Characteristic outflows with optimal transverse terms: the edges and corners coupling algorithm

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1. Motivation and objectives

The accuracy of compressible solvers is known to be, in general, strongly sensitive to the boundary solution. The flow properties at the boundaries are obtained by combining (a) numerical boundary conditions pertaining the information coming from the inside of the computational domain, and (b) physical boundary conditions obtained from some approximations of the outside flow features (Hirsch 1990). Based on the above classification, the overall performances of the boundary conditions in terms, in particular, of numerical reflection is affected, for the most part, by what is prescribed by the physical conditions and how the information is combined with their numerical counterpart.

Several different techniques have been proposed to deal with the mathematical formulation of the boundary problem, and the definition of open frontiers for the most general set of flow conditions is particularly challenging (see Givoli 2004; Colonius 2004; Nicoud 1999 and references cited therein for a review). The present study focuses on the class of characteristic boundary conditions which are obtained by reformulating the system of conservation laws in terms of the characteristic waves traveling in a fixed direction (generally, the normal to the boundary is chosen, even though different directions may be used as proposed by Bayliss & Turkel 1982; Roe 1989). Once a distinction is made between waves that leave and enter the computational domain, the boundary problem can be formulated as a set of numerical and physical boundary conditions to be imposed on outgoing and incoming waves, respectively.

The identification of incoming waves allows some level of control over boundary reflection, as the boundary condition can be designed to minimize incoming perturbations or to damp their amplitude while allowing smooth transients (Hedstrom 1979; Thompson 1987, 1990; Rudy & Strikwerda 1980). This technique was extended to the Navier-Stokes equations by Poinsot & Lele (1992) by providing the additional boundary conditions necessary for the viscous terms. This procedure is commonly referred to as Navier-Stokes Characteristic Boundary Conditions (NSCBC) and makes explicit use of the hypothesis that the flow field at the boundary may be approximated as inviscid and one-dimensional.

In order to overcome one of the main weaknesses of the NSCBC approach, i.e., the one-dimensional hypothesis, which may lead to significant numerical reflection for strongly three-dimensional flow fields, a series of modifications was discussed by different authors. Addressing directly the role played by the transverse convective and viscous terms, as well as the source terms in the case of reactive flows, Yoo et al. (2005) and Yoo & Im (2007) showed that an appropriate treatment of these terms in the computation of incoming waves improves the accuracy and convergence rate toward target values for selected relaxed quantities while reducing flow distortion. Following the above idea, in the specific context of structured codes on cartesian grids, a systematic procedure to solve transverse terms-induced wave coupling at the edges and corners of the computational domain was proposed by Lodato et al. (2008), leading to the 3D-NSCBC formulation.
A thorough analysis of the subsonic characteristic outflow boundary condition with transverse terms has been recently carried out by Lodato et al. (2010, 2011). By properly identifying the different contributions to the transverse terms in the characteristic equations, namely the transverse terms relevant to the material derivatives along the bi-characteristics and the coupling terms between the characteristic equations, a unifying formulation involving three relaxation parameters was presented. Resorting to the normal mode analysis, a parametric analysis of well-posedness was performed in order to establish the admissible bounds of the relaxation coefficients. Furthermore, inspection of the relevant matrix of reflection demonstrated that an optimal formulation—similar to the second-order subsonic outflow condition developed by Giles (1990)—that minimizes coupling between outgoing vorticity modes and incoming acoustic modes can be obtained.

Numerical tests on the inviscid convected vortex test case over different computational grids, for different base flow directions and at different Mach numbers, demonstrated that by using the above-mentioned optimal formulation, the boundary-induced numerical errors can be maintained relatively constant, regardless of the base flow direction with respect to the boundary plane. Comparable errors were also measured even when the computational mesh was characterized by a curved boundary, thus suggesting that even if boundary numerical perturbations cannot be completely removed due to the intrinsic limitations of the characteristic boundary conditions approach, the optimal formulation can be used to keep the errors under control even when the character of the flow at the boundary is difficult or impossible to be anticipated. The edges and corners coupling procedure for cartesian meshes proposed by Lodato et al. (2008) is here extended to this optimal formulation.

2. The NSCBC approach

Let us consider the local formulation of the Euler equations involving the vector of conservative variables $\tilde{U}$ and the flux vector in the $k$th direction $\tilde{F}^k$. Using Einstein summation convention for repeated indices, these may be written as

$$ \frac{\partial \tilde{U}}{\partial t} + \frac{\partial \tilde{F}^k}{\partial x_k} = 0, \quad (2.1) $$

with

$$ \tilde{U} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho e \end{pmatrix}, \quad \tilde{F}^k = \begin{pmatrix} \rho u_k \\ \rho u_1 u_k + \delta_{1k} p \\ \rho u_2 u_k + \delta_{2k} p \\ \rho u_3 u_k + \delta_{3k} p \\ (\rho e + p) u_k \end{pmatrix}. \quad (2.2) $$

In the above equations, $\rho$ represents the fluid density, $u_k$ is the velocity component in the $k$th direction, $p$ is the pressure, $\delta_{ik}$ is the Kronecker’s delta and $e$ is the total energy, defined by the following relation:

$$ \rho e = \frac{1}{2} \rho u_k u_k + \frac{p}{\gamma - 1}. \quad (2.3) $$

Introducing the Jacobians $\partial \tilde{F}^k / \partial \tilde{U}$ and the transformation matrix $P = \partial \tilde{U} / \partial U$ between primitive variables $U$ and conservative variables $\tilde{U}$, Eq. (2.1) may be reduced in
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quasi-linear form as (Hirsch 1990; Thompson 1987, 1990; Warming et al. 1975):

\[ \frac{\partial U}{\partial t} + A^k \frac{\partial U}{\partial x_k} = 0. \tag{2.4} \]

In particular, if the density, the three components of velocity and the pressure are chosen as primitive variables, the Jacobians \( P \) and \( A^k \) become

\[ P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
u_1 & \rho & 0 & 0 & 0 \\
u_2 & 0 & \rho & 0 & 0 \\
u_3 & 0 & 0 & \rho & 0 \\
\frac{1}{2}u_k u_k & \rho u_1 & \rho u_2 & \rho u_3 & 1/\kappa
\end{pmatrix}, \tag{2.5} \]

\[ P^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-u_1/\rho & 1/\rho & 0 & 0 & 0 \\
-u_2 & 0 & 1/\rho & 0 & 0 \\
-u_3/\rho & 0 & 0 & 1/\rho & 0 \\
\frac{1}{2}u_k u_k + \kappa & -\kappa u_1 & -\kappa u_2 & -\kappa u_3 & \kappa
\end{pmatrix}, \tag{2.6} \]

where \( \kappa = \gamma - 1 \), and

\[ A^k = \begin{pmatrix}
u_k & \delta_{1k} \rho & \delta_{2k} \rho & \delta_{3k} \rho & 0 \\
0 & u_k & 0 & 0 & \delta_{1k} / \rho \\
0 & 0 & u_k & 0 & \delta_{2k} / \rho \\
0 & 0 & 0 & u_k & \delta_{3k} / \rho \\
0 & \delta_{1k} \rho a^2 & \delta_{2k} \rho a^2 & \delta_{3k} \rho a^2 & u_k
\end{pmatrix}, \tag{2.7} \]

where \( a = \sqrt{\gamma p/\rho} \) is the speed of sound.

In order to build characteristic boundary conditions, Eq. (2.4) is first rewritten into a form that makes explicit the characteristic signals propagating along a specified direction, e.g. normal to the boundary. Physical boundary conditions are then imposed on characteristic waves coming from the outside of the domain, based on some information about the exterior domain (Thompson 1987, 1990; Poinsot & Lele 1992; Yoo & Im 2007; Lodato et al. 2008). In order to do so, considering for simplicity a plane boundary orthogonal to the \( x_1 \) direction, the \( A^1 \) matrix is first diagonalized using the following transformation:

\[ \Lambda = S_1^{-1} A^1 S_1, \tag{2.8} \]

where \( \Lambda = \text{diag}(u_1 - a, u_1, u_1, u_1 + a) \) and \( S_1^{-1} \) and \( S_1 \) are the relevant transformation matrices built with the left and right eigenvectors of \( A^1 \):

\[ S_1 = \begin{pmatrix}
1/(2a^2) & 1/a^2 & 0 & 0 & 1/(2a^2) \\
-1/(2\rho a) & 0 & 0 & 0 & 1/(2\rho a) \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1/2 & 0 & 0 & 0 & 1/2
\end{pmatrix}, \tag{2.9} \]

\[ S_1^{-1} = \begin{pmatrix}
0 & -\rho a & 0 & 0 & 1 \\
0 & a^2 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & \rho a & 0 & 0 & 1
\end{pmatrix}. \tag{2.10} \]

Hence, the characteristic waves may be made explicit by combining Eqs. (2.4) and (2.8)
as

\[
\frac{\partial U}{\partial t} + S_1 \mathbf{L} + A^t \frac{\partial U}{\partial x_t} = 0, \tag{2.11}
\]

where we have introduced the wave amplitude variations \( \mathbf{L} = \Lambda S_1 \frac{\partial U}{\partial x_1} \) and the repeated \( t \) index means summation over the transverse directions \( x_2 \) and \( x_3 \).

The vector \( \mathbf{L} \) is a function of the gradient of the solution along \( x_1 \) only; furthermore, each wave \( L_i \) is characterized by a certain propagation velocity \( \lambda_i \), i.e. the corresponding diagonal component of the matrix \( \Lambda \). For a subsonic flow, three of these waves travel with the local velocity \( u_1 \), namely the entropy and the two shear waves (in the present case, \( L_2, L_3 \), and \( L_4 \), respectively), whereas the two acoustic signals are always left- and right-going (\( L_1 \) and \( L_5 \), respectively). If we focus our attention to an outflow boundary on the right of the domain, i.e. \( u_1 \geq 0 \) and \( x_1 = L \), there is one incoming wave \( L_1 \) and four outgoing waves. The boundary condition problem is reduced to the determination of the incoming one from some knowledge about the exterior domain.

Assuming, for instance, that the flow is one-dimensional at the boundary, Eq. (2.11) may be simplified as

\[
S_1^{-1} \frac{\partial U}{\partial t} + \mathbf{L} = 0, \tag{2.12}
\]

which may be used to derive useful relations between outgoing and incoming waves and the time derivative of the primitive variables (Thompson 1987, 1990; Poinset & Lele 1992). Using Eq. (2.12) and the results from Rudy & Strikwerda (1980), Poinset & Lele (1992) derived the subsonic outflow boundary condition

\[
\mathbf{L}_1 = \sigma \frac{a(1 - \text{Ma}^2)}{L} (p - p_\infty), \tag{2.13}
\]

where \( \sigma \) is a relaxation coefficient, \( \text{Ma} \) is a typical Mach number for the flow field under study, \( L \) is a reference length and \( p_\infty \) is the reference outflow pressure. The Thompson’s non-reflecting outflow condition may be recovered from the above relation by setting \( \sigma = 0 \).

If transverse effects are not negligible, Eq. (2.12) becomes too stringent to correctly reproduce the flow field at the boundary without numerical reflection; hence, a more general formulation may be obtained based on the use of Eq. (2.11), as proposed by Yoo et al. (2005); Yoo & Im (2007). Eq. (2.11) is rewritten as

\[
S_1^{-1} \frac{\partial U}{\partial t} + \mathbf{L} = \mathbf{T}, \tag{2.14}
\]

where \( \mathbf{T} = -S_1^{-1} A^t \frac{\partial U}{\partial x_t} \) (\( t = 2, 3 \)). Using again the condition proposed by Rudy & Strikwerda (1980), the boundary condition for \( \mathbf{L}_1 \) then becomes

\[
\mathbf{L}_1 = \sigma \frac{a(1 - \text{Ma}^2)}{L} (p - p_\infty) + (1 - \beta_t) T_1^t + \beta_t T_1^{tx}, \tag{2.15}
\]

where \( \beta_t \) is a transverse relaxation coefficient and \( T_1^{tx} \) is a known estimation of the transverse term \( T_1^t \) toward which this last is relaxed.

Eq. (2.15) is quite effective in preventing significant reflection at low Mach number, giving interesting results in the classical inviscid vortex convection test case, as well as in other more complex flows (Yoo et al. 2005; Yoo & Im 2007; Lodato et al. 2008). Nonetheless, as pointed out by Lodato et al. (2008), when the Mach number is increased, Eq. (2.15) becomes less and less effective, and stronger perturbations are observed at the outflow.
3. Optimal inclusion of transverse terms

In a recent work by Lodato et al. (2010), the role played by different transverse terms in the characteristic equations was carefully analyzed in the most general case of non-orthogonal reference frames. The main idea was to make a distinction between the transverse terms relevant to the material derivatives along the bi-characteristics and the coupling terms between the characteristic equations. Grounded on this fundamental distinction between different contributions to the transverse terms, it was demonstrated by normal mode analysis (Kreiss 1970) that there exists an optimal way to include the transverse terms in the formulation of the subsonic non-reflecting outflow condition such that coupling between outgoing vorticity waves and incoming acoustic waves can be minimized, thus allowing a significant reduction of numerical reflection, especially when the base flow is not orthogonal to the boundary. The idea behind the above-mentioned classification of transverse effects is briefly summarized below for the simplest case of an orthogonal reference frame, which is particularly relevant for the generalization of the edges and coupling algorithm for cartesian grids that will be detailed in the next section.

Premultiplying Eq. (2.4) by \( S_1^{-1} \) we get

\[
\frac{\partial W}{\partial t} + \mathcal{L} + \mathcal{L}_t - T = 0, \tag{3.1}
\]

where

\[
\delta W = S_1^{-1} \delta U = \begin{pmatrix}
\delta p - \rho a \delta u_1 \\
a^2 \delta p - \delta p \\
\delta u_2 \\
\delta u_3 \\
\delta p + \rho \delta u_1
\end{pmatrix}, \tag{3.2}
\]

represents a small increment of the characteristic variables \( W \) and \( \mathcal{L} \) is defined as before, i.e.

\[
\mathcal{L} = S_1^{-1} A \frac{\partial U}{\partial x_1} = \begin{pmatrix}
(u_1 - a) \left( \frac{\partial p}{\partial x_1} - \rho a \frac{\partial u_1}{\partial x_1} \right) \\
u_1 \left( a^2 \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_1} \right) \\
u_1 \frac{\partial u_2}{\partial x_1} \\
u_1 \frac{\partial u_3}{\partial x_1} \\
(u_1 + a) \left( \frac{\partial p}{\partial x_1} + \rho a \frac{\partial u_1}{\partial x_1} \right)
\end{pmatrix} = \Lambda \frac{\partial W}{\partial x_1}. \tag{3.3}
\]

The other two terms are defined as

\[
\mathcal{L}_t = \begin{pmatrix}
\frac{\partial}{\partial x_1} \left( u_1 \left( \frac{\partial p}{\partial x_1} - \rho a \frac{\partial u_1}{\partial x_1} \right) \right) \\
\frac{\partial}{\partial x_1} \left( u_1 \left( a^2 \frac{\partial p}{\partial x_1} - \frac{\partial p}{\partial x_1} \right) \right) \\
\frac{\partial}{\partial x_1} \left( u_1 \frac{\partial u_2}{\partial x_1} \right) \\
\frac{\partial}{\partial x_1} \left( u_1 \frac{\partial u_3}{\partial x_1} \right) \\
\frac{\partial}{\partial x_1} \left( u_1 \left( \frac{\partial p}{\partial x_1} + \rho a \frac{\partial u_1}{\partial x_1} \right) \right)
\end{pmatrix} = u_1 \frac{\partial W}{\partial x_1}, \quad T = \begin{pmatrix}
-\rho a^2 \frac{\partial u_1}{\partial x_1} \\
0 \\
-1 \frac{\partial p}{\partial x_2} \\
-1 \frac{\partial p}{\partial x_3} \\
-\rho a^2 \frac{\partial u_1}{\partial x_1}
\end{pmatrix}, \tag{3.4}
\]

where the repeated index \( t \) stands for summation over the transverse components 2 and 3. The advantage of the above reformulation is that if we now introduce the material derivative along the bi-characteristic \( (u \pm an) \) (Hirsch 1990)

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (u_j \pm an_j) \frac{\partial}{\partial x_j}, \tag{3.5}
\]
and the usual material derivative \( \frac{d}{dt} \) along a streamline, Eq. (3.1) can be rearranged as

\[
\begin{align*}
\frac{d}{dt} W_1 &= T_1 = -\rho a_2^2 \frac{\partial u_t}{\partial x_t}, \\
\frac{d}{dt} W_2 &= T_2 = 0, \\
\frac{d}{dt} W_3 &= T_3 = -\frac{1}{\rho} \frac{\partial p}{\partial x_2}, \\
\frac{d}{dt} W_4 &= T_4 = -\frac{1}{\rho} \frac{\partial p}{\partial x_3}, \\
\frac{d}{dt} W_5 &= T_5 = -\rho a_2^2 \frac{\partial u_t}{\partial x_1},
\end{align*}
\]

where all the transverse effects connected with the material derivatives are grouped inside the vector \( L_t \), whereas the vector \( T \) represents the coupling term between the characteristic equations.

Given the profoundly different nature of the transverse terms represented by \( L_t \) and \( T \), it is argued that different amounts of relaxation may be needed when defining characteristic boundary condition with transverse relaxation. Therefore, starting from Eq. (2.15) and supposing for simplicity that \( T_{\text{ex}} \) is identically zero, such a distinction is readily obtained by splitting \( T_1 \) as \( T_{1L} - L_t \) and by using different relaxation coefficients. The boundary condition for \( L_1 \) then becomes

\[
L_1 = K(p - p_{\infty}) - (1 - \beta_t)L_{1L} + (1 - \beta_t)T_1, \quad \text{(3.7)}
\]

where we have set \( K = \sigma a(1 - Ma^2)/L \) and we have defined \( \beta_t \) as the relaxation coefficient for \( L_{1L} \).

In the work by Lodato et al. (2010, 2011), the analysis of well-posedness and reflection, and a series of numerical tests on the convected vortex at different angles and different Mach numbers led to the conclusion that, in order to minimize the coupling between outgoing vorticity modes and incoming acoustic modes, the relaxation coefficient \( \beta_t \) should be set equal to 1, thus leading to the optimal definition of the incoming wave for the subsonic outflow boundary condition

\[
L_1 = K(p - p_{\infty}) + (1 - \beta_t)T_1. \quad \text{(3.8)}
\]

Replacing \( L_1 \) from the above definition into the first component of Eq. (3.1) leads to the optimal boundary condition below:

\[
\left[ S^{-1} \frac{\partial U}{\partial t} \right]_1 + K(p - p_{\infty}) + L_{1L} - \beta_t T_1 = 0. \quad \text{(3.9)}
\]

The above distinction between different transverse terms carries along modifications to the edges and coupling algorithm proposed by Lodato et al. (2008). These modifications are detailed in the next section.

4. Edges and corners coupling algorithm

The edges and corners coupling procedure (Lodato et al. 2008) is here adapted to the optimal formulation from Eq. (3.8). In order to derive the coupling relations for edges and corners, the Jacobians \( A^k \) are first conveniently decomposed such that the two different
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Contributions to the transverse terms are readily obtained. In fact, it is readily verified that if $A^k$ is decomposed as

$$A^k = A^k_d + A^k_t,$$

(4.1)

with $A^k_d = u_k I$, where $I$ is the identity matrix, and $A^k_t = A^k - A^k_d$, then the two different contributions to the transverse terms are obtained as [see Eq. (3.4)]

$$L_t = S_1^{-1} \left( A_2^d \frac{\partial U}{\partial x_2} + A_3^d \frac{\partial U}{\partial x_3} \right),$$

(4.2)

$$-T^1 = S_1^{-1} \left( A_1^t \frac{\partial U}{\partial x_1} + A_3^t \frac{\partial U}{\partial x_3} \right).$$

(4.3)

Let us consider the edge joining two subsonic outflows that are orthogonal to $x_1$ and $x_2$, and let us define the wave amplitude variations relevant to modes propagating along $x_1$ and $x_2$ as

$$L = S_1^{-1} A_1 \frac{\partial U}{\partial x_1} \text{ and } M = S_2^{-1} A_2 \frac{\partial U}{\partial x_2},$$

(4.4)

respectively. Premultiplying Eq. (2.4) by $S_1^{-1}$ and $S_2^{-1}$, after some manipulation involving the use of Eqs. (4.1) and (4.4), the following relations are obtained:

$$S_1^{-1} \frac{\partial U}{\partial t} + L + L_t + S_1^{-1} \left( S_2 M - A_2^d \frac{\partial U}{\partial x_2} + A_3^d \frac{\partial U}{\partial x_3} \right) = 0,$$

(4.5)

$$S_2^{-1} \frac{\partial U}{\partial t} + M + M_t + S_2^{-1} \left( S_1 L - A_1^d \frac{\partial U}{\partial x_1} + A_3^d \frac{\partial U}{\partial x_3} \right) = 0,$$

(4.6)

where

$$L_t = S_1^{-1} \left( A_2^d \frac{\partial U}{\partial x_2} + A_3^d \frac{\partial U}{\partial x_3} \right),$$

(4.7)

$$M_t = S_2^{-1} \left( A_1^d \frac{\partial U}{\partial x_1} + A_3^d \frac{\partial U}{\partial x_3} \right).$$

(4.8)

Assuming that the location of the boundaries is such that $L_1$ and $M_1$ are the unknown wave amplitude variations (see Lodato et al. 2008, Section 3.2.1.), the actual boundary conditions from Eq. (3.9) are

$$\left[ S_1^{-1} \frac{\partial U}{\partial t} \right]_1 + K(p - p_\infty) + L_{t1} - \beta_1 T^1_1 = 0,$$

(4.9)

$$\left[ S_2^{-1} \frac{\partial U}{\partial t} \right]_1 + K(p - p_\infty) + M_{t1} - \beta_2 T^2_1 = 0,$$

(4.10)

which, coupled with the first components of Eqs. (4.5) and (4.6), lead to the following system in the two unknown waves for the edge:

$$L_1 = K(p - p_\infty) + (1 - \beta_1) T^1_1,$$

(4.11)

$$M_1 = K(p - p_\infty) + (1 - \beta_2) T^2_1,$$

(4.12)
with
\[ T_1^1 = -\frac{1}{2}(M_1 + M_5 - M_4) + \frac{u_2}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_2} - \frac{\partial u_1}{\partial x_2} \right] - a \frac{\partial u_3}{2 \partial x_3}, \]
\[ T_1^2 = -\frac{1}{2}(L_1 + L_5 + L_4) + \frac{u_1}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_1} - \frac{\partial u_2}{\partial x_1} \right] - a \frac{\partial u_3}{2 \partial x_3}. \]

After grouping the unknown terms to the left, the linear system to be solved at the edge is
\[ \begin{cases} \mathcal{L}_1 + \frac{1 - \beta_2}{2} M_1 = K(p - p_\infty) + (1 - \beta_1) \tilde{T}_1^1 \\ \frac{1 - \beta_2}{2} \mathcal{L}_1 + M_1 = K(p - p_\infty) + (1 - \beta_1) \tilde{T}_1^2 \end{cases} \]
where \( \tilde{T}_1^1 \) and \( \tilde{T}_1^2 \) are now defined as
\[ \tilde{T}_1^1 = -\frac{1}{2}(M_5 - M_4) + \frac{u_2}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_2} - \frac{\partial u_1}{\partial x_2} \right] - a \frac{\partial u_3}{2 \partial x_3}, \]
\[ \tilde{T}_1^2 = -\frac{1}{2}(L_5 + L_4) + \frac{u_1}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_1} - \frac{\partial u_2}{\partial x_1} \right] - a \frac{\partial u_3}{2 \partial x_3}. \]

The corner coupling system can be easily obtained by analogy to what is done for the edge and is summarized below. After defining the wave amplitude variations relevant to modes propagating along \( x_3 \) as
\[ \mathcal{N} = S_3^{-1} A^3 \frac{\partial U}{\partial x_3}, \]
and supposing that the location of the boundaries is such that \( \mathcal{L}_1, M_1 \) and \( N_1 \) are the unknown wave amplitude variations, the modified transverse terms are
\[ \bar{T}^1 = -S_1^{-1} \left( S_2 M + S_3 N - A_2^3 \frac{\partial U}{\partial x_2} - A_3^3 \frac{\partial U}{\partial x_3} \right), \]
\[ \bar{T}^2 = -S_2^{-1} \left( S_1 \mathcal{L} + S_3 N - A_3^1 \frac{\partial U}{\partial x_1} - A_3^3 \frac{\partial U}{\partial x_3} \right), \]
\[ \bar{T}^3 = -S_3^{-1} \left( S_1 \mathcal{L} + S_2 M - A_1^d \frac{\partial U}{\partial x_1} - A_3^d \frac{\partial U}{\partial x_3} \right), \]
and the linear system to be solved at the corner is
\[ \begin{cases} \mathcal{L}_1 + \frac{1 - \beta_2}{2} M_1 + \frac{1 - \beta_2}{2} N_1 = K(p - p_\infty) + (1 - \beta_1) \tilde{T}_1^1 \\ \frac{1 - \beta_2}{2} \mathcal{L}_1 + M_1 + \frac{1 - \beta_2}{2} N_1 = K(p - p_\infty) + (1 - \beta_1) \tilde{T}_1^2 \\ \frac{1 - \beta_2}{2} \mathcal{L}_1 + \frac{1 - \beta_2}{2} M_1 + N_1 = K(p - p_\infty) + (1 - \beta_1) \tilde{T}_1^3 \end{cases} \]
where \( \tilde{T}_1^1, \tilde{T}_1^2 \) and \( \tilde{T}_1^3 \) are now defined as
\[ \tilde{T}_1^1 = -\frac{1}{2}(M_5 + N_5 - M_4 + N_3) + \sum_{k=2,3} \frac{u_k}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_k} - \frac{\partial u_1}{\partial x_k} \right], \]
\[ \tilde{T}_1^2 = -\frac{1}{2}(L_5 + N_5 + L_4 - N_2) + \sum_{k=1,3} \frac{u_k}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_k} - \frac{\partial u_2}{\partial x_k} \right], \]
\[ \tilde{T}_1^3 = -\frac{1}{2}(L_5 + M_5 - L_3 + M_2) + \sum_{k=1,2} \frac{u_k}{2} \left[ \frac{1}{\rho a} \frac{\partial p}{\partial x_k} - \frac{\partial u_3}{\partial x_k} \right]. \]
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Figure 1. Evolution of the global error in pressure as a function of the normalized time $t^* = t \frac{2u_\infty}{L}$ (a) and instantaneous pressure map with vorticity isolines (b): , BC as per Eq. (3.9); , 3D-NSCBC (Lodato et al. 2008); , BC by Thompson (1990). The pressure map is represented over a quarter of the domain.

The above coupling algorithm is validated in the next section. In particular, the results obtained by using Eq. (3.9) in conjunction with the edge coupling algorithm are compared with the 3D-NSCBC formulation (Lodato et al. 2008) and the locally one-dimensional counterparts (Thompson 1990; Poinsot & Lele 1992).

5. Results and discussion

5.1. Gaussian vortex test

The present section addresses the specific problem of the transmission of vorticity through the corner of the computational domain. This is achieved by looking at the case of a convected Gaussian vortex convected at an angle of 45°. This is a very challenging test for boundary condition and it is generally advisable not to have complex flow structures crossing the edges and the corners of the grid.

The initial flow field at $t = 0$ was computed resorting to the following analytical solution for the inviscid vortex problem:

\[
\frac{p(x, t)}{p_\infty} = \left[ \frac{p(x, t)}{p_\infty} \right]^{\gamma} = \left[ 1 - C_v \exp \left( 1 - \frac{\Delta x_1^2 + \Delta x_2^2}{R_v^2} \right) \right]^{\gamma/(\gamma - 1)},
\]

\[
u_1(x_1, x_2, t) = u_\infty \left[ \frac{\sqrt{2}}{2} - \epsilon \frac{\Delta x_2}{R_v} \exp \left( \frac{1}{2} - \frac{\Delta x_1^2 + \Delta x_2^2}{2R_v^2} \right) \right],
\]

\[
u_2(x_1, x_2, t) = u_\infty \left[ \frac{\sqrt{2}}{2} + \epsilon \frac{\Delta x_1}{R_v} \exp \left( \frac{1}{2} - \frac{\Delta x_1^2 + \Delta x_2^2}{2R_v^2} \right) \right],
\]

with $C_v = 0.5 \epsilon^2 (\gamma - 1) M_{\infty}^2$, $\Delta x_1 = x_1 - x_0 - u_\infty t \sqrt{2}/2$ and $\Delta x_2 = x_2 - y_0 - u_\infty t \sqrt{2}/2$. $M_{\infty}$ was set at 0.42, whereas the vortex intensity and radius were specified setting $\epsilon = 0.08$ and $R_v = 0.1L$, respectively, where $L$ is the side of the square computational domain ($64 \times 64$ equally spaced cells extending from $-L/2$ to $L/2$ in both directions). The other reference quantities were $\rho_\infty = 1$, $p_\infty = 1$ and $\gamma = 1.4$.

The left and bottom boundaries were imposed as inviscid penalty terms (Carpenter...
et al. 1994) computed from a fixed state coincident with the far field conditions. The top and right boundaries were treated using Eq. (3.9) and the coupling algorithm from Eqs. (4.5)–(4.6). The analytical solution from Eqs. (5.1)–(5.3) was used as a benchmark solution to quantify the performance of the outflow boundary condition by means of the global normalized error

\[
E_{\text{glb}}(\phi, t) = \left( \frac{\sum_{i,j} \left( \phi_{i,j}(t) - \phi_{b,i,j}^b(t) \right)^2}{\sum_{i,j} \left( \phi_{b,i,j}(0) \right)^2} \right)^{1/2}
\]

(5.4)

where the superscript 'b' refers to the benchmark solution.

The global errors in density are plotted, as a function of the non-dimensional time \( t^* = \frac{t}{2u_\infty/L} \), in Figure 1. The errors obtained by using the boundary conditions from Thompson (1990) and Lodato et al. (2008) are also plotted for reference. An instantaneous map of pressure with vorticity contours is also shown when the vortex center reaches the corner of the domain. As expected in the cases where outgoing vorticity is involved, the optimal relaxation determines a significant reduction of the global error, which is about one order of magnitude lower than the one obtained with non-optimal relaxation, and more than one order of magnitude lower than that obtained using Thompson’s BC.

5.2. Pressure pulse test

The computational domain was a square of side \( L \), with 62 × 62 square control volumes and subsonic outflows according to Eq. (3.9) were applied on all the four faces. The corners of the computational domain were computed using Eqs. (4.5)–(4.6).

The pressure field was initialized with a circular pressure pulse of amplitude \( \delta = 0.001 \):

\[
p(r) = p_\infty \left[ 1 + \delta \exp \left( -\frac{r^2}{2R_p^2} \right) \right],
\]

(5.5)

where \( r = \sqrt{x_1^2 + x_2^2} \) is the distance from the center of the domain and \( R_p = 0.05L \) is the characteristic dimension of the pressure pulse. The temperature was set constant and equal to \( T_0 \) and the initial density distribution was computed according to the state equation \( \rho(r) = p(r)/(RT_0) \). The flow field was initialized at rest and then left to evolve.

In order to assess the performance of the formulation, a benchmark solution was computed on a computational domain four times bigger (i.e. \( 4L \times 4L \) with 248 × 248 cells). The global error in pressure was then computed with Eq. (5.4) using the benchmark solution as a reference. The time history of the global error is shown in Figure 2 (note that the time is normalized using the sound speed, hence the pressure wave crosses the boundary at \( t^* \sim 1 \)). The errors obtained by using the boundary conditions from Thompson (1990); Poinso et al. (1992): Lodato et al. (2008) are also plotted for reference.

As expected, in the case of a purely acoustic wave, Eq. (3.9) and the 3D-NSCBC (Lodato et al. 2008) produce almost identical results (the curves are actually indistinguishable). In fact, the proposed optimal relaxation for transverse terms is expected to minimize, for the most part, acoustic reflection by outgoing vorticity (see Lodato et al. 2010). In agreement with Yoo et al. (2005), Yoo & Im (2007) and Lodato et al. (2008), the inclusion of transverse terms in the formulation of the incoming waves improves the results, giving about a three fold reduction in the error compared to formulations without transverse terms.
Characteristic outflows with optimal transverse terms

Figure 2. Evolution of the global error in pressure as a function of the normalized time \( t^* = t^2a/L \) (a) and instantaneous pressure contours (b): --- , BC as per Eq. (3.9); ----- , 3D-NSCBC (Lodato et al. 2008); ------ , BC by Thompson (1990); --- --- , BC by Poinsot & Lele (1992). Pressure contours are represented over a quarter of the test domain (lower left square) and part of the benchmark solution domain.

6. Concluding remarks

The role played by different transverse terms in the characteristic equations was carefully analyzed by Lodato et al. (2010, 2011) and a modified optimal formulation for the subsonic outflow boundary condition was proposed.

In particular, this optimal formulation included transverse effects in the computation of incoming modes at the boundary in a way that minimizes coupling between outgoing vorticity modes and incoming acoustic modes in a broad range of Mach numbers and base flow directions. In applications involving turbulent flows for which noise generation and propagation are the main concern, such a decoupling between vorticity and acoustic modes has the potential to significantly reduce spurious numerical noise generated when vortical structures cross the boundary.

When structured codes and cartesian grids are used, the above-mentioned modification to the characteristic subsonic outflow with transverse terms makes it necessary to reformulate the edges and corners coupling algorithm (Lodato et al. 2008) accordingly. The modified edges/corners procedure is detailed in the present work. Despite the significant changes in the mathematical development, the modifications that the new definition of the transverse terms carries along result in minor changes to the original edges and corners coupling procedure, thus allowing a relatively easy adaptation into existing implementations. Numerical tests involving a Gaussian vortex and an acoustic wave were provided in support of the proposed edges/corners algorithm.

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REFERENCES


