tau_i)\partial_\eta + (u_\zeta \pm a\hat{\nu}_i\kappa_i)\partial_\zeta, \tag{2.23}$$

$$\frac{\mathrm{d}^{\pm}_{\mathrm{b}u}}{\mathrm{d}t} = \partial_t + (u_k \pm a\sqrt{\nu_i\nu_i}\, n_k)\partial_{x_k} = \partial_t + (u_\xi \pm a\hat{\nu}_i\nu_i)\partial_\xi + u_\eta\partial_\eta + u_\zeta\partial_\zeta. \tag{2.24}$$

From Eqs. (2.15)–(2.18), it is readily verified that the sum $\partial_t \boldsymbol{W} + \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}_{\mathrm{t}}$ make up the material derivatives of the dependent variables along the above-defined directions. In particular, Eq. (2.15) can be rewritten as

$$\begin{cases} \dfrac{1}{2\rho a}\dfrac{\mathrm{d}^{-}_{\mathrm{b}p}p}{\mathrm{d}t} - \dfrac{1}{2}\dfrac{\mathrm{d}^{-}_{\mathrm{b}u}u_\nu}{\mathrm{d}t} = \overline{T}_1, \\ \hat{\nu}_j\dfrac{\mathrm{d}s}{\mathrm{d}t} - \dfrac{\mathrm{d}\omega_j}{\mathrm{d}t} = \overline{T}_j, \quad (j = 2, 3, 4), \\ \dfrac{1}{2\rho a}\dfrac{\mathrm{d}^{+}_{\mathrm{b}p}p}{\mathrm{d}t} + \dfrac{1}{2}\dfrac{\mathrm{d}^{+}_{\mathrm{b}u}u_\nu}{\mathrm{d}t} = \overline{T}_5, \end{cases} \tag{2.25}$$

where $\mathrm{d}/\mathrm{d}t = \partial_t + u_\xi\partial_\xi + u_\eta\partial_\eta + u_\zeta\partial_\zeta$ is the material derivative along the pseudo-path line. We observe, in particular, that the dependent variables p and u_ν are differentiated along two distinct directions, both lying on the same characteristic surface, as is expected when using non-orthogonal reference frames (Parpia 1991).

This analysis suggests that (a) the two parts of the transverse terms represented by $\boldsymbol{\mathcal{L}}_{\mathrm{t}}$ and $\overline{\boldsymbol{T}}$ are profoundly different in nature; (b) if these two terms are both to be included in the definition of the characteristic boundary condition with some sort of *transverse relaxation*, it is reasonable to expect that different amounts of relaxation may be needed. According to the above considerations and following the development by Yoo & Im (2007), the subsonic non-reflecting outflow condition may be written using two distinct transverse relaxation coefficients, thus leading to the following expression for the incoming wave amplitude variation:

$$\mathcal{L}_1 = K(p - p_\infty) - (1 - \beta_{\mathrm{l}})\mathcal{L}_{\mathrm{t}1} + (1 - \beta_{\mathrm{t}})\overline{T}_1. \tag{2.26}$$

In the above relation, K is the pressure relaxation coefficient computed from the pressure relaxation parameter σ, a typical Mach number of the flow M and a characteristic length-scale L, as $K = \sigma(1 - M^2)/(2\rho L)$ (*cf.* Poinsot & Lele 1992); β_{l} and β_{t} are the relaxation coefficients for $\mathcal{L}_{\mathrm{t}1}$ and $\overline{T}_1$, respectively. Substituting this relation into the first component of Eq. (2.15), the enforced boundary condition becomes

$$\partial_t W_1 + K(p - p_\infty) + \beta_{\mathrm{l}}\mathcal{L}_{\mathrm{t}1} - \beta_{\mathrm{t}}\overline{T}_1 = 0. \tag{2.27}$$

Therefore, when β_{l} is set to 1, the whole contribution from the transverse components of the material derivative along the bicharacteristic is retained when solving the equation for the characteristic variable W_1. As it will be shown in Section 2.1, well-posedness does not depend on the value of β_{l}; nonetheless, choosing $\beta_{\mathrm{l}} \neq 1$, as in the formulation proposed by Yoo & Im (2007), determines a direct coupling between outgoing vorticity modes and incoming acoustic modes, thus negatively affecting the boundary condition's reflection coefficient (see Section 2.2). Conversely, by setting $\beta_{\mathrm{l}} = 1$, reflected acoustic modes by outgoing vorticity modes can be avoided at the leading order, and Eq. (2.26) becomes

$$\mathcal{L}_1 = K(p - p_\infty) + (1 - \beta_\mathrm{t})\overline{T}_1, \tag{2.28}$$

which, written over an orthonormal reference frame with $K = 0$ and β_t equal to the base flow Mach number, is analogous to the approximate second-order unsteady boundary condition proposed by Giles (1988, 1990).

Both formulations, *viz.* $\beta_\mathrm{l} = 1$ and β_l specified as in Yoo & Im, will be tested in the next section. For convenience, the two formulations are summarized below in the two-dimensional case:

$$\mathcal{L}_1 = \sigma \frac{1 - \overline{M}_\nu^2}{2\rho L} \Delta p - \left(1 - \overline{M}_\nu\right) \left[\frac{a}{2} \tau_i \partial_\eta u_i + \frac{u_\eta - a\hat{\nu}_i \tau_i}{2\rho a} \partial_\eta p - \frac{u_\eta}{2} \partial_\eta u_\nu\right], \tag{2.29}$$

$$\mathcal{L}_1 = \sigma \frac{1 - \overline{M}_\nu^2}{2\rho L} \Delta p - \left(1 - \overline{M}_\nu\right) \frac{a}{2} \tau_i \partial_\eta u_i, \tag{2.30}$$

the former obtained from Eq. (2.26) by setting $\beta_\mathrm{l} = \overline{M}_\nu$ as suggested by Yoo & Im (2007) and the latter obtained from the same equation for $\beta_\mathrm{l} = 1$. In the above relations, $M_\nu = u_\xi/(a\sqrt{\nu_i \nu_i}) = u_\nu/a$ is the local Mach number computed from the velocity projection along the characteristic direction $\boldsymbol{\nu}$ and the $\overline{\cdot}$ operator represents averaging over the boundary plane, as it was found to be optimal for both formulations.

2.1. *Analysis of well-posedness*

In this section, well-posedness, according to the theory developed by Kreiss (1970), is analytically proven for the subsonic non-reflecting outflow with transverse terms and pressure relaxation [see Eq. (2.26)]. The following analysis is grounded on the development detailed in Giles (1988) and the same formalism will be adopted hereafter, hence the boundary condition problem will be regarded as related to small perturbations around a uniform steady state around which the Euler equations are linearized. For simplicity, the analysis will be performed in the two-dimensional case over an orthonormal reference frame.

Supposing that the outflow boundary is located on the right of the computational domain and is orthogonal to the x_1 axis, the boundary condition from Eq. (2.27) is

$$\frac{1}{2}\left[\frac{1}{\rho a}\partial_t p - \partial_t u_1\right] + K(p - p_\infty) + \beta_\mathrm{l}\frac{u_2}{2}\left[\frac{1}{\rho a}\partial_{x_2} p - \partial_{x_2} u_1\right] + \beta_\mathrm{t}\frac{a}{2}\partial_{x_2} u_2 = 0. \tag{2.31}$$

Using the optimal value for the transverse relaxation coefficient β_t, *i.e.*, the Mach number, and normalizing with respect to the density and speed of sound of the steady uniform base flow, the above relation becomes

$$[\partial_{t^*}\hat{p}^* - \partial_{t^*}\hat{u}^*] + \alpha\hat{p}^* + \beta_\mathrm{l} v^* [\partial_{y^*}\hat{p}^* - \partial_{y^*}\hat{u}^*] + u^*\partial_{y^*}\hat{v}^* = 0, \tag{2.32}$$

where starred variables represent non-dimensional quantities, $\alpha = 2\rho L\, K$ is a real pressure relaxation coefficient and the *circumflex* accent indicates perturbation quantities. Also note that $u^* = u_1/a$ and $v^* = u_2/a$ and that, as mentioned above, we have set $\beta_\mathrm{t} = u^*$. In order to simplify the notation in the following analysis, the star superscript will be omitted under the implicit assumption that all the quantities are normalized.

Following the procedure described by Giles (1988), we consider the generalized incoming mode, which in the case of a subsonic outflow is a purely acoustic mode in the form†

$$\hat{\boldsymbol{U}}(x,y,t) = a_1 \boldsymbol{u}_1^R \mathrm{e}^{jk_1x}\mathrm{e}^{j(ly-\omega t)}, \tag{2.33}$$

where

$$\boldsymbol{u}_1^R = \frac{1}{2(1+u)}\begin{pmatrix}(1-v\lambda)(1+uS)\\ -(1-v\lambda)(u+S)\\ \lambda(1-u^2)\\ (1-v\lambda)(1+uS)\end{pmatrix}, \tag{2.34}$$

is the relevant right eigenvector,

$$S = \sqrt{1-\frac{(1-u^2)l^2}{(\omega-vl)^2}}, \tag{2.35}$$

$\lambda = l/\omega$, $j^2=-1$ and $\hat{\boldsymbol{U}}$ is the vector of perturbation variables $\hat{\rho}$, $\hat{u}$, $\hat{v}$ and $\hat{p}$ (see Giles 1988, 1990). The corresponding wave number is

$$k_1 = -\frac{(\omega-vl)(S+u)}{1-u^2}. \tag{2.36}$$

Substituting Eq. (2.33) into Eq. (2.32), we obtain the so-called critical matrix, which, in this case, where only one incoming mode is present, reduces to a single scalar value:

$$C = \overline{\boldsymbol{v}}_1^L \cdot \boldsymbol{u}_1^R = 0, \tag{2.37}$$

where $\overline{\boldsymbol{v}}_1^L$ is the approximate left eigenvector corresponding to Eq. (2.32), namely

$$\overline{\boldsymbol{v}}_1^L = \begin{pmatrix}0 & -(1-\beta_{\mathrm{l}}v\lambda) & -u\lambda & \left(1-\beta_{\mathrm{l}}v\lambda-\dfrac{\alpha}{j\omega}\right)\end{pmatrix}. \tag{2.38}$$

Well-posedness is then ensured if C is non-zero for all real l and complex ω with a non-negative imaginary part, *i.e.* if there is no incoming mode satisfying exactly the boundary condition with the exception of the trivial one.

Let us consider a reference frame which moves in the y direction with speed v; as observed by Giles, well-posedness in this reference frame is a necessary and sufficient condition for well-posedness in the original reference frame. Rewriting Eq. (2.37) with $v=0$, we obtain

$$C = \frac{1}{2(1+u)}\left[u+S-u\lambda^2(1-u^2)+(1+uS)\left(1-\frac{\alpha}{j\omega}\right)\right] = 0, \tag{2.39}$$

with $S=\sqrt{1-(1-u^2)\lambda^2}$. As it can be observed, Eq. (2.39) for C does not show any explicit dependence on the value of the transverse relaxation coefficient β_{l}; as a consequence, if well-posedness is proved, this will hold true regardless of the value of β_{l} and, in particular, for the formulation proposed by Yoo & Im (2007) [*cf.* Eq. (2.26)].

After some algebra, Eq. (2.39) may be rewritten as

$$S = \frac{\alpha + j\omega u\lambda^2(1-u^2) - j\omega(1+u)}{j\omega(1+u)-\alpha u}; \tag{2.40}$$

then, using the definition of S and taking the square, the following biquadratic equation

† Note the different indexing convention from Giles' analysis for the modes, which in the present paper are ordered as acoustic (left running for $M<1$), entropic, vortical and acoustic (right running), respectively.

in λ is obtained:

$$\left[u^2\omega^2(1-u^2)\right]\lambda^4 + \left[\omega^2(1-u^2) + \alpha u^2(2j\omega - \alpha)\right]\lambda^2 + \alpha(2j\omega - \alpha) = 0, \tag{2.41}$$

where the relevant discriminant is equal to $\Delta = [\omega^2(1-u^2) - \alpha u^2(2j\omega - \alpha)]^2$. As a consequence, λ^2 may be written as

$$\lambda^2 = \frac{\pm[\omega^2(1-u^2) - \alpha u^2(2j\omega-\alpha)] - [\omega^2(1-u^2) + \alpha u^2(2j\omega - \alpha)]}{2u^2\omega^2(1-u^2)}, \tag{2.42}$$

and the two relevant solutions are readily found as

$$\lambda^2 = -\frac{1}{u^2}, \quad \text{and} \quad \lambda^2 = -\frac{\alpha(2j\omega - \alpha)}{\omega^2(1-u^2)}. \tag{2.43}$$

As mentioned above, well-posedness will be checked for the two roots of λ^2, by proving that Eq. (2.39), with a non-negative imaginary part $\Im(\omega)$, is never satisfied.

Case 1: for $\lambda^2 = -1/u^2$, from the definition of λ and assuming $\Im(\omega) \geq 0$, we get $\omega = ju|l| \in \mathbb{C}$. Hence, the correct branch of the complex square root defining S is $S = 1/u$, *viz.* the one that makes the imaginary part of the wave number k_1 negative, as it can be readily verified by direct substitution into Eq. (2.36). Substituting $S = 1/u$ and $\lambda^2 = -1/u^2$ into Eq. (2.39) we find

$$C = \frac{1}{(1+u)}\left[\frac{1+u}{u} - \frac{\alpha}{j\omega}\right] = 0 \quad \Leftrightarrow \quad \omega = -j\frac{\alpha u}{1+u}. \tag{2.44}$$

Since $u/(1+u) \geq 0$, in agreement with the analysis by Rudy & Strikwerda (1980) for the one-dimensional case, well-posedness is ensured provided that $\alpha > 0$.

Case 2: substituting $\lambda^2 = -\alpha(2j\omega - \alpha)/[\omega^2(1-u^2)]$ into Eq. (2.35) we get

$$S = \sqrt{(\omega + j\alpha)^2/\omega^2}. \tag{2.45}$$

Again, the correct branch must be selected so that $\Im(k_1) < 0$. From the definition of λ and the value of λ^2, we find

$$\omega = j\left[\frac{l^2(1-u^2) - \alpha^2}{2\alpha}\right]. \tag{2.46}$$

Using the above values of S and ω and substituting them into Eq. (2.36), we obtain the two wave numbers

$$k_1^+ = -j\left[\frac{l^2(1+u^2) + \alpha^2}{2\alpha(1+u)}\right] \quad \text{and} \quad k_1^- = j\left[\frac{l^2(1-u^2) + \alpha^2}{2\alpha(1-u)}\right], \tag{2.47}$$

where k_1^+ and k_1^- refer to the positive and negative branches of S, respectively. Taking into account the result from Case 1, *i.e.* $\alpha > 0$, and considering that in the subsonic regime $u \leq 1$, we find that $\Im(k_1^+) < 0$ and $\Im(k_1^-) > 0$, hence the positive branch for S must be considered in order to select an incoming mode:

$$S = (\omega + j\alpha)/\omega. \tag{2.48}$$

Upon substitution of λ^2 and Eq. (2.48) into Eq. (2.39), after some algebra, we arrive at the following result:

$$C = \frac{(\omega + j\alpha)\left[\omega(1+u) + j\alpha u\right]}{\omega^2(1+u)} = 0. \tag{2.49}$$

As a result, $C = 0$ if and only if $\omega = -j\alpha$ or $\omega = -j\alpha u/(1+u)$, thus implying that, again, no incoming mode with $\Im(\omega) \geq 0$ will satisfy exactly the boundary condition from Eq. (2.32), provided that $\alpha > 0$, *i.e.*, the relaxation coefficient for pressure is positive.

In conclusion, we have proved that, provided that a positive relaxation coefficient K is used, the subsonic characteristic boundary condition with transverse effects [see Eq. (2.26)] is well-posed, independently from the value of β_l. The effect of β_l on the behavior of the boundary condition in terms of reflection will be addressed in the next section.

2.2. *Analysis of reflection*

Considering a general solution where both incoming and outgoing modes are present, the matrix of reflection coefficients, relating the amplitudes of incoming modes to those of outgoing modes, may be computed as (Giles 1988)

$$\boldsymbol{R} = -\boldsymbol{C}^{-1}\boldsymbol{D}, \tag{2.50}$$

where $\boldsymbol{C}$ is the critical matrix [see Eq. (2.37)] and the matrix $\boldsymbol{D}$, in particular, is computed as

$$\boldsymbol{D}_{mn} = \overline{\boldsymbol{v}}_m^L \cdot \boldsymbol{u}_n^R, \tag{2.51}$$

from the m approximate left eigenvectors connected to the boundary condition approximation and the n right eigenvectors relevant to the outgoing modes. In the present case of subsonic outflow, the matrix $\boldsymbol{D}$ reduces to a vector of three components representing reflection from outgoing entropic, vortical and acoustic modes, respectively.

In particular, the right eigenvectors of the outgoing modes are (Giles 1988)

$$\boldsymbol{u}_2^R = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_3^R = \begin{pmatrix} 0 \\ -u\lambda \\ 1 - v\lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_4^R = \frac{1}{2(1-u)} \begin{pmatrix} (1-v\lambda)(1-uS) \\ (1-v\lambda)(S-u) \\ \lambda(1-u^2) \\ (1-v\lambda)(1-uS) \end{pmatrix}, \tag{2.52}$$

and the matrix of reflection for the boundary condition from Eq. (2.32) is readily obtained using Eq. (2.38):

$$\boldsymbol{R} = -\frac{1}{C} \begin{pmatrix} 0 & uv\lambda^2(1-\beta_l) & \dfrac{R_p}{2(1-u)} \end{pmatrix}, \tag{2.53}$$

with

$$R_p = (1 - \beta_l v\lambda)(1 - v\lambda)(1 + u)(1 - S) - u\lambda^2(1 - u^2) - \frac{\alpha(1 - v\lambda)(1 - uS)}{j\omega}. \tag{2.54}$$

It is interesting to note, in particular, that entropy waves are transmitted by the boundary condition without any reflection (the first component of $\boldsymbol{R}$ is identically zero). On the other hand, reflection from vorticity and acoustic waves is to be expected. Reflection of vortical modes, in particular, can be avoided by setting $\beta_l = 1$, thus canceling the second component of $\boldsymbol{R}$.

3. Results

We focus our attention on the problem of a convected inviscid vortex, as this simple test case represents a useful source of information about the behavior of the boundary condition when the flow field is characterized by transport of vortical structures as happens, for instance, in turbulent flows. The tests presented have been computed using the CDP code (second-order in space, third-order in time and first-order at the boundary) developed at CTR Stanford (Mahesh *et al.* 2004; Ham & Iaccarino 2004; Ham *et al.* 2007; Moureau *et al.* 2007).

The initial flow field at $t = 0$ is computed resorting to the following analytical solution for the inviscid vortex problem:

$$\frac{p(x_1, x_2, t)}{p_\infty} = \left[\frac{\rho(x_1, x_2, t)}{\rho_\infty}\right]^\gamma = \left[1 - C_v \exp\left(1 - \frac{\Delta x_1^2 + \Delta x_2^2}{R_v^2}\right)\right]^{\gamma/(\gamma-1)}, \tag{3.1}$$

$$u_1(x_1, x_2, t) = u_\infty \left[\cos\theta - \varepsilon_{2\pi} \frac{\Delta x_2}{R_v} \exp\left(\frac{1}{2} - \frac{\Delta x_1^2 + \Delta x_2^2}{2R_v^2}\right)\right], \tag{3.2}$$

$$u_2(x_1, x_2, t) = u_\infty \left[\sin\theta + \varepsilon_{2\pi} \frac{\Delta x_1}{R_v} \exp\left(\frac{1}{2} - \frac{\Delta x_1^2 + \Delta x_2^2}{2R_v^2}\right)\right], \tag{3.3}$$

with

$$C_v = \frac{\varepsilon_{2\pi}^2}{2}(\gamma - 1) M_\infty^2, \tag{3.4}$$

$$\Delta x_1 = x_1 - x_0 - u_\infty t \cos\theta, \tag{3.5}$$

$$\Delta x_2 = x_2 - y_0 - u_\infty t \sin\theta. \tag{3.6}$$

In the above equations, the subscript ∞ indicates the value of the corresponding quantities in the far field, (x_0, y_0) is the initial location of the vortex center, R_v is a measure of the vortex radius and θ defines the direction of the base flow convecting the vortex. The parameter $\varepsilon_{2\pi}$ is used to set the vortex strength.

The results that will be presented below refer to three different values of M_∞, 0.042, 0.42 and 0.8, respectively. With regards to the other parameters of the simulations, the vortex intensity and radius are specified setting $\varepsilon_{2\pi} = 0.08$ and $R_v = 0.1L$, respectively, with $L = 10$ being the side of the square computational domain (64×64 equally spaced cells extending from $-L/2$ to $L/2$ in both directions). The other reference quantities are $\rho_\infty = 1$, $p_\infty = 1$ and $\gamma = 1.4$.

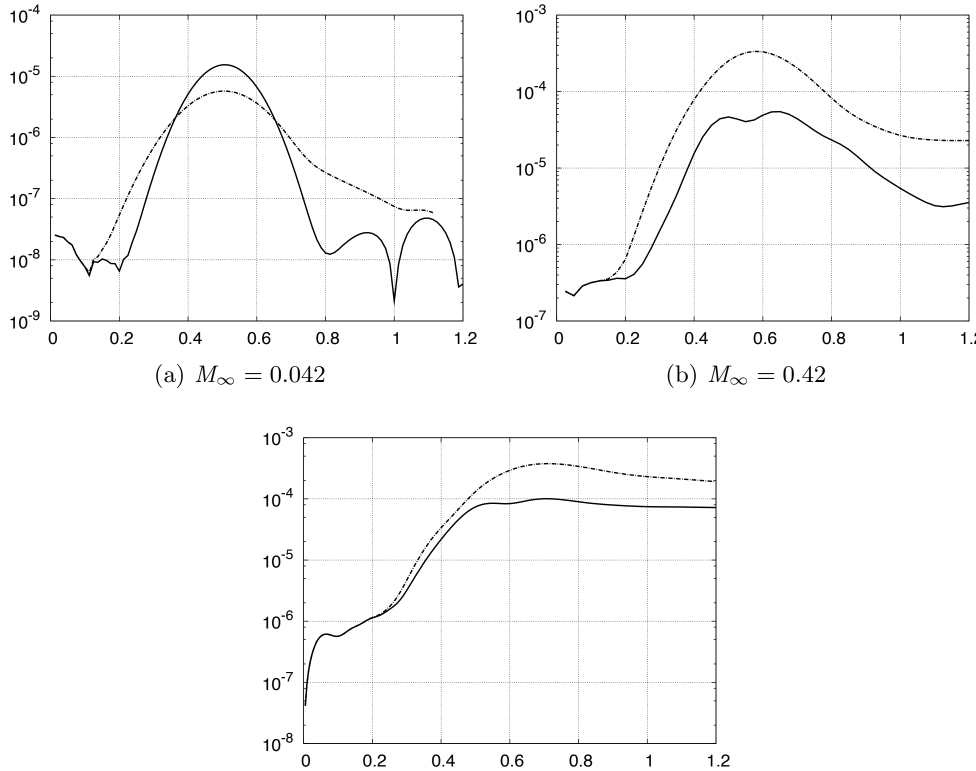
The left, top and bottom boundaries are imposed as inviscid penalty terms (Carpenter *et al.* 1994; Ham *et al.* 2007). The analytical solution from Eqs. (3.1)–(3.6) are used as a benchmark solution to quantify the performance of the outflow boundary condition by means of the global normalized error measure as defined in Prosser (2005) and Lodato *et al.* (2008):

$$\mathcal{E}_{\mathrm{glb}}(\phi, t) = \frac{\left[\sum_{i,j} \left(\phi_{i,j}(t) - \phi_{i,j}^{\mathrm{a}}(t)\right)^2\right]^{1/2}}{\left[\sum_{i,j} \left(\phi_{i,j}^{\mathrm{a}}(0)\right)^2\right]^{1/2}}, \tag{3.7}$$

where the superscript 'a' refers to the analytical solution.

The global error in density is plotted, as a function of the non-dimensional time $t^* = tL/u_\infty$, in Figure 2. The two different formulations for the transverse terms, as per Eqs. (2.29) and (2.30), are compared at the three mentioned Mach numbers with the base flow direction θ set to 12°. In order to better isolate the behavior of the transverse terms' formulation, the pressure relaxation coefficient σ was set to zero. In agreement with the results shown in Section 2.2, the global error at mid/high Mach number is generally reduced when the transverse terms are accounted for according to Eq. (2.30), *i.e.*, when $\beta_l = 1$. At $M = 0.42$, in particular, the error is reduced by about one order of magnitude.

When the base flow is horizontal, no significant differences between the two formu-



(a) $M_\infty = 0.042$

(b) $M_\infty = 0.42$

(c) $M_\infty = 0.8$

FIGURE 2. Evolution of the global error in density as a function of the normalized time $t^* = tL/u_\infty$ for the inviscid convected vortex with $\theta = 12°$: solid line, BC from Eq. (2.30); dashed line, BC from Eq. (2.29).

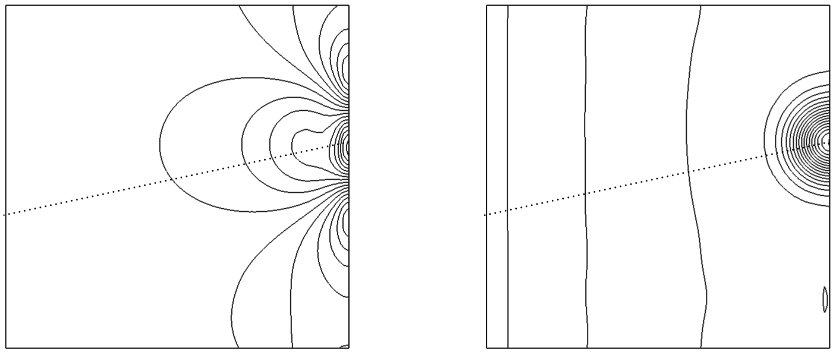
lations of the transverse terms are observed. This result is clearly a direct consequence of the fact that for $\theta = 0°$ the transverse term $\mathcal{L}_{t1}$ becomes negligible and the two formulations tend to be equivalent.

At low Mach [see Figure 2(a)], analysis of the global error does not allow a fair assessment of the two different formulations. The qualitative inspection of the density map, in this case, is much more revealing. In Figure 3, the density map and contours at $t^* = 0.5$, for the test case at $M = 0.042$, show that, despite the slightly higher global error measured, Eq. (2.30) produces a more physical solution, whereas the transverse terms proposed by Yoo & Im (2007) cause significant perturbations to the flow field.

4. Summary and conclusions

A comparative study was presented between different formulations of the subsonic characteristic outflow boundary condition with transverse terms, *i.e.*, a formulation similar to the approximate second-order developed by Giles (1990) and the one proposed by Yoo & Im (2007).

In particular, by properly identifying the different contributions to the transverse terms in the characteristic equations, namely the transverse terms relevant to the material derivatives along the bicharacteristics and the coupling terms between the characteristic



(a) (b)

FIGURE 3. Instantaneous contours of density at $M_\infty = 0.042$ and $t^* = 0.5$ using BC as per Eq. (2.29) (a) and Eq. (2.30) (b). The dotted line indicates the base flow direction.

equations, it was shown that the Giles-like formulation can be obtained by minimizing the reflection coefficient, *viz.*, avoiding coupling between outgoing vorticity modes and incoming acoustic modes. Accordingly, it was concluded that, from a theoretical standpoint, Giles's formulation for the transverse terms represents an optimal solution. Numerical tests on the inviscid convected vortex test case supported this conclusion.

Moreover, the presented formulation of the characteristic equations over generalized non-orthogonal reference frames is readily applicable to unstructured grids with arbitrary alignment of the boundary surface.

Future work will be devoted to the non-trivial reformulation of the edge/corner coupling problem (Lodato *et al.* 2008) when using cartesian structured meshes in conjunction with the redefined transverse terms as described in the present analysis.

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REFERENCES

BAYLISS, A. & TURKEL, E. 1982 Far field boundary conditions for compressible flows. *J. Comput. Phys.* **48** (2), 182–199.

CARPENTER, M., GOTTLIEB, D. & ABARBANEL, S. 1994 Time-stable boundary conditions for finite-difference schemes solving hyperbolic systems: Methodology and application to high-order compact schemes. *J. Comput. Phys.* **111**, 220–236.

ENGQUIST, B. & MAJDA, A. 1977 Absorbing boundary conditions for numerical simulation of waves. *Proc. Natl. Acad. Sci. USA* **74** (5), 1765–1766.

GILES, M. 1988 Non-reflecting boundary conditions for the unsteady Euler equations. *Tech. Rep.* CFDL-TR-88-1. Massachusetts Institute of Technology.

GILES, M. 1990 Nonreflecting boundary conditions for Euler equation calculations. *AIAA J.* **28** (12), 2050–2058.

HAM, F. & IACCARINO, G. 2004 Energy conservation in collocated discretization schemes on unstructured meshes. In *Annual Research Briefs*, pp. 3–14. Center for Turbulence Research, Stanford University.

Ham, F., Mattsson, K., Iaccarino, G. & Moin, P. 2007 Towards time-stable and accurate LES on unstructured grids. *Lect. Notes Comp. Sci.* **56**, 235.

Hedstrom, G. 1979 Nonreflecting boundary conditions for nonlinear hyperbolic systems. *J. Comput. Phys.* **30**, 222–237.

Hirsch, C. 1990 *Numerical Computation of Internal and External Flows, Computational Methods for Inviscid and Viscous Flows,* vol. 2. John Wiley & Sons Ltd.

Kim, J. & Lee, D. 2000 Generalized characteristic boundary conditions for computational aeroacoustics. *AIAA J.* **38** (11), 2040–2049.

Kim, J. & Lee, D. 2004 Generalized characteristic boundary conditions for computational aeroacoustics, part 2. *AIAA J.* **42** (1), 47–55.

Kreiss, H. 1970 Initial boundary value problems for hyperbolic systems. *Commun. Pure Appl. Math.* **23** (3), 277–298.

Lodato, G., Domingo, P. & Vervisch, L. 2008 Three-dimensional boundary conditions for direct and large-eddy simulation of compressible viscous flows. *J. Comput. Phys.* **227** (10), 5105–5143.

Mahesh, K., Constantinescu, G. & Moin, P. 2004 A numerical method for large-eddy simulation in complex geometries. *J. Comput. Phys.* **197** (1), 215–240.

Moureau, V., Bérat, C. & Pitsch, H. 2007 An efficient semi-implicit compressible solver for large-eddy simulations. *J. Comput. Phys.* **226** (2), 1256–1270.

Parpia, I. 1991 Finite-difference method-of-characteristics scheme for the equations of gasdynamics. *AIAA J.* **29** (4), 567–571.

Poinsot, T. & Lele, S. 1992 Boundary conditions for direct simulations of compressible viscous flows. *J. Comput. Phys.* **101**, 104–129.

Prosser, R. 2005 Improved boundary conditions for the direct numerical simulation of turbulent subsonic flows. I. Inviscid flows. *J. Comput. Phys.* **207**, 736–768.

Roe, P. 1989 Remote boundary conditions for unsteady multidimensional aerodynamic computations. *Comput. Fluids* **17** (1), 221–231.

Rudy, D. & Strikwerda, J. 1980 A nonreflecting outflow boundary condition for subsonic Navier-Stokes calculations. *J. Comput. Phys.* **36**, 55–70.

Thompson, K. W. 1990 Time dependent boundary conditions for hyperbolic systems, II. *J. Comput. Phys.* **89**, 439–461.

Warming, R., Beam, R. & Hyett, B. 1975 Diagonalization and simultaneous symmetrization of the gas-dynamic matrices. *Math. Comput.* **29** (132), 1037–1045.

Yoo, C. & Im, H. 2007 Characteristic boundary conditions for simulations of compressible reacting flows with multi-dimensional, viscous and reaction effects. *Combust. Theor. Model.* **11** (2), 259–286.