# An evaluation of a conservative fourth order DNS code in turbulent channel flow

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## 1. Motivation and objectives

Direct numerical simulation (DNS) and large eddy simulation (LES) of turbulent flows require a numerical method that is able to capture a wide range of turbulent length scales. In DNS, all the turbulent length and time scales are resolved. In LES, the large energy carrying length scales of turbulence are resolved and the small structures are modeled. The separation of large and small scales is done using a filtering procedure that is applied to the Navier-Stokes equations. The effect of the small scale turbulence on the resolved scales is modeled using a subgrid scale (SGS) model.

Widely used SGS models are the scale similarity models by Bardina *et al.* (1980) and Liu *et al.* (1994) and the eddy viscosity based dynamic Smagorinsky model proposed by Germano *et al.* (1991). In general, scale similarity models do not dissipate enough energy, and eddy viscosity models do not carry enough stress (Baggett *et al.* (1997) and Jiménez (1998)). SGS models tend to over- and underpredict velocity fluctuations (Kravchenko & Moin (1997), Domaradzki & Loh (1999), Sarghini *et al.* (1999)).

SGS models typically use information from the smallest resolved length scales to model the stresses of the unresolved scales. Therefore, it is of great importance that these resolved length scales are captured accurately. This requires that the numerical error of the scheme is sufficiently small. One approach is to use high order finite difference schemes. However, high order schemes require that the differentiation and filtering operations commute. This is generally not the case in inhomogeneous flow fields where the required smallest resolved length scales vary throughout the flow field. For this situation, the filter width varies introducing commutation errors of $O(\Delta^2)$ where $\Delta$ represents the filter width (Ghosal & Moin (1995) and Ghosal (1996)).

High order finite difference schemes with good conservation properties and three-dimensional commutative filters have been developed by Morinishi *et al.* (1998), Vasilyev *et al.* (1998) and Vasilyev (2000). These discretization schemes can conserve energy, which ensures a stable simulation free of numerical dissipation. The proposed commutative filters are discrete and can be constructed to commute up to any desired order.

The objective of this work is to investigate the SGS shear stresses predicted by different SGS models and investigate the influence of numerical errors and filtering in three dimensions. This is done using a fourth order finite difference channel flow code. The code was developed by Morinishi *et al.* (1998), Vasilyev *et al.* (1998) and Vasilyev (2000). It conserves kinetic energy and uses commutative filters. Most channel flow simulations have been made using either spectral codes or second order finite differences using filtering in the homogeneous directions only.

To validate the fourth order algorithm, both DNS and LES have been performed of a turbulent channel flow for different Reynolds numbers. Mean velocity profiles, fluctuation velocities and energy spectra are compared to results from a second order finite difference code and to DNS data computed with a spectral code.

## 2. Numerical method

### 2.1. *Governing equations*

The governing equations for incompressible flow are the continuity equation and the Navier-Stokes equations

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re_\tau}\frac{\partial^2 u_i}{\partial x_j^2}. \tag{2.2}$$

Here, $u_i$ is the velocity component in the $x_i$ direction, $t$ denotes time, and $p$ denotes pressure. All terms are normalized with the friction velocity, $u_\tau$, and channel half width, $h$. $Re_\tau = u_\tau h/\nu$ is defined as the Reynolds number.

In LES, a filter function is applied to the flow variable $f$

$$\overline{f}(x,\Delta,t) = \int_{-\infty}^{\infty} G(x,x',\Delta) f(x',t)dx' \tag{2.3}$$

where $G$ is a filter function and $\Delta$ the filter width. Filtering Eqs. 2.1 and 2.2 in space yields

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{2.4}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i \overline{u}_j}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{Re_\tau}\frac{\partial^2 \overline{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j}. \tag{2.5}$$

where $\tau_{ij}$ is the SGS stress tensor defined as $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$. This stress tensor describes the interaction between the large resolved grid scale (GS) and the small unresolved SGS. The SGS stress tensor contains the unknown velocity correlation $\overline{u_i u_j}$, which cannot be expressed in the resolved flow quantities. This term has to be modeled.

### 2.2. *Subgrid scale models*

The goal of SGS modeling is to account for the SGS stresses using resolved flow field variables. The most important requirement of a SGS model is that it has to be dissipative.

The most widely used SGS models are the scale similarity models by Bardina *et al.* (1980) and Liu *et al.* (1994) and the Smagorinsky model (Smagorinsky (1963)) with the dynamic approach proposed by Germano *et al.* (1991) to calculate the model coefficient. The scale similarity models and the dynamic procedure uses the assumption that similar behavior exists between the resolved and unresolved stresses. When the filter function can be expressed as a polynomial of differentials, there exist an explicit relation between unfiltered and filtered flow variables (Fuchs (1996)). This approach results in an explicit expression of the SGS stress in terms of the resolved flow variables, *i.e.*, an exact differential SGS model.

### 2.3. *Solution algorithm*

In the code used, the continuity and Navier-Stokes equations are discretized using a fourth order finite difference scheme on a staggered grid. The convective term is discretized in a skew-symmetric form

$$\frac{\partial u_i u_j}{\partial x_j} = \frac{1}{2}\frac{\partial u_i u_j}{\partial x_j} + \frac{1}{2} u_j \frac{\partial u_i}{\partial x_j}.$$

This ensures conservation of kinetic energy (Morinishi *et al.* (1998) and Vasilyev (2000)). The no slip boundary condition is applied at the walls and the flow is assumed to be periodic in the streamwise and spanwise directions. A semi-implicit time marching algorithm is used in which the diffusion terms in the wall normal direction are treated implicitly with a Crank-Nicolson scheme. All other terms use a third order Runge-Kutta scheme described by Spalart *et al.* (1991). The splitting method of Dukowicz & Dvinsky (1992) is used to enforce the solenoidal condition. The resulting discrete Poisson's equation of pressure is solved using a penta diagonal direct matrix solver in the wall normal direction and a discrete Fourier transform in the periodic directions. A fixed mean pressure gradient is used in the streamwise direction.

The initial flow field is set to a parabolic profile in the streamwise direction with randomly generated fluctuations. The initial flow field in the wall normal direction and in the spanwise direction consist of these random fluctuations. The initial guess of the flow field is advanced in time to a statistically stationary solution before statistics are sampled.

To obtain a commutative system, a general class of discrete filters applied to nonuniform filter widths was proposed by Vasilyev *et al.* (1998). The procedure applies mapping of the nonuniform grid onto a uniform one in computational space in which the filtering is applied. The filters are constructed by applying a number of constraints to the filter weights to achieve both commutation and an acceptable filter shape. The filter weights are calculated by forcing the zeroth moment to be one and a number of higher moments to be zero. This determines the order of the commutation error. Other constraints are added to adjust the filter shape.

## 3. Turbulent channel flow simulations

In order to validate the fourth order finite difference (FD) scheme, numerical simulations of turbulent channel flow were performed using both DNS and LES. The same simulations were made with a second order FD code by Morinishi (1995) to compare the influence of the numerical scheme on the results. The FD results are compared to DNS data obtained using a spectral code (Kim *et al.* (1987) and Moser *et al.* (1999)).

The DNS and LES were carried out for Reynolds numbers $Re_\tau$=180 and $Re_\tau$=395, respectively. In both computations, the grid is stretched in the direction normal to the wall according to the following hyperbolic-tangent function,

$$x_j = -\frac{tanh(\gamma(1-\frac{2j}{N_2}))}{tanh(\gamma)} \qquad j = 0, ..., N_2 \tag{3.1}$$

Here, $N_2$ is the number of grid points in the $j$ direction and $\gamma$ is the stretching parameter.

### 3.1. *DNS results*

DNS was performed for $Re_\tau$=180. The grid resolution is $128^3$ and the computational domain is $(4\pi h, 2h, 4/3\pi h)$, which is the same as in the spectral simulations. The stretching parameter is $\gamma = 2.8$.

The mean velocity profile normalized with the friction velocity as a function of the

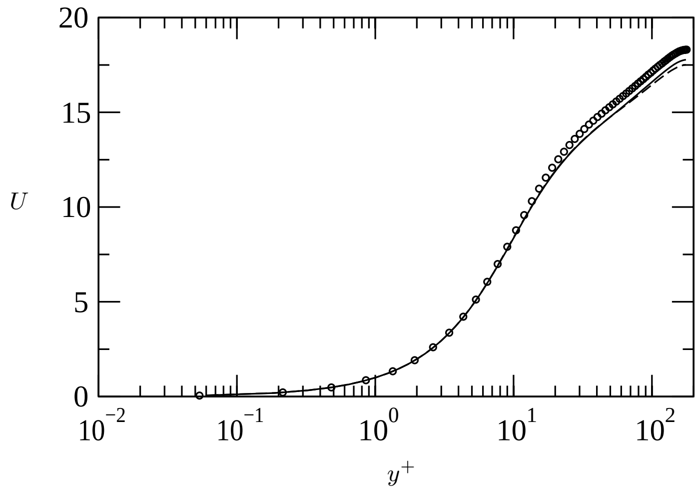


FIGURE 1. Mean velocity profile $U$ as a function of the distance to the wall $y^+$. $Re_\tau$=180. DNS ○: spectral code, ——— : 4th order FD code and - - - - : 2nd order FD code.

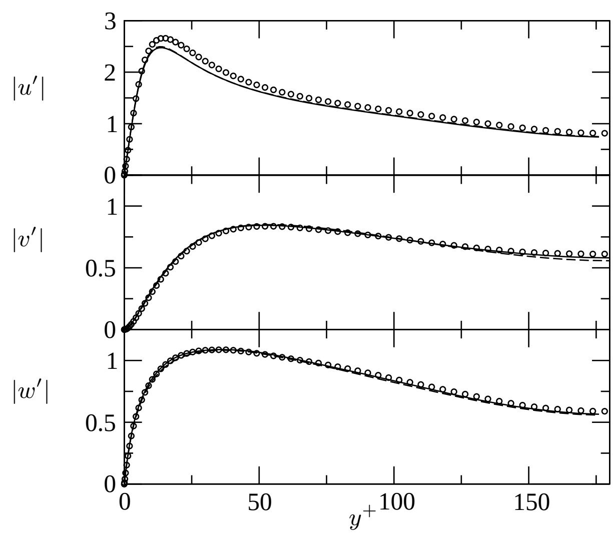


FIGURE 2. Velocity fluctuations in streamwise $|u'|$, wall normal $|v'|$ and spanwise $|w'|$ direction as a function of the distance to the wall $y^+$. $Re_\tau$=180. DNS ○: spectral code, ——— : 4th order FD code and - - - - : 2nd order FD code.

dimensionless distance to the wall is shown in Fig. 1. The comparison with the spectral code shows an underprediction of the mean velocity profiles by the FD codes.

Velocity fluctuations are plotted in Fig. 2. Again, the difference between the second and fourth order codes is negligible. Note that the streamwise velocity fluctuation is underpredicted by the FD codes, which is to be expected due to the drop in the modified wavenumber (Kravchenko & Moin (1997)).

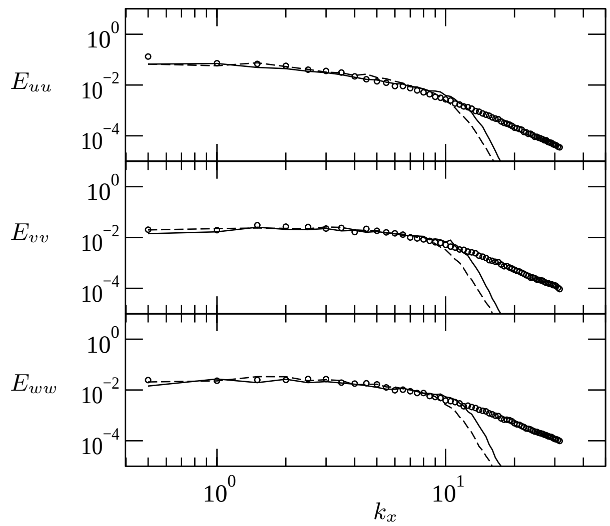


FIGURE 3. Energy spectrum of the streamwise $E_{uu}$, wall normal $E_{vv}$ and spanwise $E_{ww}$ velocity correlation as a function of the streamwise wavenumber $k_x$. $Re_\tau$=180 at $y^+$ ≈180. DNS ∘: spectral code, ——— : 4th order FD code and – – – – : 2nd order FD code.

The energy spectra as a function of the wavenumber in the streamwise direction are shown in Fig. 3. The second order code deviates from the energy spectra from the spectral code for smaller wavenumbers than compared with the fourth order code. Both FD results show a steep slope of the energy spectra in the high wavenumber part of the spectra.

### 3.2. *LES results*

LES was performed for $Re_\tau$=395 using the dynamic Smagorinsky model proposed by Germano *et al.* (1991) to model the SGS stresses. To calculate the model coefficient dynamically in the Smagorinsky model, the least square approximation by Lilly (1992) and averaging in the homogeneous directions as in Germano *et al.* (1991) are used. The grid resolution is (64,49,48), which is a quarter of the grid points in each direction compared to the DNS grid. The domain is $(2\pi h, 2h, \pi h)$. The mesh is stretched normal to the wall with $\gamma = 2.75$.

The mean velocity profiles are shown in Fig. 4. The fourth order code predicts the mean velocity profile closer to the DNS data. The second order code predicts a larger mass flow.

Velocity fluctuations are plotted in Fig. 5. The streamwise components are overpredicted using LES, while the wall normal and spanwise fluctuations are underpredicted when compared to the DNS results. Note that the overprediction of the streamwise velocity fluctuation is larger with the second order FD code than with the fourth order code. Kravchenko & Moin (1997) reported that this is due to the truncation error of the second order scheme.

The energy spectra in the streamwise direction are shown in Fig. 6. As for $Re_\tau$=180, the second order code deviates for smaller wavenumbers from the spectral results compared to the fourth order code. The slopes of the energy spectra in the high wavenumber part of the spectra are steep for the FD codes. This is of concern because most SGS models use

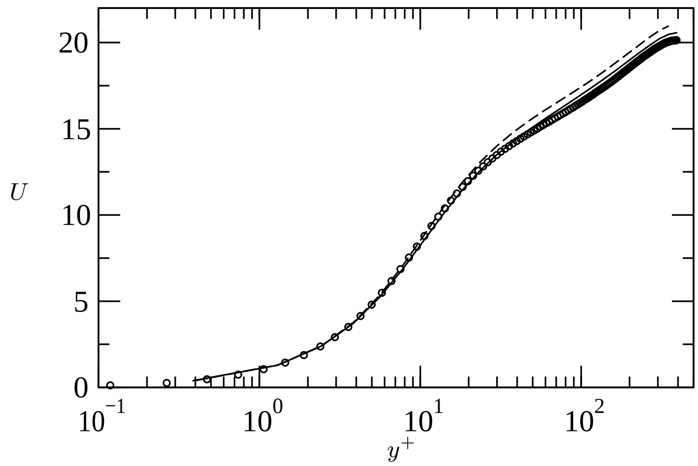


FIGURE 4. Mean velocity profile $U$ as a function of the distance to the wall $y^+$. $Re_\tau$=395. ∘: DNS spectral code, ——— : LES 4th order FD code and - - - - : LES 2nd order FD code.

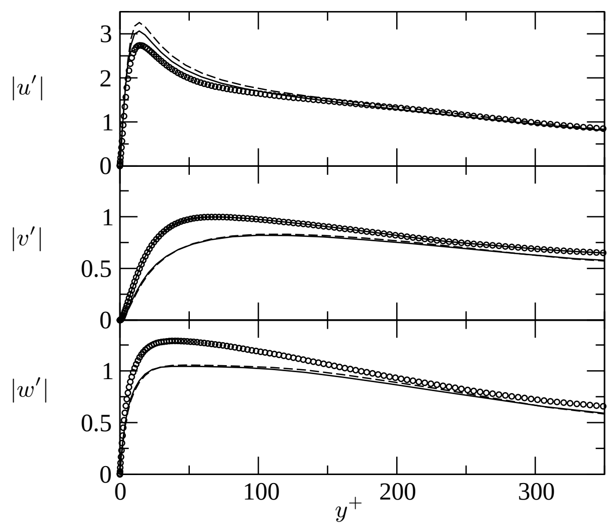


FIGURE 5. Velocity fluctuations in streamwise —u'—, wall normal $|v'|$ and spanwise direction $|w'|$ as a function of the distance to the wall $y^+$. $Re_\tau$=395. ∘: DNS spectral code, ——— : LES 4th order FD code and - - - - : LES 2nd order FD code.

information from the smallest resolved length scales (highest wavenumbers) to model the SGS stresses. Fig. 6 clearly shows that even when a high order scheme such as the fourth order FD scheme is used, the high wavenumber part of the spectra is contaminated with numerical errors.

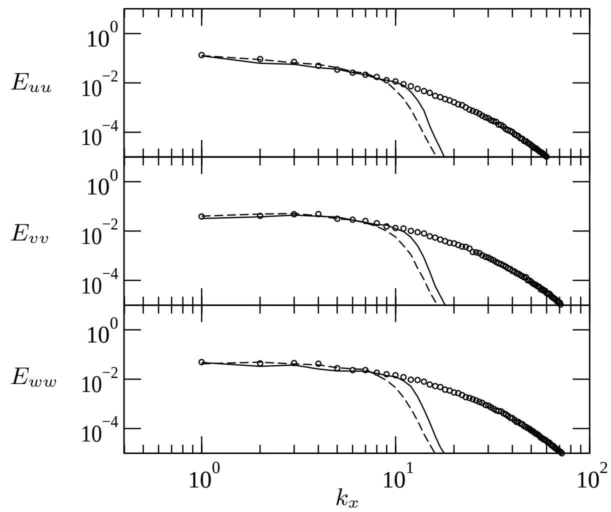


FIGURE 6. Energy spectrum of the streamwise $E_{uu}$, wall normal $E_{vv}$ and spanwise $E_{ww}$ velocity correlation as a function of the streamwise wavenumber $k_x$. $Re_\tau$=395 at $y^+$ ≈395. ∘: DNS spectral code, ——— : LES 4th order FD code and ---- : LES 2nd order FD code.

## 4. Current work

The fourth order finite difference scheme requires further evaluation with the use of the three-dimensional commutative filters in order to perform a "true" LES, defined as a a solution that converges to an LES solution and not to a DNS solution as the computational grid is refined. This can be made by separating the filter width from the computational grid size. By keeping the filter width constant while the computational grid is refined, the solution should converge to the "true" LES solution.

Different SGS models will also be investigated as well as the influence of numerical errors. The influence of the finite difference schemes on the high wavenumber part of the energy spectra and its influence on the predicted SGS stresses need to be determined. An approach to reduce the influence from the numerical error might be to use information from the length scales (or wavenumbers) that are not contaminated with large numerical errors and still is in the inertial subrange of the energy spectrum.

The cause of the over- and underprediction of the velocity fluctuations needs to be determined. The problem depends upon the SGS model, the grid resolution and the numerical method.

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