Bifurcation analysis of scramjet unstart

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1. Motivation and objectives

Scramjet unstart occurs when a strong shock initiated from disturbances in the engine propagates upstream and finally spills out of the engine inlet. When unstart occurs, the airflow into the engine is greatly diminished, leading to loss-of-thrust and engine stall. Because the probability of unstart increases with increasing heat-release, the danger of unstart is an important limiting factor on the performance of scramjet engines. In the Predictive Science Academic Alliance Program (PSAAP), a large computational and experimental effort has been made at Stanford to quantify the likelihood of unstart for various and potentially uncertain scramjet operating conditions as a way to determine safe operability limits for scramjet engines.

The investigation of the dynamics of unstart in a scramjet combustor is motivated by observations of hysteresis effects in both the experiment of Wagner et al. (2010) and in steady Reynolds-averaged Navier-Stokes (RANS) calculations of the same geometry (Jang et al. 2010). For these cases, unstart is a catastrophe in the sense that once unstart is initiated, the scramjet system proceeds irreversibly towards the unstarted state. According to the mathematical study of dynamical systems, this occurs because the branch of steady, started solutions undergoes a fold bifurcation at a critical heat-release rate. Above the critical heat-release rate, the branch of steady started solutions no longer exists, and instead the system is attracted to the branch of solutions corresponding to the unstarted state.

In addition to providing physical insight into the dynamics of unstart, bifurcation analysis can also help in the quest to quantify the likelihood of unstart for sub-critical heat-release rates. As we will see below, heat-release rates in the hysteresis zone support both the started and unstarted branches of steady solutions, and these two branches are necessarily separated by a third branch of steady but unstable solutions. The third branch of unstable solutions lies exactly on the boundary between the basins of attraction of the started and unstarted solutions. In this brief, we argue that in the neighborhood of the critical heat-release rate, the branch of unstable solutions can be used to define a measure of how close a steady, started solution is to the basin of attraction of the unstarted branch. This new metric represents a combination and refinement of the Rayleigh (thermal choking) and Korkegi (boundary layer separation) limits (Korkegi 1975, 1985) that have been previously used in determining the unstart bound. The new metric based on the unstable solution branch is a function of the heat-release rate.

The remainder of this brief is organized as follows. First, Section 2 discusses details of the numerical method used to trace the bifurcation structure of scramjet unstart. Bifurcation curves for a two-dimensional geometry are presented in Section 3.1. To investigate the influence of flow separation caused by oblique shock/boundary layer interaction, the calculations are repeated with and without wall friction. Once the bifurcation structure is determined, Section 3.2 considers linearized dynamics about the solution branches. For the stable, started branch, the least-stable global mode indicates the spatial shape of the
most dangerous perturbation – the one that will push the system closest to unstart. In Section 3.3, the structural sensitivity of the scramjet system to base-flow modifications is computed through a multiplication of the least-stable direct and adjoint global modes. This information indicates how the scramjet should be modified in order to delay the onset of unstart. The final section of the brief provides conclusions and discusses directions for future research.

2. Numerical methodology

We consider a dynamical system of the form

$$\frac{du}{dt} = F(\phi; u), \quad u, F \in \mathbb{R}^n,$$

(2.1)

where $\phi$ is an adjustable system parameter and $u$ is a vector describing the system state. $F$ is a non-linear vector function describing the temporal evolution of the system state.

For this study, $F$ is the compressible RANS equations with the Wilcox $k-\omega$ turbulence model (Wilcox 2006), $u$ contains the primary flow variables (density, momentum, total energy, $k$, and $\omega$) at every point in the computational domain, and $\phi$ is a heat release coefficient. $F$ is computed using our in-house RANS solver called JOE (Pečnik et al. 2012). The solver is a finite-volume compressible flow solver using the HLLC shock-capturing scheme (Toro et al. 1994) and has the capability to handle unstructured meshes.

A steady-state solution $u_0$ for a given $\phi$ satisfies

$$F(\phi; u_0) = 0.$$

(2.2)

This equation implicitly defines the set of the steady-state solutions $u_0$ at each $\phi$, i.e., $X = \{ (\phi, u_0) \mid F(\phi, u_0) = 0 \}$. In general, the solution set $X$ forms a curve in phase space, and may be multi-valued with respect to $\phi$. A steady solution belonging to the set $X$ can either be stable or unstable, depending on the locations of the eigenvalues of the Jacobian matrix $J = \nabla_u F$.

Figure 2 shows a schematic of the steady solution curve from a scramjet engine. The horizontal axis represents the heat-release rate $\phi$ and the vertical axis corresponds to a quantity of interest (QoI) or a metric selected to represent the flow field. For sufficiently low heat-release rates ($\phi < \phi_{\text{restart}}$), only one branch of solutions exists, corresponding to started scramjet operation. At the other extreme, for sufficiently high heat-release ($\phi_{\text{unstart}} < \phi$), the steady unstarted solution is the only possibility. Physically, the steady unstarted solution corresponds to a bow shock upstream of the scramjet inlet. Both the started branch and unstarted branch are at least Lyapunov stable. As Figure 2 highlights, the started and unstarted branches overlap in the hysteresis zone $\phi_{\text{restart}} \leq \phi \leq \phi_{\text{unstart}}$. Mathematically, in this region, the two stable branches must be separated by an unstable branch, and all three branches should connect in a continuous S-shaped curve. In this picture, the unstable branch meets the stable branches at points corresponding to fold bifurcations. Physically, the unstable branch of solutions corresponds to a normal unstart shock at various streamwise locations inside the scramjet combustor.

Solutions belonging to the unstable branch cannot be obtained in practice using regular RANS calculations, because small perturbations, such as those introduced by round-off error, will drive the system away from the unstable branch, and towards either of the started or unstarted branches. Instead the unstable portion of the set $X$ should be obtained by numerically continuing the trajectory of the solution curve past the bifurcation point. In this study, the pseudo-arclength continuation method (Keller 1977;
Figure 1. Schematic of a scramjet bifurcation structure.

Chan & Keller (1982) is used as a “wrapper” around the RANS solver. The pseudo-arclength continuation algorithm is basically a predictor-corrector method, and it uses Newton-Raphson iterations with the pseudo-arclength constraint in the corrector step (Allgower & Georg 1997).

The Newton-Raphson procedure requires the evaluation of Jacobian matrix $J$ as well as the vector field $F$. To compute $J$, taking into account complex geometry using unstructured meshes as well as shock-capturing via the HLLC scheme, we employ the technique of automatic differentiation (AD) (Griewank 2000). When the AD technique is applied to the RANS solver, it yields a fully-discrete Jacobian.

3. Numerical experiments

To verify our numerical methodology, a two-dimensional converging-diverging nozzle is selected as a base flow configuration as shown in Figure 2. The area ratio between the inlet and the throat is $A_{\text{throat}}/A_{\text{inlet}} = 0.99$. This slight convergence was selected in order to model the deceleration of the supersonic flow caused by slow growth of a boundary layer that is present in laboratory experiments, but which was otherwise absent for our initial inviscid calculations. Downstream of the throat, the nozzle rapidly diverges until an area ratio of $A_{\text{outlet}}/A_{\text{inlet}} = 1.195$ is met at the nozzle exit. The distance between the inlet and the throat is $1.689H$ where $H$ is the inlet height, and the outlet is located at $2.5H$ downstream of the inlet. The incoming flow at the inlet is uniform Mach 2 flow, and the specified turbulence quantities at the inlet are $k = 0.001$ and $\omega = 1.0$ which are normalized by the velocity and the kinematic viscosity at the inlet. The lower boundary is symmetric (slip), and the Neumann boundary condition is used for the outlet boundary. Two different boundary conditions are tested for the upper boundary to distinguish the Rayleigh mechanism and the Korkegi mechanism: a slip wall (Figure 2(a)) and a no-slip wall (Figure 2(b)). In the case of the no-slip wall, a boundary layer grows rapidly on the top wall due to the wall friction, and the rapidly diverging section causes massive boundary layer separation behind the throat. Adiabatic wall boundary conditions with the same temperature as the inlet profile are used for both cases. The Reynolds number based on the inlet height and inlet velocity is 2000,
and power law is used for viscosity. Heat is added to the system through a volumetric term specified by a simple one-dimensional heat release model

\[ h(x, y) = \begin{cases} \phi & \text{if } x \leq x_{\text{throat}} \\ 0 & \text{otherwise} \end{cases}, \quad (3.1) \]

where \( x \) is the streamwise coordinate, \( y \) is the vertical coordinate, and \( x_{\text{throat}} \) is the streamwise location of the throat.

### 3.1. Bifurcation structure

The bifurcation structure of the flow with and without wall friction is computed using the techniques discussed in Section 2. The calculation of the bifurcation curves starts from a converged RANS solution on the started branch. Thanks to pseudo-arclength continuation, the method can proceed past the bifurcation point and continue onto the unstable branch of the solution curve. As the numerical continuation approaches the restart limit, the Newton-Raphson iterations become increasingly difficult to converge. This is probably because the true, steady unstarted solution corresponds to a bow shock upstream of the inlet, outside of our computational domain. Our computational domain, however, is sufficient to investigate the bifurcation structure near to the unstart bifurcation point because the unstart shock initially forms near to the nozzle throat.

Figure 3 compares the bifurcation curves obtained with and without wall friction. The difference in the bifurcation point \( \phi_{\text{unstart}} \) for the two cases provides an exact measurement of the difference between the Rayleigh and Korkegi limits, as well as their relative effects on the margin to unstart. In the following sections, we will focus upon the bifurcation curve obtained with wall friction.

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**Figure 2.** Mach number contours for the two cases at \( \phi = 0 \): (a) without friction on the top wall (b) with friction on the top wall.

**Figure 3.** Bifurcation curves of the two test cases: — with friction —— without friction.
3.2. Linearized dynamics

From the implicit function theorem, a necessary condition for a steady-state (fold) bifurcation is that the Jacobian matrix $J$ is singular. In other words, one of the eigenvalues $\lambda$ of $J$ is zero when a steady-state bifurcation occurs. Figure 4 shows 30 eigenvalues with largest real part at the bifurcation point obtained by using an Arnoldi method (Lehoucq & Sorensen 1996). The real part of the eigenvalue $\lambda_r$ is the growth rate (or decay rate) of the associated global mode, and the imaginary part $\lambda_i$ is its frequency. At the bifurcation point, the least stable $\lambda$ (with the largest $\lambda_r$) is zero, verifying that this is indeed a steady-state bifurcation.

The eigenfunctions corresponding to the eigenvalues $\lambda$ are known as global modes. Figure 5 shows the global modes corresponding to the least stable eigenvalue in the case of wall friction. Close to unstart, this global mode may be linked to a “most dangerous” direction leading to unstart.

As previously mentioned, small perturbations made to a solution belonging to the unstable branch will drive the flow either to the started branch or to the unstarted branch. To illustrate this behavior, we consider two such perturbations: one above and one below the unstable branch. The initial field of Figure 6(b) is obtained by the linear combination of the started and unstable solutions at the same $\phi = 0.44$, i.e., $u(\phi) = 0.95u_{\text{unstable}}(\phi) + 0.05u_{\text{started}}(\phi)$ so that the slightly perturbed field is located closer to the started branch than the unstarted branch (below the unstable branch). As the system evolves in time by the unsteady RANS equations, the flow field approaches the started solution at the same $\phi = 0.44$ (Figure 6(a)). However, if the perturbed field is closer to the unstarted branch $u(\phi) = 1.05u_{\text{unstable}}(\phi) - 0.05u_{\text{started}}(\phi)$, the system undergoes the unstart event as shown Figure 6(c). This result shows that solutions on the unstable branch can be used to separate unstarting and restarting behaviors, and in this way provides a refined, solution-dependent definition the unstart limit.

3.3. Delaying unstart

The sensitivity of the eigenvalue associated with each direct global mode was found through a multiplication with its corresponding adjoint global mode. The sensitivity to base-flow modifications $\nabla_u \lambda$ is quantified by following the discussion by Giannetti & Luchini (2007). The authors showed that the region in space where the least stable
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Figure 5. The least stable global mode: (a) density (b) streamwise momentum (c) vertical momentum (d) total energy.

Figure 6. Normalized pressure ($p/p_{inlet}$) contours: (a) steady-state solution on the started branch (b) steady-state solution of the perturbed initial field below the unstable branch in Figure 3 (c) a snapshot of the unstart event starting from the perturbed initial field above the unstable branch in Figure 3.

eigenvalue is sensitive to a local base flow modification can be found by overlapping the direct global mode $\hat{u}$ and the adjoint global mode $\hat{u}^+$. Figure 7 shows the product of the direct and adjoint magnitudes $\delta(x, y) = ||\hat{u}(x, y)||||\hat{u}^+(x, y)||$. In locations where $\delta(x, y)$ is high, a modification of the base flow is able to produce the greatest shift of the least stable eigenvalue.

Figure 7 depicts two possible mechanisms by which unstart may be prevented. In Figure 7(b), $\delta(x, y)$ shows a peak in streamwise momentum near the wall at the throat. This is related to the Korkegi limit: if the base flow is accelerated in this region, flow separation is reduced, which delays unstart. In Figure 7(d), $\delta(x, y)$ is the maximum for total energy in the bulk region. If heat is removed from the system in the region, unstart can be also delayed. The peak near the outlet is spurious, and derives from an imperfect boundary treatment for subsonic flow.

4. Conclusions and future work

The bifurcation structure of scramjets is numerically investigated along with eigendecomposition of the linearized system to find the stability boundaries. By comparing the solution curves from slip wall and friction wall cases, the Rayleigh and Korkegi mecha-
nisms are clearly separated. The dynamics of the perturbed field near the unstable branch show that the unstable branch may provide a refined, solution dependent unstart bound.

Structural sensitivity to base flow modifications is found through a multiplication of corresponding direct and global modes. For the least-stable mode, this clearly identified separate regions of the flow associated with both the Rayleigh and Korkegi mechanisms, and indicated how the base flow should be changed to delay the onset of unstart.

Future work is required to understand the bifurcation structure and unstart dynamics of a more realistic scramjet. The next step will be to extend the developed techniques to thermal choking of the HyShot II configuration in which chemical reaction is involved (Gardner et al. 2004; Smart et al. 2006).

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REFERENCES


