1. Motivation and objectives

Intermittency is a ubiquitous feature of a wide range of dynamic systems that arise in engineering and nature. For example, the motion of the edge of free shear-flows is sometimes turbulent and sometimes non-turbulent. Classically, this type of intermittency is quantified in terms of a binary intermittency function or an intermittency factor that describes the probability of the flow being in one state or the other. Once known, the latter factor can be used to obtain conditional statistics for each state independently (Pope 2000). These statistics might accurately describe the random process in a quantitative manner. The underlying physics, however, are often better understood from a dynamical systems perspective. In this light, sporadic deviations from the statistical equilibrium can, for many systems, be characterized as instabilities that are intermittently triggered through external stochastic excitation (Mohamad & Sapsis 2015).

In the context of fluid mechanics, transient growth provides such a mechanism. Transient growth is associated with the non-normality of the underlying linearized dynamics (Trefethen et al. 1993; Farrell & Ioannou 1996; Schmid & Henningson 2001). The linear theory that governs transient growth allows for the computation of modal solutions that identify spatial structures that undergo significant energy growth over short periods of time. Pipe flow is a prominent example of a flow in which a transient linear amplification process plays a crucial role as the flow transitions from the laminar to an intermittently turbulent state with increasing flow rate (Kerswell 2005; Avila et al. 2011). In particular, patches of turbulence, so-called turbulent puffs, are observed despite the fact that the laminar state is linearly stable to infinitesimal disturbances. In other systems, nonlinear interactions may sporadically conspire in such a way that responses of exceptionally large amplitudes are observed. Such statistical outliers are commonly referred to as extreme or rare events due to their isolated nature. An example of this class of intermittent effects is oceanographic surface gravity waves of extreme magnitude, or rogue waves (Dysthe et al. 2008).

In this paper, we present a data-driven approach to isolate and identify intermittent structures in a statistical manner by leveraging both their temporal and spatial coherence. As an example, we chose the highly intermittent super-directive acoustic radiation of a hot turbulent jet. The jet noise problem is of eminent technical interest for the aviation industry, in particular naval aviation. Aircraft carrier flight crew personnel operate in one of the world’s loudest work environments and regularly suffer from hearing loss because of this (Aubert & McKinley 2011). In Figure 1, we contrast two time instances of the flow field obtained from a high-fidelity numerical simulation (Brès et al. 2017). As in Koenig
et al. (2013), we use a wavelet transform of the pressure signal at a probe location in the far-field to identify loud events, or acoustic bursts, and distinguish them from quieter periods in which the pressure fluctuations are closer to their statistical mean.

A loud event is identified from the scaleogram in Figure 1(a) and the corresponding perturbation velocity and pressure fields for the $m=1$ azimuthal Fourier component are shown in Figure 1(b). The acoustic burst manifests itself as a high-amplitude wave in the pressure field that is emitted at a low angle (relative to the jet axis) from the jet. Shortly after, the pressure field at the probe location appears quiet in Figure 1(c). It is well established that large-scale coherent structures, or wavepackets, are the dominant source of low aft-angle jet noise and that these structures can be modeled as spatial linear stability modes (Jordan & Colonius 2013) or resolvent modes (Schmidt et al. 2018). Their footprint can be seen in the pressure field in the developing jet region with $x \lesssim 10$ in Figure 1(b,c). Jet noise models based on such linear theories, however, tend to significantly underpredict jet acoustics. The main reason for their failure is the intermittency of the shear-layer instability wavepackets and therefore their radiated sound, which has been observed experimentally (Juvé et al. 1980; Kearney-Fischer et al. 2013), quantified and modeled (Cavalieri et al. 2011) but is not fully understood as of now.

In Section 2, we first formulate a conditional space-time proper orthogonal decomposition (POD) problem. This allows us to infer intermittent or rare events as space-time modes that optimally capture the conditional variance of the event’s ensemble of realizations. As an example, we then specialize the general framework to acoustic bursts in turbulent jets in Section 3, and summarize our findings in Section 4.
2. A conditional space-time POD formulation

We start by defining the space-time inner product

\[ \langle \mathbf{q}_1, \mathbf{q}_2 \rangle_{t, \mathbf{x}} = \int_{-\infty}^{\infty} \int_{V} q_1^*(\mathbf{x}, t) W(\mathbf{x}) q_2(\mathbf{x}, t) \, dV \, dt \quad (2.1) \]

in which we wish to measure the energy of some quantity \( \mathbf{q} \). The diagonal positive-definite weight matrix \( W(\mathbf{x}) \) is introduced to accommodate space and/or variable dependent weights.

Equipped with the norm \( \langle \cdot, \cdot \rangle_{t, \mathbf{x}} \) and an expectation operator \( E \{ \cdot \} \), commonly defined as the sample average, we embed \( \mathbf{q}(\mathbf{x}, t) \) into a Hilbert space \( \mathcal{H} \), which allows us to identify the spatio-temporal structure \( \phi(\mathbf{x}, t) \in \mathcal{H} \) that maximizes

\[ \lambda = \frac{E \left\{ \langle \mathbf{q}(\mathbf{x}, t), \phi(\mathbf{x}, t) \rangle_{t, \mathbf{x}}^2 \right\}}{\langle \phi(\mathbf{x}, t), \phi(\mathbf{x}, t) \rangle_{t, \mathbf{x}}} \quad (2.2) \]

using a variational approach. The \( \phi(\mathbf{x}, t) \) and \( \lambda \) that satisfy Eq. (2.2) can be found as a solution to the Fredholm eigenvalue problem

\[ \int_{-\infty}^{\infty} \int_{V} C(\mathbf{x}, \mathbf{x}', t, t') \phi(\mathbf{x}', t') \, d\mathbf{x}' \, dt' = \lambda \phi(\mathbf{x}, t), \quad (2.3) \]

where \( C(\mathbf{x}, \mathbf{x}', t, t') = E \{ \mathbf{q}(\mathbf{x}, t) \mathbf{q}^*(\mathbf{x}', t') \} \) is the two-point space-time correlation tensor. This formulation corresponds to the classical space-time POD problem introduced by Lumley (1970), which was vastly overshadowed by its popular spatial variant (Sirovich 1987; Aubry 1991). A notable exception is the work by Gordeyev & Thomas (2013), in which the authors solve a space-time POD problem to investigate flow transients. More recently, the frequency-domain version (Towne et al. 2018; Schmidt et al. 2018), which is derived from Eq. (2.3) under the assumption of statistical stationary, has gained popularity in the analysis of turbulent flows.

In deviation from the other formulations, we introduce a conditional expectation

\[ E \left\{ \langle \mathbf{q}(\mathbf{x}, t), \phi(\mathbf{x}, t) \rangle_{t, \mathbf{x}}^2 \right| H \} \quad (2.4) \]

with respect to the event \( H \). As we are interested in the average evolution of the rare event in space and time, we further recast the space-time inner product Eq. (2.5) in the form of a weighted space-time inner product

\[ \langle \mathbf{q}_1, \mathbf{q}_2 \rangle_{t, \mathbf{x}, \Delta T} = \int_{\Delta T} \int_{V} q_1^*(\mathbf{x}, t) W(\mathbf{x}) q_2(\mathbf{x}, t) \, dV \, dt, \quad (2.5) \]

over some finite time horizon \( \Delta T \) in the temporal neighborhood of the rare event \( H \). The quantity to maximize,

\[ \lambda = \frac{E \left\{ \langle \mathbf{q}(\mathbf{x}, t), \phi(\mathbf{x}, t) \rangle_{t, \mathbf{x}, \Delta T}^2 \right\}| H}{\langle \phi(\mathbf{x}, t), \phi(\mathbf{x}, t) \rangle_{t, \mathbf{x}, \Delta T}}, \quad (2.6) \]

is defined analogously to Eq. (2.2), and its solution is obtained, analogous to Eq. (2.3), from the corresponding weighted Fredholm eigenvalue problem

\[ \int_{\Delta T} \int_{V} C(\mathbf{x}, \mathbf{x}', t, t') W(\mathbf{x}') \phi(\mathbf{x}', t') \, d\mathbf{x}' \, dt' = \lambda \phi(\mathbf{x}, t). \quad (2.7) \]

In discrete time and space, the eigenvectors of the two-point space-time correlation tensor
are approximated from the eigendecomposition

\[ \mathbf{Q} \mathbf{Q}^\ast \mathbf{M} \Phi = \Phi \Lambda, \quad (2.8) \]

where

\[
\mathbf{Q} = \begin{bmatrix}
q(t_0^{(1)} - t^-) & q(t_0^{(2)} - t^-) & \cdots & q(t_0^{(N_{\text{peaks}})} - t^-) \\
\vdots & \vdots & \ddots & \vdots \\
q(t_0^{(1)}) & q(t_0^{(2)}) & \cdots & q(t_0^{(N_{\text{peaks}})}) \\
\vdots & \vdots & \ddots & \vdots \\
q(t_0^{(1)} + t^+) & q(t_0^{(2)} + t^+) & \cdots & q(t_0^{(N_{\text{peaks}})} + t^+) 
\end{bmatrix}
\quad (2.9)
\]

is the space-time data matrix of realizations of events occurring at times \( t_0^{(j)} \). The \( j \)-th column of \( \mathbf{Q} \) hence contains the \( j \)-th realization of \( H \) evolving from time \( t_0^{(j)} - t^- \) to \( t_0^{(j)} + t^+ \), i.e., over \( \Delta T/\Delta t \) time steps. Here, \( \Delta t \) is the temporal separation between two consecutive snapshots, and \( t^- \) and \( t^+ \) define the time interval

\[ \Delta T^{(j)} \in [t_0^{(j)} - t^-, t_0^{(j)} + t^+] \]

over which the \( j \)-th event evolves. The matrix \( \mathbf{M} \) accounts for both the weights \( \mathbf{W}(x) \) and numerical quadrature weights stemming from the spatial integration in equation (2.7). The matrices \( \Phi = [\phi^{(1)}(x, t), \phi^{(2)}(x, t), \ldots, \phi^{(N_{\text{peaks}})}(x, t)] \) and \( \Lambda = \text{diag}[\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N_{\text{peaks}})}] \) contain the eigenvectors and eigenvalues, i.e., space-time POD modes and their corresponding energies in the space-time inner product, respectively. In practice, the large eigenvalue problem Eq. (2.8) is solved by computing the cheaper eigendecomposition of the smaller matrix \( \mathbf{Q}^\ast \mathbf{M} \mathbf{Q} \) first, in the same manner as for all other POD variants (see, e.g., Towne et al. 2018). Note that the eigenvectors \( \phi(x, t) \) of the space-time correlation tensor, i.e., the conditional space-time POD modes, are dependent on time. In fact, they are coherent in space and over the time horizon \( \Delta T \) by construction.

3. Example: acoustic bursts in a round supersonic jet

As an example, we consider acoustic bursts that are intermittently emitted from a high Reynolds number round jet. We analyze a total of 10,000 snapshots separated by \( \Delta t = \Delta t^*/c_\infty = 0.1 \) acoustical time units of a high-fidelity LES by Brès et al. (2017) of a heated ideally expanded turbulent jet with the jet Mach number \( M_j = U_j/c_j = 1.5 \) and nozzle temperature ratio \( T_0/T_\infty = 2.53 \). The subscripts \( j, \infty, \) and 0 indicate jet, stagnation condition, free stream condition, the superscript \( * \) marks dimensional quantities, and \( c \) and \( T \) indicate the speed of sound and temperature, respectively. The data studied in what follows corresponds to the pressure field of case \( A2 \) in Brès et al. (2017), and the reader is referred to that paper for more details. For cylindrical coordinates with
\( \mathbf{x} = (x, r) \) and \( dV = rdxdr \), where \( x \) and \( r \) are the streamwise and radial coordinates, respectively, the rotational symmetry allows for a Fourier decomposition of the pressure field
\[
\mathbf{p}(x, r, \theta, t) = \sum_{m} \hat{p}_m(x, r, t) e^{im\theta}
\] (3.1)
into azimuthal Fourier modes \( \hat{p}_m(x, r, t) \). For this demonstration, we focus on the \( m = 1 \) component of the data as shown in Figure 1, which contains the largest fraction of the fluctuating energy of unheated jets (Schmidt et al. 2018). In the following, we drop the \( \hat{} \), with the understanding that \( \mathbf{p} \) and \( p \) denote the \( m = 1 \) Fourier component of the pressure. The pressure 2-norm
\[
|\langle \mathbf{p}(x, t), \mathbf{p}(x, t) \rangle_{x, \Delta T}|^2
\]
is chosen as a convenient measure for the acoustic energy of the flow.

In order to focus on the acoustic emissions to the far-field plus the outer part of the shear-layer, we let
\[
W(x) = \begin{cases} 
1, & \text{if } \frac{U(x)}{U_j} \leq 0.1 \\
0, & \text{otherwise},
\end{cases}
\] (3.2)
to mask out the highly energetic pressure fluctuations in the jet column. Here, \( U \) is the mean streamwise velocity and \( U_j \) the jet velocity. The threshold \( U(x) \leq 0.1U_j \) is chosen as a convenient way to distinguish these different parts of the flow. The inclusion of the outer part of the shear-layer facilitates backtracking of the acoustic burst event to its precursor in the mixing region close to the nozzle. The final results were found qualitatively independent of the exact choice of thresholding.

The event \( H \) is specified to identify loud events, or acoustic bursts, in the far-field of the jet. In particular, we detect such events as local maxima in time from a single-point measurement of the far-field pressure \( p(x_0, t) \) at some probe location \( x_0 = (x_0, r_0) \). For the case at hand, we detect loud events at \((x_0, r_0) = (19, 6) \) (see Figure 1). This location corresponds to the peak location of the power spectral density of the pressure along the edge of the domain, i.e. the dominant aft-angle noise. A local maximum that identifies an event is said to be detected at time \( t_0 \) if \( p(x_0, t_0) \) is larger than its two neighboring samples at \( t = t_0 - \Delta t \) and \( t = t_0 + \Delta t \) in discrete time. The event is hence defined as
\[
H : \left\{ t_0^{(j)} \in t_{\text{sim}} \mid p(x_0, t_0^{(j)} - \Delta t) < p(x_0, t_0^{(j)}) > p(x_0, t_0^{(j)} + \Delta t) \right\},
\] (3.3)
where \( t_{\text{sim}} = [\Delta t, 2\Delta t, \ldots, 10^4\Delta t] \) is the discrete simulation time at which the LES snapshots are saved. We further restrict the conditional expectation
\[
E\left( |\langle \cdot, \cdot \rangle_{x, \Delta T}|^2 \right| H_{\text{peaks}})
= \frac{1}{N_{\text{peaks}}} \sum_{j=1}^{N_{\text{peaks}}} \left\{ |\langle \mathbf{p}(x, \Delta T^{(j)}), \phi(x, \Delta T^{(j)}) \rangle_{x, \Delta T}|^2 \right| H_{\text{peaks}} \},
\]
with \( H_{\text{peaks}} = \left\{ H : |p(x_0, t_0^{(1)})| \geq |p(x_0, t_0^{(2)})| \geq \cdots \geq |p(x_0, t_0^{(N_{\text{peaks}})})| \right\} \), sorted such that the first \( N_{\text{peaks}} \) largest peaks are selected, i.e., we threshold the cardinality \(|H|\) of the event.

Figure 2 shows the time trace of the pressure signal at the probe location and its
Figure 2. Time trace (left) and histogram (right) of the pressure at the probe location $(x_0, r_0) = (19, 6)$: (a) $m = 1$ component with $t^- = 150\Delta t$ and $t^+ = 149\Delta t$, such that $\Delta T = 30$ spans 300 snapshots centered about any $t_0$, and $N_{\text{peaks}} = 200$; (b) $m = 1$ component with $t^- = 100\Delta t$, $t^+ = 60\Delta t$, $N_{\text{peaks}} = 35$. The inset in (b) shows an example of an isolated event. Gray indicates the full signal, red events occurring at times $t_0$, and blue their temporal neighborhoods $t_0 - \Delta T \leq t \leq t_0 + \Delta T$.

Figure 3. Conditional space-time POD energy spectrum for $t^- = 150\Delta t$, $t^+ = 149\Delta t$, $N_{\text{peaks}} = 200$.

histogram for two different choices of space-time POD parameters $t^-$, $t^+$, and $N_{\text{peaks}}$. The parameter combination $t^- = 150$, $t^+ = 149$, and $N_{\text{peaks}} = 200$ shown in Figure 2a is used in the reminder of this paper. This choice centers the rare event within a time interval of 30 acoustic units, which approximates the time it takes an acoustic pulse to transverse the domain. The combination shown in Figure 2b solely serves as a less dense example, with the insert highlighting a single isolated event (filled red circle) occurring at $t_0 = 288.1$ and its temporal neighborhood, or time segment, $278.1 \leq \Delta T \leq 294.6$ (blue line segment). Note that every snapshot might well contain multiple rare events at different stages of their temporal evolution, which results in overlapping time segments. The plot shown on the right side of Figure 2 uses the same color representation as used for the time trace to distinguish the histogram of the entire signal (gray) from those of the event (red) and segments (blue). As we are considering the absolute value of a presumably Gaussian-distributed random signal, the time trace histogram can be approximated by a $\chi^2$-distribution. By the definition of $H$ in Eq. (3.3), the rare acoustic events occupy the tail of this distribution.

In Figure 3, we present the eigenvalue, or energy spectrum of the conditional space-
temporal POD problem defined by the parameters presented in Figure 2(a). Two features of the spectrum stand out: first, the dominance of the first eigenvalue which is $\approx 40\%$ larger than the second eigenvalue; second, the slow decay of the rest of the spectrum starting with mode 2. These two observations are crucial to our analysis as they indicate that the particular space-time structure associated with the first eigenvalue plays a dominant role in the space-time statistics.

We next investigate the corresponding structure, i.e., the leading space-time POD mode $\phi^{(1)}(x, t)$, in Figure 4. Twelve representative time instances $t^{(j)} = [55\Delta T, 70\Delta T, 85\Delta T, \ldots, 220\Delta T]$ that track the event’s evolution from its formation in the near-nozzle shear-layer (top left) to the time it exits the computation domain (lower right) are shown. The first instance at $t = 5.5$ represents the earliest time for which we observe a distinct localized wavepacket structure. This structure is identified as the initial seed, or precursor, of the acoustic burst. Over time, the initial seed evolves into an acoustic burst with a distinct wavenumber, spatial extent, and ejection angle. The existence of such a statistically prevalent mode was \textit{a priori} not obvious and suggests the existence of an underlying physical mechanism.

4. Conclusions

We present a data-driven approach based on a conditional space-time POD formulation that is tailored to the eduction and statistical description of intermittent or rare events from data. The resulting modes are orthonormal in a space-time inner product and optimally capture the conditional variance of an ensemble of realizations of the intermittent event. By construction, the modes are coherent in space and over a finite time horizon. The formalism hence bridges the gap between standard POD modes, which are coherent at zero time lag only, and spectral POD modes, which are perfectly coherent over all time.

As an example, we apply conditional space-time POD to the example of superdirective acoustic radiation of a turbulent jet. We find a statistically dominant space-time mode...
that is energetically well separated from all other modes. It takes the form of a spatially
cconfined wavepacket that originates in the thin initial shear-layer close to the nozzle. Over
time, this initial seed evolves into an acoustic burst of distinct wavenumber and spatial
support that is emitted to the far-field. The structure and dynamics of this prototype
acoustic burst suggest the existence of an underlying physical mechanism, for example a
finite-time instability. We plan to test this hypothesis in future work.

The orthogonality and optimality properties of the conditional space-time POD modes
make them potential candidates for basis functions for use in reduced-order models. In
fluid mechanics, for example, Galerkin models (see, e.g., Noack et al. 2011) are widely
successful and build on exactly these properties. The temporal coherence of the modes
furthermore enables the identification of precursors of extreme events, such as the initial
seed of the acoustic burst event shown in the upper-left panel of Figure 4. We plan to
leverage this for model-predictive control in the future.

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