Nondissipative variable density turbulence: Routes toward statistical equilibrium

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In the dissipativeless limit of no molecular viscosity and no diffusivity, the dynamics of a variable density turbulent flow converge toward a statistical equilibrium state characterized by a stable coarse-grained buoyancy field (Venaille *et al.* 2017). In this brief, we develop a simple Langevin model for the coarse-grained dynamics, aiming to describe their time evolution toward their statistical equilibrium state.

1. Introduction

Mixing is a key process in a large variety of settings with length scales and timescales spanning orders of magnitude. Of particular significance is the mixing of density in oceanic flows, where the rate at which turbulence mixes the water column is known to have a leading-order impact on the large-scale global ocean circulation (Wunsch & Ferrari 2004), biological productivity (Marra *et al.* 1990) and tracer pathways (Cimoli *et al.* 2023). It is currently impossible to resolve mixing directly in observational measurements and large-scale models, and appropriate mixing proxies and parameterizations are required.

One such proxy is the mixing efficiency (Caulfield 2021), that is the amount of energy that effectively goes into mixing relative to the total amount of energy injected into the system. It can be related to the probability distribution of buoyancy levels in the flow (Tseng & Ferziger 2001), pinpointing the necessity to accurately model the time evolution of the distribution of density levels at a given point in space. As a first step, it is instructive to understand how this distribution evolves under stirring only (i.e., in the absence of molecular diffusion and hence mixing), providing insight into the buoyancy levels that fluid parcels encounter along their stochastic paths and will mix into if molecular diffusion is turned on. This is the point of view of Venaille *et al.* (2017). In particular, they show that in the limit of no diffusivity and viscosity, the probability of finding a buoyancy level at a given point in space reaches a steady state that arises as a balance between the competing effects of the flow energetics that displace fluid parcels away from their neutrally buoyant position and restoring buoyancy forces. In this work, we aim to provide insight into the route toward this steady state.

Before moving forward, we summarize the findings of Venaille *et al.* (2017) on which the present work is based. The inviscid and adiabatic dynamics evolving through a turbulent stirring field satisfying the incompressible Navier-Stokes equations under the Boussinesq approximation are shown to converge toward a statistical equilibrium state characterized by a probability density function (PDF) $p_\infty(\boldsymbol{x},\boldsymbol{u},b)$ of finding a buoyancy level b and velocity $\boldsymbol{u}=(u,v,w)$ in the vicinity of $\boldsymbol{x}=(x,y,z)$ of the form

$$p_\infty(\boldsymbol{x},\boldsymbol{u},b)=\frac{1}{\mathcal{Z}(z)}\exp(-\beta^{-1}\mathcal{H}),\quad \mathcal{H}(\boldsymbol{x},\boldsymbol{u},b):=\frac{1}{2}\boldsymbol{u}^2-bz+\gamma(b). \tag{1.1}$$

This distribution maximized the entropy $\mathcal{S}=\int p\log(p)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u}\mathrm{d}b$ under the constraints of energy conservation (β^{-1} being the Lagrange multiplier associated with this constraint; it can be interpreted as a temperature associated with the turbulence that excites the erratic movement of fluid parcels) and buoyancy histogram conservation [with Lagrange multiplier $\gamma(b)$]. Here, $\mathcal{Z}(z) = \int \exp(-\beta^{-1}\mathcal{H})\mathrm{d}b\mathrm{d}\boldsymbol{u}$ ensures normalization such that

$$\forall \boldsymbol{x}, \quad \int p_\infty(\boldsymbol{x}, \boldsymbol{u}, b)\mathrm{d}b\mathrm{d}\boldsymbol{u} = 1, \tag{1.2}$$

that is, at any point in space, there is a fluid parcel. This statistical steady state is associated with an averaged (coarse-grained) buoyancy field

$$\overline{b}_\infty(z) := \int b p_\infty(\boldsymbol{x}, \boldsymbol{u}, b)\mathrm{d}b\mathrm{d}\boldsymbol{u} = \frac{1}{\beta^{-1}}\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}z}. \tag{1.3}$$

Whereas the statistical equilibrium state is known, the time evolution of p toward its long-time limit p_∞ is less clear. Venaille *et al.* (2017) suggested that the probability of finding a density level b in the vicinity of $\boldsymbol{x}$, $p_b(\boldsymbol{x}, b) = \int p(\boldsymbol{x}, \boldsymbol{u}, b)\mathrm{d}\boldsymbol{u}$ satisfies an equation of the form

$$\frac{\partial p_b}{\partial t} = D\frac{\partial}{\partial z}\left[\frac{\partial p_b}{\partial z} + \beta^{-1}(\overline{b} - b)p_b\right]. \tag{1.4}$$

Here $\overline{b} = \int b p(\boldsymbol{x}, \boldsymbol{u}, b; t)\mathrm{d}b\mathrm{d}\boldsymbol{u}$ is the instantaneous buoyancy profile. The choice of the diffusivity D does not alter the steady-state solution of Eq. (1.4) but rather how fast this steady state solution is reached. Here, we show that this equation arises as a distinguished limit of the Fokker-Planck equation corresponding to a particular Langevin model associated with the stochastic (Lagrangian) paths of coarse-grained fluid parcels.

2. A Langevin model for coarse-grained variable-density turbulence

2.1. *The Langevin model and its associated Fokker-Planck equation*

Let us follow a fluid parcel at (stochastic) position (X, Y, Z) with velocity (U, V, W) and buoyancy level B. We assume that the stochastic processes $X(t), Y(t), Z(t), U(t), V(t), W(t)$ and $B(t)$ follow the set of stochastic differential equations

$$\begin{cases} \mathrm{d}X = U\mathrm{d}t (= \frac{\partial \mathcal{H}}{\partial u}\mathrm{d}t), \ \mathrm{d}Y = V\mathrm{d}t (= \frac{\partial \mathcal{H}}{\partial v}\mathrm{d}t), \ \mathrm{d}Z = W\mathrm{d}t (= \frac{\partial \mathcal{H}}{\partial w}\mathrm{d}t), \\ \mathrm{d}U = -\gamma_u U\mathrm{d}t + \sqrt{2\beta\gamma_u}\mathrm{d}\mathcal{W}_u, \\ \mathrm{d}V = -\gamma_v V + \sqrt{2\beta\gamma_v}\mathrm{d}\mathcal{W}_v, \\ \mathrm{d}W = -\frac{\partial \mathcal{H}}{\partial z}\mathrm{d}t - \overline{b}(Z, t)\mathrm{d}t - \gamma_w W\mathrm{d}t + \sqrt{2\beta\gamma_w}\mathrm{d}\mathcal{W}_w, \\ \mathrm{d}B = -\gamma_b\frac{\partial \mathcal{H}}{\partial b}\mathrm{d}t + \sqrt{2\beta\gamma_b}\mathrm{d}\mathcal{W}_b. \end{cases} \tag{2.1}$$

Here, the γs are constants, the $\mathcal{W}$s are Wiener processes (modeling turbulent fluctuations) and $\overline{b}(z, t) := \int b p(\boldsymbol{x}, \boldsymbol{u}, b; t)\mathrm{d}b\mathrm{d}\boldsymbol{u} \xrightarrow[t \to +\infty]{} \overline{b}_\infty(z)$ is the (instantaneous) background buoyancy field. Hence, the term $-\partial_z\mathcal{H} - \overline{b} = B - \overline{b}$ is nothing but the Archimedes forces acting on the fluid parcel. Note that since we are considering the dissipativeless limit of no molecular diffusivity, the buoyancy level carried by a (Lagrangian) fluid parcel is constant, and hence we consider $\gamma_b = 0$.

The Fokker-Planck equation associated with Eq. (2.1) reads [considering only the coordinates (z, w, b) for simplicity]:

$$\frac{\partial p}{\partial t} = -\left[\frac{\partial}{\partial z}\left(\frac{\partial \mathcal{H}}{\partial w}p\right) + \frac{\partial}{\partial w}\left(-\gamma_w\frac{\partial \mathcal{H}}{\partial w}p - \frac{\partial \mathcal{H}}{\partial z}p - \overline{b}p\right)\right] + \frac{\partial^2}{\partial w^2}(\gamma_w\beta p). \tag{2.2}$$

For consistency, let us check that the distribution in Eq. (1.1) is a stationary solution of Eq. (2.2). First, recall that, in the absence of the background buoyancy field $\overline{b}$, $\exp(-\beta^{-1}\mathcal{H})/\mathcal{Z}$ (with the normalization constant $\mathcal{Z} = \int \exp(-\beta^{-1}\mathcal{H})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{u}\mathrm{d}b$ independent of z for now) is a stationary solution of Eq. (2.2) with $\overline{b} = 0$ [indeed, in that case, Eq. (2.1) corresponds to the generalized Langevin equation associated with the energy $\mathcal{H}$]. When replacing $\mathcal{Z}$ by $\mathcal{Z}(z)$ and adding $\overline{b}_\infty \neq 0$, the only terms affected in (2.2) are (with primes denoting differentiation with respect to a function's argument)

$$\frac{\partial}{\partial z}\left(\frac{\partial \mathcal{H}}{\partial w}p_\infty\right) = w\frac{\partial}{\partial z}\left[\frac{1}{\mathcal{Z}(z)}\exp(-\beta^{-1}\mathcal{H})\right] \tag{2.3}$$

$$= -w\frac{\mathcal{Z}'(z)}{\mathcal{Z}(z)}p_\infty - w\beta^{-1}\frac{\partial \mathcal{H}}{\partial z}p_\infty \tag{2.4}$$

$$= -w\frac{\mathrm{dlog}\mathcal{Z}}{\mathrm{d}z}p_\infty - \beta^{-1}\frac{\partial \mathcal{H}}{\partial w}\frac{\partial \mathcal{H}}{\partial z}p_\infty, \tag{2.5}$$

$$= -w\frac{\mathrm{dlog}\mathcal{Z}}{\mathrm{d}z}p_\infty - \underbrace{\frac{\partial}{\partial w}\left[-\frac{\partial \mathcal{H}}{\partial z}p_\infty\right]}_{\text{Cancels first term in Eq. (2.2)}}, \tag{2.6}$$

and

$$\frac{\partial}{\partial w}\left(-\overline{b}_\infty p_\infty\right) = \beta^{-1}\overline{b}_\infty w p_\infty \overset{(1.3)}{=} w\frac{\mathrm{dlog}\mathcal{Z}}{\mathrm{d}z}p_\infty. \tag{2.7}$$

Hence, all the terms affected by these changes cancel out, and p_∞ as defined in Eq. (1.1) is indeed a stationary solution of Eq. (2.2).

2.2. *The overdamped limit* $\gamma_w \gg 1$

We define $\epsilon := 1/\gamma_w$ and consider the limit $\epsilon \ll 1$ (overdamped limit). Let us consider the change of variables

$$\overline{t} = \epsilon t,\ \overline{p}(z, w, b, \overline{t}) = p(z, w, b, t). \tag{2.8}$$

Note that large friction ($\epsilon \ll 1$) leads to fast relaxation of momentum. We can therefore consider w as a fast variable and so as a good candidate for averaging.

In this new coordinate system, the Fokker-Planck equation[Eq. (2.2)] reads as

$$\begin{cases} \epsilon^2 \frac{\partial \overline{p}}{\partial \overline{t}} & = \epsilon G[\overline{p}] + L[\overline{p}], \\ L[\overline{p}] & := \frac{\partial}{\partial w}\left(\frac{\partial \mathcal{H}}{\partial w}\overline{p}\right) + \beta\frac{\partial^2 \overline{p}}{\partial w^2}, \\ G[\overline{p}, \overline{b}] & := -\frac{\partial \mathcal{H}}{\partial w}\frac{\partial \overline{p}}{\partial z} + \frac{\partial \mathcal{H}}{\partial z}\frac{\partial \overline{p}}{\partial w} + \overline{b}\frac{\partial \overline{p}}{\partial w}. \end{cases} \tag{2.9}$$

We now seek a solution of the form

$$\overline{p}(z, w, b, \overline{t}) = \overline{p}^{(0)}(z, w, b, \overline{t}) + \epsilon\overline{p}^{(1)}(z, w, b, \overline{t}) + \cdots . \tag{2.10}$$

A similar expansion for $\overline{b}$ is used. At order $\mathcal{O}(1)$, we have $L[\overline{p}^{(0)}] = 0$ and hence

$$\overline{p}^{(0)}(z, w, b, \overline{t}) = \overline{p_b}(z, b, \overline{t})f(w, \overline{t}),\ \text{with}\ f(w, \overline{t}) := \frac{1}{\sqrt{2\pi\beta}}\exp\left[-\frac{w^2}{2\beta}\right]. \tag{2.11}$$

The order $\mathcal{O}(\epsilon)$-equation reads as

$$0 = G[\overline{p}^{(0)}, \overline{b}^{(0)}] + L[\overline{p}^{(1)}] \Rightarrow L[\overline{p}^{(1)}] = wf\frac{\partial \overline{p_b}}{\partial z} - \frac{\partial \mathcal{H}}{\partial z}p_b\frac{\partial f}{\partial w} - \overline{b}^{(0)}p_b\frac{\partial f}{\partial w} \tag{2.12}$$

$$\Rightarrow L[\overline{p}^{(1)}] = wf(w, \overline{t})\overline{h}, \tag{2.13}$$

where $\overline{h}(z,b,\overline{t}) := \frac{\partial \overline{p_b}}{\partial z} + \beta^{-1}\frac{\partial \mathcal{H}}{\partial z}\overline{p_b} + \beta^{-1}\overline{b}^{(0)}\overline{p_b}$. We check that $\overline{p}^{(1)} = -wf(w,\overline{t})\overline{h}$ is a solution of Eq. (2.13). At order $\mathcal{O}(\epsilon^2)$, we have

$$\frac{\partial \overline{p}^{(0)}}{\partial \overline{t}} = G[\overline{p}^{(0)}, \overline{b}^{(1)}] + G[\overline{p}^{(1)}, \overline{b}^{(0)}] + L[\overline{p}^{(2)}]. \tag{2.14}$$

Integrating Eq. (2.14) with respect to the fast variable w (and assuming that $\overline{p}$ decreases toward 0 fast enough as $|w| \to +\infty$ and recalling that $\int wf \mathrm{d}w = 0$), we have

$$\int \frac{\partial \overline{p}^{(0)}}{\partial \overline{t}} \mathrm{d}w = -\int \frac{\partial \mathcal{H}}{\partial w}\frac{\partial \overline{p}^{(1)}}{\partial z} \mathrm{d}w. \tag{2.15}$$

Computing the various terms gives the diffusion equation

$$\begin{aligned} \frac{\partial \overline{p_b}}{\partial \overline{t}} &= \overline{D}\frac{\partial \overline{h}}{\partial z} \\ &= \overline{D}\frac{\partial}{\partial z}\left[\frac{\partial \overline{p_b}}{\partial z} + \beta^{-1}\frac{\partial \mathcal{H}}{\partial z}\overline{p_b} + \beta^{-1}\overline{b}^{(0)}\overline{p_b}\right] \\ &= \overline{D}\frac{\partial}{\partial z}\left[\frac{\partial \overline{p_b}}{\partial z} + \beta^{-1}\left(\overline{b}^{(0)} - b\right)\overline{p_b}\right]. \end{aligned} \tag{2.16}$$

Here $\overline{D} := \int w^2 f(w,\overline{t})\mathrm{d}w = \beta$. Going back to the initial coordinates, we get

$$\boxed{\frac{\partial p_b}{\partial t} = D\frac{\partial}{\partial z}\left[\frac{\partial p_b}{\partial z} + \beta^{-1}(\overline{b} - b)p_b\right],\ D := \frac{\beta}{\gamma_w}.} \tag{2.17}$$

This is Eq. (3.7) in Venaille *et al.* (2017). The first term on the right-hand side of Eq. (2.17) corresponds to the dispersive effect of turbulent fluctuations on the fluid parcels' position. The second term represents restratification toward the mean buoyancy field $\overline{b}$.

2.3. *Parameter estimation*

The model in (2.17) involves two free parameters (the dispersivity D and effective temperature β). We here speculate on scalings for these quantities with respect to bulk, measurable quantities in the flow.

The competing effects of turbulent fluctuations that vertically displace fluid parcels and buoyancy that brings them back to their neutrally buoyant position constrain the vertical distance fluid parcels travel (Kimura & Herring 1996) to a layer of depth L_S. We here infer scalings for the screening length L_S from a force balance argument between inertia and buoyancy. For simplicity, we here consider an initial two-layer density profile consisting of two density levels, ρ_0 and ρ_1. For a fluid parcel of density ρ_0 and volume V_0 evolving in the layer of density ρ_1, force balance gives, for the vertical displacement z of the parcel,

$$\rho_0 V_0 \mathrm{d}_t^2 z = -gV_0[\rho_0 - \rho_1] \Rightarrow d_t^2 z = -g\left[1 - \frac{\rho_1}{\rho_0}\right]. \tag{2.18}$$

Here g is the acceleration of gravity. The solution to the above reads [assuming $z(t = 0) = 0$ and $\mathrm{d}_t z(t = 0) = \sqrt{2\mathcal{E}_k}$, with $\mathcal{E}_k$ the initial kinetic energy (per unit mass); recall that since we assumed no molecular diffusivity, the density of the fluid parcel remains constant in time]

$$z(t) = -\frac{1}{2}g\left[1 - \frac{\rho_1}{\rho_0}\right]t^2 + \sqrt{2\mathcal{E}_k}t. \tag{2.19}$$

The maximum excursion of the parcel then satisfies (with H the total depth of the water column)

$$\frac{L_S}{H} \sim \frac{\mathcal{E}_k}{gH}\left[1-\frac{\rho_1}{\rho_0}\right]^{-1} := \frac{1}{Ri}, \tag{2.20}$$

with Ri the Richardson number. Hence, we expect the steady solution of Eq. (2.17) to exhibit a characteristic (dimensionless) length scale $1/Ri$, as formally shown by Venaille *et al.* (2017). Steady-state balance (at characteristic scale L_S) in Eq. (2.17) gives

$$\frac{1}{L_S} \sim \frac{g(\rho_0-\rho_1)}{\rho_0\beta} \Rightarrow \beta \sim \mathcal{E}_k, \tag{2.21}$$

as expected from the formula $\beta = \int w^2 f(w,t)\mathrm{d}w$ derived in the previous section.

To constrain D, we need to estimate the time to reach the steady-state solution of Eq. (2.17). We assume that it corresponds to the time for a parcel of fluid to come back to its neutrally buoyant position, that is, the time for a parcel to explore the entire depth that is accessible energetically (i.e., L_S). From Eq. (2.19), we see that this time is of order

$$t_R \sim \frac{\sqrt{\mathcal{E}_k}}{g}\left[1-\frac{\rho_1}{\rho_0}\right]^{-1} \Rightarrow D \sim \frac{L_S^2}{t_R} \sim \frac{\mathcal{E}_k^{3/2}}{g}\left[1-\frac{\rho_1}{\rho_0}\right]^{-1} = \frac{\sqrt{\mathcal{E}_k}H}{Ri}. \tag{2.22}$$

The stronger the stratification (large Richardson number Ri), the weaker the effect of turbulence on dispersing fluid parcels. Note that we considered a two-layer system in this section. In the case of linear stratification, we expect the above scalings to change, as the buoyancy forces the fluid parcels experience would be altered. In particular, using similar arguments, we can show $L_S/H \sim \sqrt{1/Ri}$ in that case.

3. Conclusions

In this brief, we derived a set of stochastic differential equations describing the stochastic paths of fluid parcels in a variable density turbulent flow with no diffusivity or viscosity. The dynamics are shown to converge toward a statistical steady state consistent with the work of Venaille *et al.* (2017). Our work sheds light on the evolution of the flow statistics toward the equilibrium state. In particular, we derived a simple differential equation for the probability distribution of finding a buoyancy level at a given height that shows that the equilibrium state arises as a competition between turbulent dispersion and restratification toward a (dynamically evolving) mean buoyancy profile.

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