The linear acoustic equations solved as a finite domain boundary value problem by eigenfunction expansion

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1. Motivation and objectives

This brief is motivated by the goal of understanding linear mechanisms preceding transition in supersonic external aerodynamics, specifically the receptivity process. One is faced with the question of what the linear disturbance field looks like on the body as well as how its amplitude is related to that of incoming disturbances in the freestream. Mathematically this can be posed as a boundary value problem in which freestream disturbances are known on an inflow boundary and the details of the response everywhere else in the domain are solved for. The solution is time-harmonic in the linear limit, though details of the freestream disturbances will determine how many frequencies are required to represent the solution. The desired outcome, however, is not a single solution to the problem given a specific inflow boundary condition but instead, some universal features of the solution that are applicable to those of arbitrary inflow disturbances. Doing so may reveal some universal physical insight that will aid understanding and design. One way to approach the problem is to find an appropriate basis onto which the solution to the boundary value problem can be projected, along with an appropriate projection operation, thus providing information on the response as well as its receptivity to freestream disturbances.

The receptivity problem in high-speed external aerodynamics has been studied in hypersonic cones, for example, through linear time-resolving calculations (Zhong & Ma 2006) and input-output analysis (Cook & Nichols 2022). It may also be useful to ask if there is an appropriate eigenfunction expansion of the governing system of equations as a basis onto which to project the solution. The weakness of time-resolving calculations is that the results are specific to a particular inflow boundary condition. Resolvent-type analyses are a natural way to study the forced response problem of non-normal operators and provide an optimal, orthogonal basis for the forced response. The resolvent modes are solutions to the forced problem. This is in contrast to the eigenmodes of the operator, which are solutions to the unforced equations. The resolvent modes could be considered a nonmodal description consisting of superposition of eigenmodes, potentially obscuring interpretation by conflating non-normal effects, propagation and modal amplification mechanisms. It is well-known that the utility of eigenvalue analysis in the response of highly non-normal problems is limited (Trefethen 1991; Trefethen *et al.* 1993). Fluid dynamics problems with boundary layers are highly non-normal, making the eigenvalue problem ill-posed, and significant nonmodal effects can manifest in the temporal response (Trefethen 1991; Trefethen *et al.* 1993; Schmid & Henningson 2001). While the eigenfunctions may not form an optimal basis, their application to a freestream acoustic field will be explored in this brief.

Solving the global eigenvalue problem for supersonic aerodynamics applications re-

quires the answer to several questions, specifically, what are the appropriate boundary conditions to apply, are the resulting eigenmodes useful and what is the best way to interpret the modes. In order to address some of these questions in an easily controlled environment, the problem of the one-dimensional linear disturbance equations in an inviscid fluid is studied. This problem is relevant because it is representative of the types of dynamics present in the supersonic freestream upstream of an aerodynamic body of interest, which is included in the global eigenvalue problem (Flint & Hack 2020). We will see that numerical eigenmodes of this problem require careful interpretation that depends on the imposed boundary conditions.

2. Formulation of the continuous problem

2.1. *Homogeneous operator with inhomogeneous boundary conditions*

This section is based on the book by Lanczos (1996); the reader is referred to Chapter 8 of that work for further reading on this subject. Consider a one-dimensional boundary value problem that is defined by a homogeneous differential equation and inhomogeneous boundary conditions. Eigenfunction expansion is a useful concept, though it can only be applied directly to problems with homogeneous boundary conditions. However, we can convert the problem of a homogeneous equation with inhomogeneous boundary conditions to one of an inhomogeneous equation with homogeneous boundary conditions, to which we can subsequently apply the method of eigenfunction expansion.

We begin by introducing a function, $s_0(x)$, which is not constrained in any way except that it satisfies the given boundary conditions and that it is sufficiently smooth that our differential operator can operate on it (it must be differentiable to the proper degree). Here, x indicates the independent variables, including both space and time.

Given a differential operator in one-dimension, $\mathcal{D}$, we wish to solve the homogeneous problem

$$\mathcal{D}s = 0 \tag{2.1}$$

with inhomogeneous boundary conditions. We begin by writing the solution as

$$s(x) = s_0(x) + S(x). \tag{2.2}$$

Substituting this solution into Eq. (2.1) yields

$$\mathcal{D}S(x) = -\mathcal{D}s_0(x), \tag{2.3}$$

and with the inhomogeneous boundary conditions accounted for by $s_0(x)$, the problem reduces to finding $S(x)$, which satisfies homogeneous boundary conditions. The problem is now one of an inhomogeneous differential equation with homogeneous boundary conditions.

2.2. *The linear acoustic equations*

We consider the linear acoustic equations in a homogeneous medium moving with constant ambient velocity, $\mathbf{U}$, (Pierce 2019, p. 58)

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right) p + \rho c^2 \nabla\cdot\mathbf{v}' = 0, \tag{2.4}$$

$$\rho\left[\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{U}\cdot\nabla)\,\mathbf{v}'\right] = -\nabla p, \tag{2.5}$$

where t, ρ and c are time, ambient density and ambient sound speed, respectively. The disturbance pressure and velocity are denoted by p and $\mathbf{v}'$, respectively. In one dimension, x, using the ansatz $p = \hat{p}(x)e^{-i\omega t}$, $v' = \hat{v}(x)e^{-i\omega t}$, defines the temporal eigenvalue problem

$$U\frac{\partial \hat{p}(x)}{\partial x} + \rho c^2 \frac{\partial \hat{v}(x)}{\partial x} = i\omega \hat{p}(x), \tag{2.6}$$

$$U\frac{\partial \hat{v}(x)}{\partial x} + \frac{1}{\rho}\frac{\partial \hat{p}(x)}{\partial x} = i\omega \hat{v}(x), \tag{2.7}$$

or in operator form

$$\begin{pmatrix} U\frac{\partial}{\partial x} & \rho c^2 \frac{\partial}{\partial x} \\ \frac{1}{\rho}\frac{\partial}{\partial x} & U\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \hat{p}(x) \\ \hat{v}(x) \end{pmatrix} = \lambda \begin{pmatrix} \hat{p}(x) \\ \hat{v}(x) \end{pmatrix}, \tag{2.8}$$

$$\mathcal{A}\mathbf{y} = \lambda \mathbf{y}, \tag{2.9}$$

where $\lambda = i\omega$.

2.3. *The adjoint linear acoustic equations*

The adjoint governing equations and their corresponding boundary conditions will be required later (Section 2.4) to complete the solution to the continuous boundary value problem while simultaneously providing receptivity information about the direct modes, so they are given here. Equation (2.8) in operator form is

$$\begin{pmatrix} U\frac{\partial}{\partial x} - i\omega & \rho c^2 \frac{\partial}{\partial x} \\ \frac{1}{\rho}\frac{\partial}{\partial x} & U\frac{\partial}{\partial x} - i\omega \end{pmatrix} \begin{pmatrix} \hat{p}(x) \\ \hat{v}(x) \end{pmatrix} = 0, \tag{2.10}$$

$$\mathcal{L}\mathbf{y} = 0, \tag{2.11}$$

separating the derivative coefficients

$$\mathcal{L} = \mathcal{A}_0 + \mathcal{A}_1 \frac{d}{dx} \tag{2.12}$$

with

$$\mathcal{A}_0 = \begin{pmatrix} -i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \tag{2.13}$$

$$\mathcal{A}_1 = \begin{pmatrix} U & \rho c^2 \\ \frac{1}{\rho} & U \end{pmatrix}. \tag{2.14}$$

The corresponding adjoint operator is (Flint & Hack 2019)

$$\tilde{\mathcal{L}} = \tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_1 \frac{d}{dx}, \tag{2.15}$$

with

$$\tilde{\mathcal{A}}_0 = \mathcal{A}_0^H - \frac{d\mathcal{A}_1^H}{dx}, \tag{2.16}$$

$$\tilde{\mathcal{A}}_1 = -\mathcal{A}_1^H, \tag{2.17}$$

or

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} i\omega & 0 \\ 0 & i\omega \end{pmatrix}, \tag{2.18}$$

$$\tilde{\mathcal{A}}_1 = \begin{pmatrix} -U & -\frac{1}{\rho} \\ -\rho c^2 & -U \end{pmatrix}, \tag{2.19}$$

recalling that in this case $d\mathcal{A}_1/dx = 0$. The superscript $(\cdot)^H$ denotes conjugate transpose. The adjoint operator is

$$\begin{pmatrix} i\omega - U\frac{\partial}{\partial x} & -\frac{1}{\rho}\frac{\partial}{\partial x} \\ -\rho c^2 \frac{\partial}{\partial x} & i\omega - U\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \hat{q}(x) \\ \hat{u}(x) \end{pmatrix} = 0, \tag{2.20}$$

$$\tilde{\mathcal{L}}\tilde{\mathbf{y}} = 0. \tag{2.21}$$

Green's identity for this case is given by

$$\langle \mathcal{L}\mathbf{a}, \mathbf{b} \rangle - \langle \mathbf{a}, \tilde{\mathcal{L}}\mathbf{b} \rangle = F \tag{2.22}$$

for the states $\mathbf{a}$ and $\mathbf{b}$, where the inner product is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_{x=x_1}^{x_2} \mathbf{a}^H \mathbf{b} dx \tag{2.23}$$

and the boundary term is

$$F = -\left[\mathbf{b}^H \mathcal{A}_1 \mathbf{a}\right]_{x=x_2}^{x=x_1}. \tag{2.24}$$

Expanding the boundary term gives

$$F = -\left[U\hat{p}\hat{q}^* + \rho c^2 \hat{v}\hat{q}^* + \frac{1}{\rho}\hat{p}\hat{u}^* + U\hat{v}\hat{u}^*\right]_{x=x_1}^{x=x_2}, \tag{2.25}$$

where * indicates conjugation, $\hat{q}$ the adjoint pressure and $\hat{u}$ the adjoint velocity.

2.4. *Solution procedure*

The homogeneous equations that we want to solve are

$$(\mathcal{A} - i\omega\mathcal{I})\mathbf{s} = 0, \tag{2.26}$$

with inhomogeneous boundary conditions. Splitting the solution as in Eq. (2.2),

$$\mathbf{s}(x) = \mathbf{S}(x) + \mathbf{s}_0(x), \tag{2.27}$$

and substituting yields

$$(\mathcal{A} - i\omega\mathcal{I})\mathbf{S}(x) = -(\mathcal{A} - i\omega\mathcal{I})\mathbf{s}_0(x). \tag{2.28}$$

The problem has been reduced to finding $\mathbf{S}(x)$, which satisfies the homogeneous boundary conditions. We choose to perform an expansion of $\mathbf{S}(x)$ in an appropriate basis. Any reasonable basis can be used for this expansion; we chose the eigenfunctions of $\mathcal{A}$, $\mathcal{A}\mathbf{y} = \lambda\mathbf{y}$, which are solutions to the homogeneous operator of interest, $(\mathcal{A} - i\omega\mathcal{I})\mathbf{y} = 0$.

Expanding $\mathbf{S}(x)$ in the eigenfunctions of $\mathcal{A}$, $\mathcal{A}\mathbf{y} = \lambda\mathbf{y}$ and with knowledge of the adjoint eigenfunctions $\tilde{\mathcal{A}}\tilde{\mathbf{y}} = \lambda^*\tilde{\mathbf{y}}$,

$$\mathbf{S}(x) = \sum_{i=1}^{\infty} c_i \mathbf{y}_i(x), \tag{2.29}$$

the coefficients, c_i, are

$$c_i = \frac{\beta_i}{\lambda_i - i\omega}, \tag{2.30}$$

with

$$\beta_i = -\int_{x=x_1}^{x_2} \tilde{\mathbf{y}}_i(x)(\mathcal{A} - i\omega\mathcal{I})\mathbf{s}_0(x)dx, \tag{2.31}$$

where we have made use of the bi-orthogonality between the direct and adjoint eigenfunctions

$$\int_{x=x_1}^{x_2} \tilde{\mathbf{y}}_i^H(x)\mathbf{y}_j(x)dx = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{2.32}$$

noting specifically that the eigenfunctions are normalized such that this inner product gives unity when $i = j$, and that the eigenfunctions $\tilde{\mathbf{y}}$ and $\mathbf{y}$ satisfy boundary conditions such that the boundary term, Eq. (2.25), vanishes. Green's identity

$$\int_{x=x_1}^{x_2} \tilde{\mathbf{y}}_i^H(x)\mathcal{D}\mathbf{s}_0(x)dx - \int_{x=x_1}^{x_2} \mathbf{s}_0^H(x)\tilde{\mathcal{D}}\tilde{\mathbf{y}}_i(x)dx = F(\mathbf{s}_0, \tilde{\mathbf{y}}_i) \tag{2.33}$$

gives

$$\beta_i = -\int_{x=x_1}^{x_2} \mathbf{s}_0^H(x)(\tilde{\mathcal{A}} + i\omega\mathcal{I})\tilde{\mathbf{y}}_i(x) - F(\mathbf{s}_0, \tilde{\mathbf{y}}_i), \tag{2.34}$$

$$\beta_i = -(\lambda_i^* + i\omega)\int_{x=x_1}^{x_2} \mathbf{s}_0^H(x)\tilde{\mathbf{y}}_i(x) - F(\mathbf{s}_0, \tilde{\mathbf{y}}_i), \tag{2.35}$$

identifying the role of the adjoint eigenfunctions, $\tilde{\mathbf{y}}_i$. The projection of the forcing field, $\mathbf{s}_0$, onto the adjoint eigenfunctions determines the expansion coefficient, c_i, hence directly informing the receptivity of the corresponding direct mode, $\mathbf{y}_i$, in the solution of the system.

3. Formulation of the discrete problem

The semi-discretised counterpart to the governing equations, Eqs. (2.4) and (2.5), can be formulated as

$$\frac{\partial \mathbf{y}}{\partial t} + \mathrm{A}\mathbf{y} = 0, \tag{3.1}$$

where the inhomogeneous boundary conditions can be naturally included by removing the equations for the values on the boundary and moving the coefficients of the boundary values (columns of A) to a forcing vector in which the boundary condition has been evaluated, resulting in an equation for all interior points, $\mathbf{y}_{in}$,

$$\frac{\partial \mathbf{y}_{in}}{\partial t} + \mathrm{A}_{in}\mathbf{y}_{in} = -\mathrm{A}'\mathbf{f}, \tag{3.2}$$

where $\mathbf{f}$ is only nonzero on the boundary points, being analogous to $\mathbf{s}_0(x)$ in the continuous solution. Matrix A' is a rectangular matrix equivalent to A with the rows corresponding to boundary equations removed. Matrix A_{in} is a square matrix equivalent to A with both rows and columns corresponding to boundary points removed. After Fourier

transforming in time,

$$(\mathrm{A}_{in} - i\omega \mathrm{I})\hat{\mathbf{y}}_{in} = -\mathrm{A}'\hat{\mathbf{f}}. \tag{3.3}$$

The solution to this system is

$$\hat{\mathbf{y}}_{in} = (\mathrm{A}_{in} - i\omega \mathrm{I})^{-1}\hat{\mathbf{b}}, \tag{3.4}$$

with $\hat{\mathbf{b}} = -\mathrm{A}'\hat{\mathbf{f}}$ being the forcing vector and $(\mathrm{A}_{in} - i\omega \mathrm{I})^{-1}$ the resolvent operator. As in the continuous case, we wish to expand the solution, $\hat{\mathbf{y}}_{in}$, in terms of eigenvectors of A_{in}, $\mathrm{A}_{in}\mathbf{w} = \mu\mathbf{w}$ with knowledge of the corresponding adjoint (i.e., transpose; note the operator is real) system, $\mathrm{A}_{in}^T\mathbf{z} = \mu\mathbf{z}$,

$$\hat{\mathbf{y}}_{in} = \sum_{i=1}^{N-N_b} d_i \mathbf{w}_i, \tag{3.5}$$

where N is the dimension of the discrete system and N_b is the number of boundary equations. The expansion coefficients, d_i, are given by

$$d_i = \frac{-1}{(\mu_i - i\omega)}\mathbf{z}_i^T \mathrm{A}'\hat{\mathbf{f}}. \tag{3.6}$$

Note that we have made use of the bi-orthogonality property

$$\mathbf{z}_i^T \mathbf{w}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{3.7}$$

noting again the normalization of the eigenfunctions such that their inner product gives unity when $i = j$. Note that the dimension of $\mathbf{z}_i$ is $N - N_b$, $\hat{\mathbf{f}}$ is N and A' is $(N - N_b) \times N$. Using the identity, $\mathbf{z}_i^T \mathrm{A}'\hat{\mathbf{f}} = \hat{\mathbf{f}}^T \mathrm{A}'^T \mathbf{z}_i$ gives

$$d_i = \frac{-1}{(\mu_i - i\omega)}\hat{\mathbf{f}}^T \mathrm{A}'^T \mathbf{z}_i. \tag{3.8}$$

Noting that A' contains A_{in}, recall that A' is A with the boundary rows removed, while A_{in} has both the boundary rows and columns removed, i.e.,

$$\mathrm{A}' = \begin{pmatrix} H & \mathrm{A}_{in} \end{pmatrix} \tag{3.9}$$

under an appropriate ordering. The coefficients of A that act on the boundary values are captured in the rectangular matrix H of size $N \times 2$. Using Eq. (3.9) in Eq. (3.8) gives

$$d_i = \frac{-1}{(\mu_i - i\omega)}\hat{\mathbf{f}}^T \begin{pmatrix} H^T \mathbf{z}_i \\ \mu \mathbf{z}_i \end{pmatrix}, \tag{3.10}$$

and recalling that $\hat{\mathbf{f}}$ is nonzero on only the boundary points,

$$d_i = \frac{-1}{(\mu_i - i\omega)}\hat{\mathbf{f}}_{bc}^T H^T \mathbf{z}_i. \tag{3.11}$$

4. Eigenfunctions

4.1. *Eigenfunctions of the continuous equations*

The continuous eigenvalue problem defined in Eq. (2.8) permits solutions of exponential type in x with both purely real or complex exponents dependent on the eigenvalue, λ. The

boundary conditions on the direct and adjoint eigenvalue problem must be chosen such that the boundary term, Eq. (2.25), equals zero. This can be done through complementary homogeneous Dirichlet and Neumann conditions applied to the direct and adjoint eigenfunctions. The choice of what boundary conditions to apply is not unique. Consider the case of applying two boundary conditions at $x = 0$ on a finite domain $x \in [0, L]$. Homogeneous Dirichlet conditions are inconsistent with the exponential solutions, i.e., a real exponential cannot be zero at a finite value of x, while a complex exponential cannot have both real and imaginary part zero at the same location. A similar argument holds if homogeneous Neumann boundary conditions are applied. To explore this inconsistency, the eigenvalue problem, Eq. (2.8), is solved numerically by discretising using a second-order central finite difference scheme with homogeneous Dirichlet and Neumann conditions. The chosen flow conditions are $U = 2$, $\rho = 0.5$ and $c = 1$.

The eigenvalue spectrum of the discretised continuous equations with homogeneous Dirichlet conditions imposed on $\hat{p}$ and $\hat{v}$ at $x = 0$ is shown in Figure 1. Eigenfunctions corresponding to the large black circles indicated in Figure 1 are plotted in Figure 2. The spectrum contains two branches that are separated by a scaling factor of 3, which corresponds to the ratio of phase speeds (and hence the ratio of the frequencies for waves of the same wavelength) of waves on each branch. Those of the outer branch are the fast acoustic waves with a phase speed of 3, while those of the inner branch are the slow acoustic waves with a phase speed of 1. Fast acoustic waves satisfy $p = \rho c v$, while slow acoustic waves satisfy $p = -\rho c v$.

The low-frequency modes [small imag(λ)] have nonzero growth rates in time, real(λ), and so exhibit exponential spatial growth [see Figure 2(a,b)]. Note that this does not correspond to acoustic waves that are spatially unstable; instead this is the required shape of a traveling undamped wave that must satisfy a decay in energy in the domain with time. In other words, at every point in the domain, a wave of lower amplitude is traveling toward the point, replacing the higher amplitude wave that was previously there but that has moved away. Note also that because the exponential solution cannot satisfy homogeneous Dirichlet boundary conditions, grid-scale artifacts manifest near $x = 0$.

The high-frequency modes [large imag(λ)] approach nondecaying waves [real(λ)$= 0$] and these modes, shown in Figure 2(c,d), resemble low-wavelength waves modulated by a sinusoid, which ensures that the boundary conditions are satisfied.

The spectrum and eigenfunctions for the case of homogeneous Neumann boundary conditions specified on $\hat{p}$ and $\hat{v}$ at $x = 0$ are similar to those for the homogeneous Dirichlet case. For completeness, one of the eigenfunctions computed with homogeneous Neumann boundary conditions is plotted in Figure 3. The artifacts at $x = 0$ are less severe than are those of the homogeneous Dirichlet case.

4.2. *Eigenvectors of the discrete operator*

The eigenvectors of the discrete operator, A_{in}, require no boundary treatment, though the selection of the operator A_{in} ensures that they are equivalent to eigenvectors of the discretised continuous operator A with homogeneous Dirichlet conditions applied.

4.3. *Alternative boundary treatment*

The results so far with homogeneous Dirichlet and Neumann conditions applied indicate that neither of these conditions provide satisfactory eigenfunctions free from grid-scale features. Instead we propose that the equations are discretised with one-sided finite-difference operators at the boundaries, and no other boundary treatment is performed. This is unconventional, but it provides complete freedom for the eigenfunctions to be

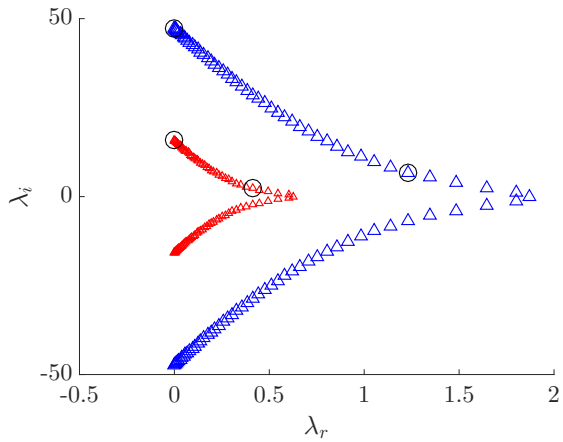

FIGURE 1. Eigenvalue spectrum of the discretised continuous equations, Eq. (2.8), with homogeneous Dirichlet conditions imposed at $x = 0$ for $\hat{p}$ and $\hat{v}$. The conditions are $U = 2$, $\rho = 0.5$ and $c = 1$. Discretised with 100 grid points. Large blue triangles indicate fast acoustic modes, small red triangles indicate slow acoustic modes and large black circles indicate eigenfunctions plotted in Figure 2.

solutions to the continuous equations at the boundaries. The operator is rank deficient, but this does not prevent one from solving the eigenvalue problem. The remaining question is whether or not these eigenfunctions are useful to the expansion of the solution to the original boundary value problem; the answer, we will see, is formally no, but they do resemble solutions to the boundary value problem, unlike the eigenfunctions shown in the previous sections.

The eigenvalue spectrum of the discretised continuous operator, $\mathcal{A}$, with one-sided gradient evaluations on the boundaries is shown in Figure 4, and an example of one of the fast acoustic modes is plotted in Figure 5. All of the eigenvalues are now purely imaginary, indicating purely wave-like modes with no exponential growth in x. The fast and slow acoustic modes now occupy similar regions in the spectrum, with differences in eigenvalue spacing and maximum frequency commensurate with their difference in phase speeds and the discrete set of supported wavelengths in the domain. For an even number of grid points, the lowest frequency modes correspond to one-quarter wavelength of a sinusoid, and each higher mode increments by one-half of a wavelength extra support in the domain. For an odd number of grid points, the lowest frequency modes correspond to one-half of a wavelength. Note that while the modes satisfy homogeneous boundary conditions separately on the real and imaginary parts, this is dependent on the method used to solve the eigenvalue problem, as the eigenfunctions to the continuous equations on an infinite domain are phase independent.

With the eigenfunctions not satisfying homogeneous Dirichlet boundary conditions, the reconstruction of the solution to arbitrary boundary conditions as presented in Section 2.4 is not possible. In fact, a boundary condition with arbitrary real frequency cannot be represented by any combination of the eigenfunctions in our basis, because they are all purely oscillatory in time, making them orthogonal, in time, to any frequency that is not an eigenvalue of the discrete system.

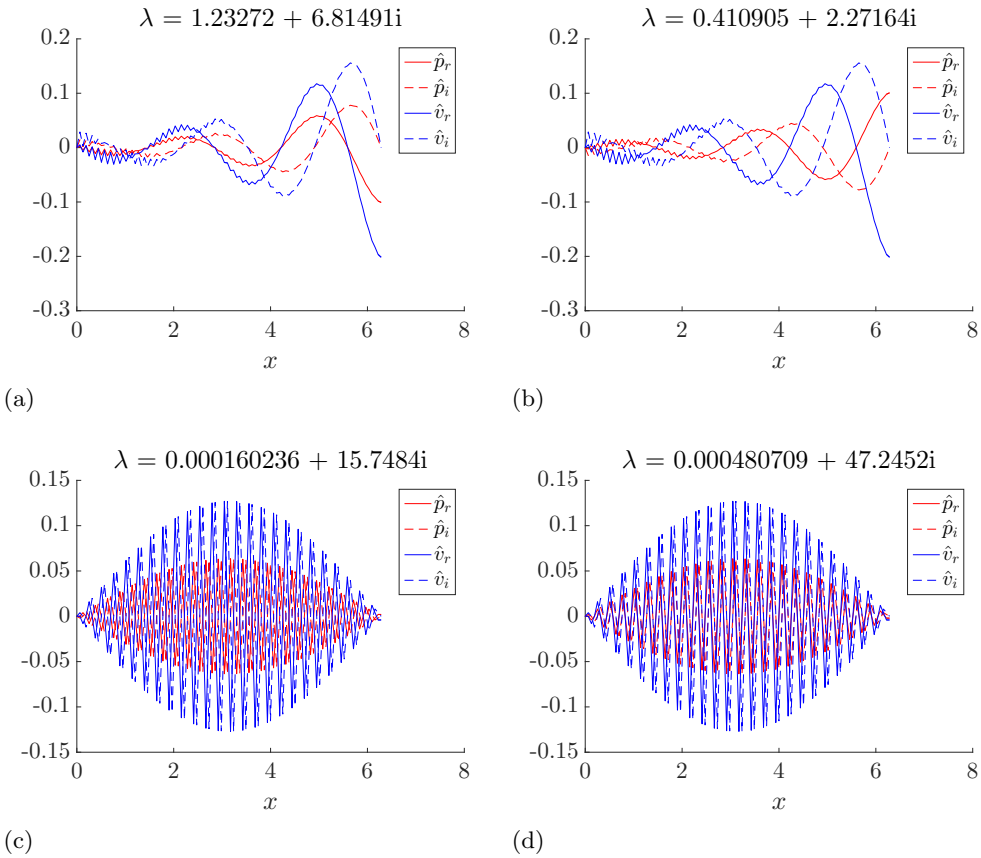

FIGURE 2. A selection of eigenfunctions of the discretised continuous equations, Eq. (2.8), with homogeneous Dirichlet conditions imposed at $x = 0$. The corresponding eigenvalues are indicated in large black circles in Figure 1. (a) and (d) are fast acoustic waves; (b) and (c) are slow acoustic waves.

5. Discussion and conclusions

In the pursuit of an easily interpretable basis with which to understand general solutions to boundary value problems involving wave-like equations, the eigenvectors of the problem with one-sided operators imposed at the boundaries, and no other boundary treatment, may provide a solution. Solving the eigenvalue problem of the one-dimensional linear acoustic equations in a moving medium with homogeneous Dirichlet boundary conditions produces eigenvectors with grid-scale artifacts at the inlet and spatial growth that is not representative of any actual solution to the boundary value problem. Problems with performing an eigenfunction expansion of similar highly non-normal problems have been documented in the past (Trefethen 1997), namely the first derivative operator (Reddy 1993), wave equations (Driscoll & Trefethen 1993) and, closer to Navier-Stokes, the advection-diffusion equation (Reddy & Trefethen 1994). The degree of non-normality depends on the boundary conditions that are imposed.

The imposition of no explicit boundary conditions and the solution of the equations with one-sided operators on the boundaries produce smooth eigenfunctions that are rep-

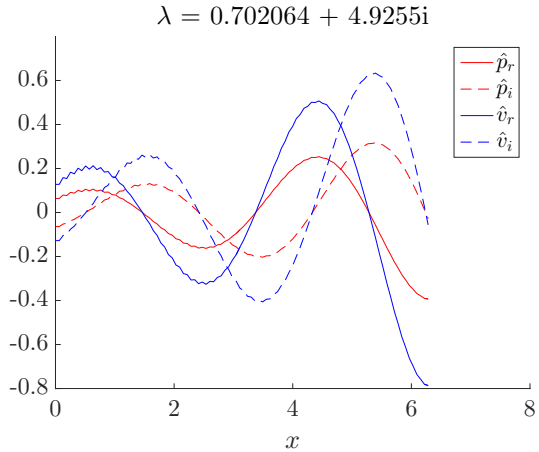

FIGURE 3. Eigenfunction of the discretised continuous equations, Eq. (2.8), with homogeneous Neumann conditions imposed at $x = 0$ for $\hat{p}$ and $\hat{v}$. The conditions are $U = 2$, $\rho = 0.5$ and $c = 1$. Discretised with 100 grid points.

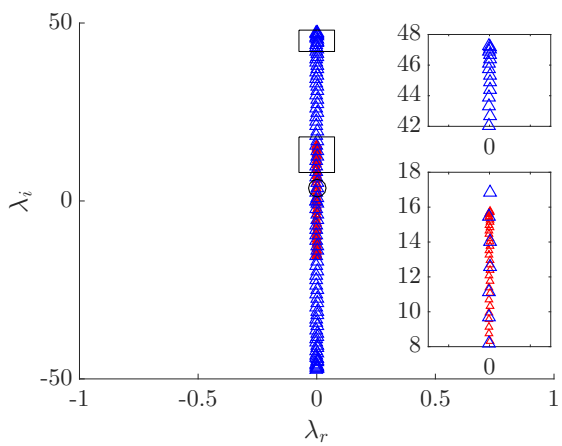

FIGURE 4. Eigenvalue spectrum of the discretised continuous equations, Eq. (2.8), with one-sided gradients applied at the boundaries. The conditions are $U = 2$, $\rho = 0.5$ and $c = 1$. Discretised with 100 grid points. Large blue triangles indicate fast acoustic modes, and small red triangles indicate slow acoustic modes. The large black circle indicates location of the mode plotted in Figure 5.

resentative of solutions to the boundary value problem. The modes resemble infinite domain solutions to the equations that are in-fact normal. Specifically, the eigenfunctions identify the expected fast and slow acoustic modes of sinusoidal nature. While the modes of the discrete problem cannot be used as a basis for the forced problem, their continuous extension can; i.e., any solution to the boundary value problem at a particular frequency can be interpreted as a superposition of one fast and one slow acoustic mode, analogous to d'Alembert's solution to the wave equation. In the context of high-speed external aerodynamics, eigenfunctions with no explicit boundary conditions imposed in the far field may be representative of solutions to the problem in an infinite medium.

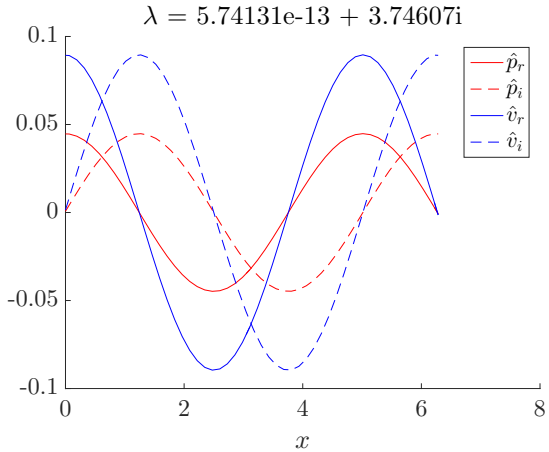

FIGURE 5. Fast acoustic wave with eigenvalue indicated in Figure 4.

While not covered here, preliminary investigation reveals that the input-output modes of this problem are unable to cleanly reproduce this basis.

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