A detailed derivation of energy conservation within a triadic set of spatiotemporal modes

By M. C. Y. Coimbra AND B. J. McKeon

The nonlinear term in the turbulent kinetic energy equation governs the redistribution of energy to different scales in a turbulent flow. This term is conservative, thus only resulting in the transfer of energy and no production or dissipation. It can be shown that this conservation property holds on a triad-by-triad basis, and not only upon accounting for all possible interactions. While this property has been known and utilized for spatial modes, i.e., modes resulting from a Fourier transform in the homogeneous spatial dimensions (see, e.g., Kraichnan 1957; Domaradzki & Rogallo 1990), the detailed derivation of this result has been difficult to find in the existing literature. The purpose of this brief is to provide a comprehensive derivation of the conservative nature of the nonlinear term within individual triadic sets of exact spatiotemporal modes describing an incompressible wall-bounded turbulent flow. Exact spatiotemporal modes extend the spatial triad-by-triad conservation result to include information describing the flow dynamics via the frequency component arising from the temporal Fourier transform. An exact spatiotemporal mode is defined as a Fourier mode resulting from a Fourier transform of a velocity field in the streamwise, spanwise, and temporal dimensions, all three of which are assumed to be periodic. The results of this derivation are not original; rather, this brief aims to clarify the derivation by providing the intermediate steps that are often omitted.

1. Introduction

This brief provides a comprehensive derivation of the energy conservation property within a triadic set of exact spatiotemporal Fourier modes for wall-bounded turbulence. It is well known that the nonlinear term in the turbulent kinetic energy (TKE) equation is purely a transfer term and, upon accounting for all interactions, is conservative. Lesser known is the property of exact energy conservation within any individual set of triadic interactions, although this too is not a new result. Kraichnan (1957) first showed that spatial Fourier modes retain this exact conservation property on a spatial triad-by-triad basis in homogeneous isotropic turbulence (HIT), and Schmid & Henningson (2001) showed that this property also held for spatial modes in wall-bounded flows. Both derivations were done in the temporal domain for modes resulting from a Fourier transform in the homogeneous spatial directions. Barthel (2022) extended this result into the frequency domain, showing that exact spatiotemporal modes retained spatiotemporal triad-by-triad energy conservation, thus incorporating temporal dynamics into the analysis. However, there are details in the aforementioned derivations that seem to be hard to find in the available literature. The results presented here meant to serve as a useful reference for those who are interested in the detailed derivation of this conservation property.

The following sections develop the TKE equation for an exact spatiotemporal mode in incompressible channel flow, then derive the more interesting result of energy conservation for a single set of triadically resonant spatiotemporal modes. We will consider the case of a wall-bounded flow, where the wall-normal coordinate is given by y and the homogeneous spatial directions are x and z. We define an exact spatiotemporal mode as a Fourier mode describing a velocity field that is periodic in the streamwise and spanwise directions and has a periodic temporal signal; the periodicity is exact in the streamwise and spanwise directions for a turbulent channel flow, but the temporal periodicity is an assumption that is made to define a valid Fourier mode of the temporal signal.

2. Derivation of the exact spatiotemporal spectral TKE equation

Here, we derive the spectral form of the TKE equation for incompressible channel flow in terms of exact spatiotemporal modes. We start from the fluctuation equations in physical space and the temporal domain with

$$\begin{aligned}\frac{\partial u_i(x,y,z,t)}{\partial t} &+ u_j(x,y,z,t)\frac{\partial U_i(y)}{\partial x_j} + U_j(y)\frac{\partial u_i(x,y,z,t)}{\partial x_j} \\ &+ \frac{\partial}{\partial x_j}\left(u_i(x,y,z,t)u_j(x,y,z,t) - \langle u_i(x,y,z,t)u_j(x,y,z,t)\rangle\right) \\ &= -\frac{\partial p(x,y,z,t)}{\partial x_i} + \frac{1}{Re_\tau}\frac{\partial^2 u_i(x,y,z,t)}{\partial x_j \partial x_j}.\end{aligned} \tag{2.1}$$

Detailed derivations of the fluctuation equation are widely available (e.g., Pope 2000) and left for the reader. Here, Re_τ is the friction Reynolds number, u_i denotes the velocity components, U_i is the mean velocity averaged in x, z and t, and p denotes the pressure. In order to reach the spectral representation of the TKE equation, we first take the spatial Fourier transform in the homogeneous directions (x and z) for each velocity component. This will be done individually for each component to emphasize the anisotropy of the wall-bounded flow. Let $\hat{u}(\boldsymbol{k},y,t), \hat{v}(\boldsymbol{k},y,t)$, and $\hat{w}(\boldsymbol{k},y,t)$ be the Fourier coefficients of the streamwise, wall-normal, and spanwise velocity components, respectively. Note that $(\hat{\cdot})$ denotes a spatial Fourier mode with corresponding spatial wavenumber $\boldsymbol{k} = [k_x, k_z]$, and $k^2 = k_x^2 + k_z^2$. This gives us

$$\begin{aligned}\frac{\partial \hat{u}(\boldsymbol{k},y,t)}{\partial t} &+ \hat{v}(\boldsymbol{k},y,t)\frac{dU}{dy} + ik_x U\hat{u}(\boldsymbol{k},y,t) + \mathcal{F}_{xz}\left[\frac{\partial}{\partial x_j}(uu_j - \langle uu_j\rangle)\right] \\ &= -ik_x\hat{p}(\boldsymbol{k},y,t) + \frac{1}{Re_\tau}\left(-k^2\hat{u}(\boldsymbol{k},y,t) + \frac{\partial^2\hat{u}(\boldsymbol{k},y,t)}{\partial y^2}\right),\end{aligned}$$

where $\mathcal{F}_{xz}[\cdot]$ denotes a Fourier transform in the streamwise and spanwise directions. Additionally, the stress terms $u_i(x,y,z,t)u_j(x,y,z,t) - \langle u_i(x,y,z,t)u_j(x,y,z,t)\rangle$ have been written as $uu_j - \langle uu_j\rangle$ and $U(y) = U$ for notational brevity. The angled brackets represent averaging across the x, z, and t dimensions such that the resulting quantity is only a function of y. The exact form of the spatially transformed (in the streamwise and spanwise directions) Navier-Stokes equations can be written out for all three velocity components as

$$\begin{aligned}\frac{\partial \hat{u}(\boldsymbol{k},y,t)}{\partial t} &+ \hat{v}(\boldsymbol{k},y,t)\frac{dU}{dy} + ik_x U\hat{u}(\boldsymbol{k},y,t) + ik_x\hat{p}(\boldsymbol{k},y,t) \\ &- \frac{1}{Re_\tau}\left(-k^2\hat{u}(\boldsymbol{k},y,t) + \frac{\partial^2\hat{u}(\boldsymbol{k},y,t)}{\partial y^2}\right) = -\mathcal{F}_{xz}\left[\frac{\partial}{\partial x_j}(uu_j) - \frac{d\langle uv\rangle}{dy}\right]\end{aligned} \tag{2.2}$$

$$\frac{\partial \hat{v}(\boldsymbol{k},y,t)}{\partial t} + ik_xU\hat{v}(\boldsymbol{k},y,t) + \frac{\partial \hat{p}(\boldsymbol{k},y,t)}{\partial y} - \frac{1}{Re_\tau}\left(-k^2\hat{v}(\boldsymbol{k},y,t) + \frac{\partial^2 \hat{v}(\boldsymbol{k},y,t)}{\partial y^2}\right) = -\mathcal{F}_{xz}\left[\frac{\partial}{\partial x_j}(vu_j) - \frac{d\langle vv\rangle}{dy}\right] \tag{2.3}$$

$$\frac{\partial \hat{w}(\boldsymbol{k},y,t)}{\partial t} + ik_xU\hat{w}(\boldsymbol{k},y,t) + ik_z\hat{p}(\boldsymbol{k},y,t) - \frac{1}{Re_\tau}\left(-k^2\hat{w}(\boldsymbol{k},y,t) + \frac{\partial^2 \hat{w}(\boldsymbol{k},y,t)}{\partial y^2}\right) = -\mathcal{F}_{xz}\left[\frac{\partial}{\partial x_j}(wu_j) - \frac{d\langle wv\rangle}{dy}\right], \tag{2.4}$$

where the nonlinear terms have been moved to the right-hand side (RHS). All Fourier coefficients are still functions of y and t at this point.

Now, assume the temporal signal is periodic in time such that a temporal Fourier transform results in an equation in terms of exact spatiotemporal modes. A Fourier transform of the temporal signal yields

$$\left(-i\omega + ik_xU + \frac{k^2}{Re_\tau} - \frac{\partial^2}{\partial y^2}\right)\tilde{\hat{u}}(\boldsymbol{k},\omega,y) + \tilde{\hat{v}}(\boldsymbol{k},\omega,y)\frac{dU}{dy} + ik_x\tilde{\hat{p}}(\boldsymbol{k},\omega,y) = -N_u(\boldsymbol{k},\omega,y) - \frac{d\widetilde{\widehat{\langle uv\rangle}}}{dy} \tag{2.5}$$

$$\left(-i\omega + ik_xU + \frac{k^2}{Re_\tau} - \frac{\partial^2}{\partial y^2}\right)\tilde{\hat{v}}(\boldsymbol{k},\omega,y) + \frac{\partial \tilde{\hat{p}}(\boldsymbol{k},\omega,y)}{\partial y} = -N_v(\boldsymbol{k},\omega,y) - \frac{d\widetilde{\widehat{\langle vv\rangle}}}{dy} \tag{2.6}$$

$$\left(-i\omega + ik_xU + \frac{k^2}{Re_\tau} - \frac{\partial^2}{\partial y^2}\right)\tilde{\hat{w}}(\boldsymbol{k},\omega,y) + ik_z\tilde{\hat{p}}(\boldsymbol{k},\omega,y) = -N_w(\boldsymbol{k},\omega,y) - \frac{d\widetilde{\widehat{\langle wv\rangle}}}{dy}, \tag{2.7}$$

where

$$N_i = \mathcal{F}_t\left[\mathcal{F}_{xz}\left[\frac{\partial}{\partial x_j}(u_iu_j)\right]\right] = \mathcal{F}_{xzt}\left[\frac{\partial}{\partial x_j}(u_iu_j)\right] \tag{2.8}$$

and

$$\frac{d\widetilde{\widehat{\langle u_iv\rangle}}}{dy} = \mathcal{F}_t\left[\mathcal{F}_{xz}\left[\frac{d\langle u_iv\rangle}{dy}\right]\right] \tag{2.9}$$

are the RHS terms after having taken the Fourier transform in the streamwise, spanwise, and temporal dimensions. Note that $\tilde{\hat{(\cdot)}}$ denotes a spatiotemporal mode. For notational compactness, we can define a spatiotemporal wavenumber as $\tilde{\boldsymbol{k}} = [k_x, k_z, \omega]$. A spatiotemporal mode is then uniquely identified by its spatiotemporal wavenumber as $\tilde{\hat{u}}_i(\tilde{\boldsymbol{k}}, y)$.

In order to derive the TKE equation, take the dot product between the conjugate of the spatiotemporal velocity modes and the fluctuation equations, i.e., multiply Eqs. (2.5)

– (2.7) by $\tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)$, $\tilde{\hat{v}}^*(\tilde{\boldsymbol{k}}, y)$, and $\tilde{\hat{w}}^*(\tilde{\boldsymbol{k}}, y)$, respectively, and sum

$$2\Big(-i\omega + ik_x U + \frac{k^2}{Re_\tau}\Big) E(\tilde{\boldsymbol{k}}, y) + \tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{v}}(\tilde{\boldsymbol{k}}, y)\frac{dU}{dy} - \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\frac{\partial^2 \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}, y)}{\partial y^2} + ik_x\tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y) + \tilde{\hat{v}}^*(\tilde{\boldsymbol{k}}, y)\frac{\partial \tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y)}{\partial y} + ik_z\tilde{\hat{w}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y) = -\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)N_j(\tilde{\boldsymbol{k}}, y) - \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\frac{d\widetilde{\widehat{\langle u_j v\rangle}}}{dy}. \tag{2.10}$$

Here, $E(\tilde{\boldsymbol{k}}, y) = \tilde{\hat{u}}_i^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_i(\tilde{\boldsymbol{k}}, y)/2$. We can add Eq. (2.10) to the corresponding equation for its conjugate to derive the real form of $E(\tilde{\boldsymbol{k}}, y)$, since

$$\Re[E(\tilde{\boldsymbol{k}}, y)] = \frac{E(\tilde{\boldsymbol{k}}, y) + E^*(\tilde{\boldsymbol{k}}, y)}{2},$$

which must be a real quantity with the property $E(\tilde{\boldsymbol{k}}, y) = E^*(\tilde{\boldsymbol{k}}, y)$. The equations then simplify to

$$\Re\Big[\tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{v}}(\tilde{\boldsymbol{k}}, y)\frac{dU}{dy}\Big] + \Re\Big[\frac{\partial}{\partial y}\big(\tilde{\hat{v}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y)\big)\Big] + \frac{2k^2}{Re_\tau}E(\tilde{\boldsymbol{k}}, y) - \frac{1}{Re_\tau}\Big[\frac{\partial^2 E(\tilde{\boldsymbol{k}}, y)}{\partial y^2} - \frac{\partial \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)}{\partial y}\frac{\partial \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}, y)}{\partial y}\Big] = -\Re\Big[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)N_j(\tilde{\boldsymbol{k}}, y)\Big] - \Re\Big[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\frac{d\widetilde{\widehat{\langle u_j v\rangle}}}{dy}\Big]. \tag{2.11}$$

Consider the last term. We can evaluate the Fourier transform of the Reynolds stress term, since it is a constant in the Fourier transformed directions and only a function of y such that

$$\Re\Big[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\frac{d\widetilde{\widehat{\langle u_j v\rangle}}}{dy}\Big] = \Re\Big[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\Big|\frac{d\langle u_j v\rangle}{dy}\Big|\delta(\omega)\Big], \tag{2.12}$$

where $\delta(\omega)$ is the Dirac delta function and is only nonzero when $\omega = 0$. In other words, the Reynolds stress term only contributes to the $\omega = 0$ mode. The TKE equation for a given mode associated with some nonzero $\boldsymbol{k}$ and ω (since we are concerned with fluctuations) is then

$$\Re\Big[\tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{v}}(\tilde{\boldsymbol{k}}, y)\frac{dU}{dy}\Big] + \Re\Big[\frac{\partial}{\partial y}\big(\tilde{\hat{v}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y)\big)\Big] + \frac{2k^2}{Re_\tau}E(\tilde{\boldsymbol{k}}, y) - \frac{1}{Re_\tau}\Big[\frac{\partial^2 E(\tilde{\boldsymbol{k}}, y)}{\partial y^2} - \frac{\partial \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)}{\partial y}\frac{\partial \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}, y)}{\partial y}\Big] = -\Re\Big[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)N_j(\tilde{\boldsymbol{k}}, y)\Big]. \tag{2.13}$$

Note that $N_j(\tilde{\boldsymbol{k}}, y)$ in the nonlinear transfer term is a convolution in Fourier space, given by

$$N_j(\tilde{\boldsymbol{k}}, y) = N_j(\boldsymbol{k}, \omega, y) = \sum_{\boldsymbol{k}'} \sum_{\omega'} \tilde{\hat{u}}_m(\boldsymbol{k}', \omega', y) \frac{\partial}{\partial x_m} \tilde{\hat{u}}_j(\boldsymbol{k} - \boldsymbol{k}', \omega - \omega', y) \tag{2.14}$$

$$= \sum_{\boldsymbol{k}'} \sum_{\omega'} \tilde{\hat{u}}_m(\boldsymbol{k}', \omega', y) \frac{\partial}{\partial x_m} \tilde{\hat{u}}_j(\boldsymbol{k}'', \omega'', y) \tag{2.15}$$

$$= \sum_{\tilde{\boldsymbol{k}}'} \tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}, y) \frac{\partial}{\partial x_m} \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y), \tag{2.16}$$

where $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}'', \omega = \omega' + \omega''$, and the index m still denotes the vector component. So the exact spatiotemporal transfer term in the TKE equation is given by

$$T(\tilde{\boldsymbol{k}}, y) = \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y) \sum_{\tilde{\boldsymbol{k}}'} \tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y) \frac{\partial}{\partial x_m} \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y). \tag{2.17}$$

The spatiotemporal wavenumber includes the frequency component so spatiotemporal triadic resonance can be compactly written as $\tilde{\boldsymbol{k}} = \tilde{\boldsymbol{k}}' + \tilde{\boldsymbol{k}}''$.

The TKE equation for a single spatiotemporal mode can finally be written out as

$$\Re\left[\tilde{\hat{u}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{v}}(\tilde{\boldsymbol{k}}, y)\frac{dU}{dy}\right] + \Re\left[\frac{\partial}{\partial y}\left(\tilde{\hat{v}}^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{p}}(\tilde{\boldsymbol{k}}, y)\right)\right] \tag{2.18}$$
$$+ \frac{1}{Re_\tau}\left[2k^2 E(\tilde{\boldsymbol{k}}, y) - \frac{\partial^2 E(\tilde{\boldsymbol{k}}, y)}{\partial y^2} + \frac{\partial \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)}{\partial y}\frac{\partial \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}, y)}{\partial y}\right] = -\Re\left[T(\tilde{\boldsymbol{k}}, y)\right].$$

It is important to note that while this governs the TKE for a single mode, the nonlinear term $T(\tilde{\boldsymbol{k}}, y)$ is a convolution over all spatiotemporal wavenumbers. As a result, this nonlinear term allows for energy to be transferred to different scales. This is purely a transfer term, in that upon integrating in y and summing over all $\tilde{\boldsymbol{k}}$ yields

$$\sum_{\tilde{\boldsymbol{k}}} \int_{-1}^{1} T(\tilde{\boldsymbol{k}}, y) dy = 0, \tag{2.19}$$

meaning it is globally conservative.

3. Triad-by-triad energy conservation for exact spatiotemporal modes

In this section, the exact spatiotemporal triad-by-triad conservation property is derived. The nonlinear energy transfer term in the TKE equation is given by

$$\Re\left[T(\tilde{\boldsymbol{k}}, y)\right] = \Re\left[\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y) \sum_{\tilde{\boldsymbol{k}}'} \tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y) \frac{\partial}{\partial x_m} \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\right], \tag{3.1}$$

which represents the net transfer of energy to $\tilde{\boldsymbol{k}}$ from all other resonant modes. Now consider a single triad, i.e., fix $\tilde{\boldsymbol{k}}'$ and integrate in y to account for the wall-normal dependence of each mode. It should be noted that upon integrating in y, the pressure term in Eq. (2.18) integrates to 0 due to homogeneous boundary conditions at the walls. The resulting nonlinear term for a single triadic set is given by

$$S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'') = \Re\left[\int_{-1}^{1} \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\frac{\partial}{\partial x_m}\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y) dy\right], \tag{3.2}$$

where $S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'')$ is the energy transferred to $\tilde{\boldsymbol{k}}$ from $\tilde{\boldsymbol{k}}''$ via $\tilde{\boldsymbol{k}}'$. The integrand of this transfer term, $S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'')$, can be rewritten as

$$S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'') = \Re\Bigg[\int_{-1}^{1} \frac{\partial}{\partial x_m}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big) - \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\frac{\partial}{\partial x_m}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\Big)dy\Bigg]. \tag{3.3}$$

Consider the first term in the integrand of Eq. (3.3) and expand the summation on m so that

$$\begin{aligned}\int_{-1}^{1} \frac{\partial}{\partial x_m}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big)dy \\ = \int_{-1}^{1} \frac{\partial}{\partial x_\beta}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big) + \frac{\partial}{\partial y}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{v}}(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big)dy.\end{aligned} \tag{3.4}$$

Note that any Roman letter index used in this brief denotes a vector component with indices that cycle through 1, 2, and 3, while the Greek index β cycles through 1 and 3. The summation on repeated indices is implied. The second term on the RHS of Eq. (3.4) evaluates to 0 because of homogeneous boundary conditions at the walls ($y = \pm 1$) in the channel flow considered here. The first term on the RHS can then be expanded as

$$\begin{aligned}\int_{-1}^{1} \frac{\partial}{\partial x_m}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big)dy = \int_{-1}^{1} & \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\Big(\frac{\partial}{\partial x_\beta}\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big) \\ & + \tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\Big(\frac{\partial}{\partial x_\beta}\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\Big)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y) \\ & + \Big(\frac{\partial}{\partial x_\beta}\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\Big)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)dy.\end{aligned}$$

Recall that we can evaluate derivatives in the spectral domain in the x ($\beta = 1$) and z ($\beta = 3$) directions as $\partial/\partial x_\beta = ik_\beta$ (note the lack of a tilde—this is the β component of a spatial wavenumber). Due to the fact that the derivatives are purely spatial derivatives, this derivation follows closely that of Schmid & Henningson (2001); while the temporal dynamics are now included in the spatiotemporal modes, hence $\tilde{\boldsymbol{k}}$ rather than $\boldsymbol{k}$ [as used in Schmid & Henningson (2001)], spatiotemporal triad-by-triad conservation holds exactly because the derivatives in the nonlinear term are only spatial. Recall that in a turbulent channel, as is considered here, the spatial modes are exact. Computing the spatial derivatives in the spectral domain yields

$$\begin{aligned}\int_{-1}^{1} \frac{\partial}{\partial x_m}\Big(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)\Big)dy = \int_{-1}^{1} & (ik''_\beta)\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y) \\ & + (ik'_\beta)\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y) \\ & + (-ik_\beta)\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}}, y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}', y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'', y)dy,\end{aligned}$$

where the $(-ik_\beta)$ in the last term comes from the differentiation acting on the complex conjugate of $\tilde{\hat{u}}_j^*(\boldsymbol{k})$. The above can be rearranged to show that

$$\int_{-1}^{1} \frac{\partial}{\partial x_m}\left(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}',y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)\right)dy$$
$$= i\int_{-1}^{1}(k''_\beta + k'_\beta - k_\beta)\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)\tilde{\hat{u}}_\beta(\tilde{\boldsymbol{k}}',y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)dy$$

and since triadic compatibility states $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}'' \to \boldsymbol{k}' + \boldsymbol{k}'' - \boldsymbol{k} = 0$,

$$\int_{-1}^{1} \frac{\partial}{\partial x_m}\left(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}',y)\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)\right)dy = 0.$$

As a result, Eq. (3.3) can be written as

$$S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') = \Re\left[\int_{-1}^{1} -\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)\frac{\partial}{\partial x_m}\left(\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}',y)\right)dy\right]. \tag{3.5}$$

We stress here that computing the spatial derivatives in the spectral domain is one of the key steps in showing triad-by-triad energy conservation—it reveals the triadic resonance condition $\boldsymbol{k}' + \boldsymbol{k}'' - \boldsymbol{k}$ that forces the first term on the RHS of Eq. (3.4) to go to 0.

Using product rule, we can show that

$$S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') = \Re\left[\int_{-1}^{1} -\tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}',y)\frac{\partial}{\partial x_m}\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)dy\right] \tag{3.6}$$

because of continuity ($\partial\tilde{\hat{u}}_m/\partial x_m = 0$ for all $\tilde{\boldsymbol{k}}$). Upon comparing Eq. (3.6) to Eq. (3.2), it is apparent that the two equations are of the same form and can be written as

$$S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') = -\Re\left[\int_{-1}^{1} \tilde{\hat{u}}_j(\tilde{\boldsymbol{k}}'',y)\tilde{\hat{u}}_m(\tilde{\boldsymbol{k}}',y)\frac{\partial}{\partial x_m}\tilde{\hat{u}}_j^*(\tilde{\boldsymbol{k}},y)dy\right]$$
$$= -S(-\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}}',-\tilde{\boldsymbol{k}}),$$

resulting in the identity

$$S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') = -S(-\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}}',-\tilde{\boldsymbol{k}}). \tag{3.7}$$

Eq. (3.7) is key in showing that energy transfer within a triadic set is conservative. The net energy transferred within a triadic set involving modes $\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}'$, and $\tilde{\boldsymbol{k}}''$, which satisfy $\tilde{\boldsymbol{k}} = \tilde{\boldsymbol{k}}' + \tilde{\boldsymbol{k}}''$ (and therefore $\tilde{\boldsymbol{k}}' = \tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}'' = \tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}'$) is given by

$$S(\tilde{\boldsymbol{k}}|\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}}'|\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}}''|\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}'), \tag{3.8}$$

where

$$S(\tilde{\boldsymbol{k}}\ \ |\tilde{\boldsymbol{k}}',\ \ \ \tilde{\boldsymbol{k}}'') = S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}}') \tag{3.9}$$
$$S(\tilde{\boldsymbol{k}}'\ |\tilde{\boldsymbol{k}}\ ,-\tilde{\boldsymbol{k}}'') = S(\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}}',-\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}}) \tag{3.10}$$
$$S(\tilde{\boldsymbol{k}}''|\tilde{\boldsymbol{k}}\ ,-\tilde{\boldsymbol{k}}'\ ) = S(\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}') + S(\tilde{\boldsymbol{k}}'',-\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}). \tag{3.11}$$

Physically, $S(\tilde{\boldsymbol{k}}|\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'')$ should be interpreted as the energy transferred to $\tilde{\boldsymbol{k}}$ from both $\tilde{\boldsymbol{k}}''$ and $\tilde{\boldsymbol{k}}'$ via $\tilde{\boldsymbol{k}}'$ and $\tilde{\boldsymbol{k}}''$, respectively. The difference in the contributions from $S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}',\tilde{\boldsymbol{k}}'')$ and $S(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}'',\tilde{\boldsymbol{k}}')$ arises from which mode the gradient operator acts on, a distinction that has been made in previous works (see, e.g., Sharma *et al.* 2017; Huang 2025). However, since $S(\tilde{\boldsymbol{k}}|\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'')$ accounts for the energy transferred from both $\tilde{\boldsymbol{k}}'$ and $\tilde{\boldsymbol{k}}''$, this distinction is not needed to prove triad-by-triad conservation; the net transfer between all modes in a triad is accounted for in Eq. (3.8). Now, using the result in Eq. (3.7), we can write the equivalent expressions for the terms present in Eqs. (3.9)–(3.11):

$$S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'') = -S(-\tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}', -\tilde{\boldsymbol{k}}) \tag{3.12}$$

$$S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}') = -S(-\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'', -\tilde{\boldsymbol{k}}) \tag{3.13}$$

$$S(\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}'') = -S(\tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}') \tag{3.14}$$

$$S(\tilde{\boldsymbol{k}}', -\tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}) = -S(-\tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}'', -\tilde{\boldsymbol{k}}') \tag{3.15}$$

$$S(\tilde{\boldsymbol{k}}'', \tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}') = -S(\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}'') \tag{3.16}$$

$$S(\tilde{\boldsymbol{k}}'', -\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}) = -S(-\tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}', -\tilde{\boldsymbol{k}}''). \tag{3.17}$$

Using Eqs. (3.9)–(3.17) and recalling that $S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'')$ is real, thus retaining the property $S(\tilde{\boldsymbol{k}}, \tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'') = S(-\tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}', -\tilde{\boldsymbol{k}}'')$, it can be shown algebraically that

$$S(\tilde{\boldsymbol{k}}|\tilde{\boldsymbol{k}}', \tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}}'|\tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}'') + S(\tilde{\boldsymbol{k}}''|\tilde{\boldsymbol{k}}, -\tilde{\boldsymbol{k}}') = 0, \tag{3.18}$$

meaning the energy transfer within any triadically resonant set is conservative.

4. Conclusion

We have shown a detailed derivation of the spatiotemporal triad-by-triad energy conservation property for exact spatiotemporal modes in a wall-bounded turbulent flow. This derivation closely followed that in Barthel (2022), filling in the steps that were omitted. We note that in showing that energy is conserved on a triad-by-triad basis, it becomes trivial to show that the transfer term, $T(\tilde{\boldsymbol{k}}, y)$, in the TKE equation is globally conservative, as stated in Eq. (2.19). As a result, this derivation serves as the proof for both the global and triad-by-triad conservation properties of exact spatiotemporal modes.

REFERENCES

Barthel, B. 2022 On the variational principles of linear and nonlinear resolvent analysis. PhD thesis, California Institute of Technology.

Domaradzki, J. A. & Rogallo, R. S. 1990 Local energy transfer and nonlocal interactions in homogeneous, isotropic turbulence. *Phys. Fluids.* **2**, 413–426.

Huang, Y. 2025 Linear and non-linear interactions involving large-scale structures in turbulence. PhD thesis, California Institute of Technology.

Kraichnan, R. H. 1957 Relation of fourth-order to second-order moments in stationary isotropic turbulence. *Phys. Rev.* **107**, 1485–1490.

Pope, S. B. 2000 *Turbulent Flows*. Cambridge University Press.

Schmid, P. J. & Henningson, D. S. 2001 *Stability and Transition in Shear Flows*. Springer.

Sharma, A. S., Moarref, R. & McKeon, B. J. 2017 Scaling and interaction of self-similar modes in models of high Reynolds number wall turbulence. *Philos. Trans. R. Soc. A* **375**, 20160089.