Generalized symmetries of the G-equation without underlying flow field

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It is shown that the admissible symmetries of the G-equation for flame fronts in premixed combustion depend essentially on whether the velocity of the underlying flow is zero or non-zero. The case of non-zero flow velocity has been exhaustively discussed by Oberlack, Wenzel & Peters (2001). If the flow velocity is zero a sixteen-dimensional Lie algebra of classical point symmetries exist. More importantly, an infinite series of generalized (Lie-Bäcklund) symmetries is derived, which includes as a special case the sixteen classical point symmetries.

1. Introduction

In recent years the G-equation, first derived by Williams (1985), has become the predominant approach for modeling premixed combustion in a very broad range of practical applications, such as spark-ignition engines and many others. A large amount of applied work has been dedicated to the G-equation. In order to make the G-equation amenable to numerical computations a diversity of numerical schemes have been derived, e.g. Adalsteinsson & Sethian (1999), Osher & Sethian (1988), Smiljanowski, Moser & Klein (1997), Sussman, Smereka & Osher (1994). Also, to make the G-approach applicable to turbulent premixed combustion a variety of model equations has been proposed, e.g. Im, Lund & Ferziger (1997), Peters (1992), Peters (1993), Ulitsky & Collins (1997), Weller, Tabor, Gosman & Fureby (1998) to name only a few.

In contrast, considerably less work has been dedicated to the mathematical properties of the G-equation. In particular, only recently have the important symmetry properties of the G-equation been explored, by Oberlack *et al.* (2001). Therein classical point symmetries of the G-equation in combination with the equations of fluid dynamics have been computed. It was shown that one particular symmetry, named "generalized scaling symmetry" by Oberlack *et al.* (2001), has important implications for the understanding and modelling of the G-equation in turbulent flows. New physically-sound modelling routes have been suggested. However, no generalized symmetries were investigated therein, since their derivation for the combined set of partial differential equations would have been formidable.

In the present approach we analyze a simplified version of the G-equation, where the flow velocity has been set to zero and the laminar burning velocity s_l is considered a constant. Physically speaking, this case describes the propagation of an infinitesimally-thin laminar flame sheet without flame-front advection due to an underlying flow field.

The G-equation with zero flow velocity is somewhat related to the eikonal equation. In Fushchich, Shtelen & Serov (1993) and Fushchich & Shtelen (1982) it is shown that the eikonal and related equations admit a wide class of symmetry transformations. Sub-

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sequently we prove that the set of classical point symmetries and generalized symmetries of the G-equation extend considerably if $\boldsymbol{u} = 0$ is imposed.

2. Symmetry groups of the G-equation

In Oberlack *et al.* (2001) the original form of the G-equation

$$\frac{\partial G}{\partial t} + u_k \frac{\partial G}{\partial x_k} = s_l \sqrt{\frac{\partial G}{\partial x_k}\frac{\partial G}{\partial x_k}}, \tag{2.1}$$

augmented by the equation of fluid dynamics, has been analyzed and discussed with respect to its classical point symmetries and the resulting physical consequences. Here $\boldsymbol{x}$ and t are space and time variables respectively, $\boldsymbol{u}$ is the velocity vector, s_l is the laminar burning velocity, and G denotes a scalar field quantity determining a instantaneous flame front at $G = G_0$. The G field has a physical meaning only at G_0.

During the derivation of the work in Oberlack *et al.* (2001) it became clear that it would be extremely difficult to extend this work to generalized symmetries as in Bluman & Kumei (1989). This task is usually considerably easier for scalar equations. For this reason the present work is limited to Eq. (2.1) where the flow velocity is set to zero

$$\frac{\partial G}{\partial t} = s_l \sqrt{\frac{\partial G}{\partial x_k}\frac{\partial G}{\partial x_k}}. \tag{2.2}$$

The latter equation has in fact close links to other equations known in mathematical physics. The first one is the eikonal equation which is the square of Eq. (2.2). In some models the squared version also contains an added constant. Though very similar in form, Eq. (2.2) and its squared version admit different reflection symmetries. Equation (2.2) admits only the reflection symmetries

$$t^* = t, \ \ x_i^* = -x_i, \ \ x_j^* = x_j, \ \ G^* = G, \ \ i = 1,2,3, \ \ j = 1,2,3/i \tag{2.3}$$

and

$$t^* = -t, \ \ x_i^* = x_i, \ \ G^* = -G, \ \ i = 1,2,3. \tag{2.4}$$

The squared version of Eq. (2.2) admits the latter time reversal where G is still unaffected. In addition it allows $G^* = -G$ with t as the identity transformation.

Equation (2.2) is also related to the usual linear wave equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_k^2} = 0$. In contrast to the one-dimensional version, in two or higher spatial dimensions one cannot give a complete analytic solution of the wave equation. However, the square of Eq. (2.2) with $s_l = 1$ may be considered as the characteristic equation of the wave equation.

In the subsections below, all classical point symmetries and the first elements of the infinite series of generalized symmetries of Eq. (2.2) are derived. It is shown how to construct arbitrarily many additional generalized symmetries.

2.1. *Classical point symmetries*

The purpose of the present analysis is to find all those continuous groups of transformation (Lie groups) which do not change the structure of the equation under investigation if written in the new variables. In the case of Eq. (2.2) the problem simplifies to obtaining the generator

$$\mathrm{X} = \xi^t(t,\boldsymbol{x},G)\frac{\partial}{\partial t} + \xi^i(t,\boldsymbol{x},G)\frac{\partial}{\partial x_i} + \eta^G(t,\boldsymbol{x},G)\frac{\partial}{\partial G}. \tag{2.5}$$

Here X is the infinitesimal form of the desired transformation, and ξ^t, ξ^i and η^G are the corresponding infinitesimals. The exponents of the infinitesimals denote the variables they refer to, and should not be mistaken for powers.

In Oberlack *et al.* (2001) it is shown that in the case of a non-zero flow velocity field, Eq. (2.1) extended by the equation of fluid dynamics admit the usual extended Galilean group. In all of these groups G is trivially contained as an identity transformation. The only symmetry group with non-zero flow velocity which non-trivially contains the transformation of G is the group

$$\mathrm{X} = \psi(G)\frac{\partial}{\partial G}, \tag{2.6}$$

where $\psi(G)$ is largely arbitrary.

Employing Lie's first theorem (see e.g. Bluman & Kumei 1989) the symmetry Eq. (2.6) may be written as the usual transformation in global form

$$G^* = \mathcal{F}(G) \quad \text{with} \quad \frac{\mathrm{d}\mathcal{F}(G)}{\mathrm{d}G} > 0, \tag{2.7}$$

where $\mathcal{F}(G)$ is connected to $\psi(G)$ by

$$\mathcal{F}(G) = \Psi^{-1}[\epsilon + \Psi(G)] \quad \text{and} \quad \Psi(G) = \int \frac{\mathrm{d}G}{\psi(G)}. \tag{2.8}$$

Here Ψ^{-1} is the inverse of Ψ. Since Ψ has to be invertible this poses a weak constraint on ψ by means of the latter integral relation Eq. (2.8).

Application of Eq. (2.5) to Eq. (2.2) leads to a considerably-extended set of groups comprising sixteen distinct Lie groups, each of which is infinite-dimensional because each contains an arbitrary function ω_i

$$\mathrm{X}_1 = \omega_1(G)\frac{\partial}{\partial G}, \tag{2.9}$$

$$\mathrm{X}_2 = \omega_2(G)\frac{\partial}{\partial t}, \tag{2.10}$$

$$\mathrm{X}_3 = \omega_3(G)\left(t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i}\right), \tag{2.11}$$

$$\mathrm{X}_{3+[j]} = \omega_{3+[j]}(G)\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right), \quad i,j = 1,2,3 \text{ and } i < j, \tag{2.12}$$

$$\mathrm{X}_{6+[i]} = \omega_{6+[i]}(G)\frac{\partial}{\partial x_i}, \quad i = 1,2,3 \tag{2.13}$$

$$\mathrm{X}_{9+[i]} = \omega_{9+[i]}(G)\left(s_l^2 t\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial t}\right), \quad i = 1,2,3 \tag{2.14}$$

$$\mathrm{X}_{13} = \omega_{13}(G)\left((s_l^2t^2 + x_kx_k)\frac{\partial}{\partial t} + 2s_l^2\, t\, x_i\,\frac{\partial}{\partial x_i}\right), \tag{2.15}$$

$$\begin{aligned}\mathrm{X}_{13+[i]} = \omega_{13+[i]}(G)\Bigg[&(s_l^2t^2 - x_kx_k + 2(x_{[i]})^2)\frac{\partial}{\partial x_{[i]}}\\ &+2x_{[i]}\left(t\frac{\partial}{\partial t} + x_k\frac{\partial}{\partial x_k} - x_{[i]}\frac{\partial}{\partial x_{[i]}}\right)\Bigg], \quad i = 1,2,3\end{aligned} \tag{2.16}$$

where the subscript $[i]$ means no summation. The enumeration in the index of several ω_i's corresponds to functions belonging to different groups. Since the task of computing Eqs.

(2.9)–(2.16) is relatively straightforward, and can in fact for the most part be aided by computer algebra systems – see Ibragimov (1996) – we omit any details of its derivation.

All ω_i are arbitrary functions of G obeying the same invertibility properties as given below Eq. (2.8). The first nine symmetries, Eqs. (2.9)–(2.13) have close relations to those admitted by Eq. (2.1) extended by the equations of fluid dynamics. In contrast, the symmetries Eqs. (2.14)–(2.16) have no counterpart in the usual G-equation (2.1) with $\boldsymbol{u} \neq 0$. It is interesting to note that all of the "new" symmetries contain the laminar flame speed s_l explicitly.

All of the symmetries Eqs. (2.9)–(2.16) may be written as global transformations. Employing Lie's first theorem, which states a unique relation between the infinitesimal transformation and the global transformation, we find the global transformation groups of Eqs. (2.9)–(2.16)

$$
\begin{aligned}
T_1 :\ & t^* = t,\ , x_i^* = x_i, \\
& G^* = \Psi^{-1}[\epsilon + \Psi(G)] \quad \text{with} \quad \Psi(G) = \int \frac{\mathrm{d}G}{\omega_1(G)}, && (2.17) \\
T_2 :\ & t^* = \varepsilon\omega_2(G) + t,\ x_i^* = x_i,\ G^* = G, && (2.18) \\
T_3 :\ & t^* = \mathrm{e}^{(\varepsilon\omega_3(G))} t,\ x_i^* = \mathrm{e}^{(\varepsilon\omega_3(G))} x_i,\ G^* = G, && (2.19) \\
T_{3+[i]} :\ & t^* = t,\ x_i^* = a_{ik}(G) x_k,\ G^* = G, \ \text{ with} \\
& \mathbf{a}\cdot\mathbf{a}^T = \mathbf{a}^T\cdot\mathbf{a} = \mathbf{I},\ |\mathbf{a}| = 1, && (2.20) \\
T_{6+[i]} :\ & t^* = t,\ x_i^* = \varepsilon\omega_{6+[i]}(G) + x_i,\ G^* = G, && (2.21) \\
T_{9+[i]} :\ & t^* = \cosh(s_l\,\omega_p(G)\,\varepsilon)\, t\, s_l + \sinh(s_l\,\omega_p(G)\,\varepsilon)\, x_{[i]}, \\
& x_i^* = \sinh(s_l\,\omega_p(G)\,\varepsilon)\, t\, s_l + \cosh(s_l\,\omega_p(G)\,\varepsilon)\, x_{[i]}, \\
& x_j^* = x_j,\ \ i = 1,2,3\,,\ \ j = 1,2,3/i \ \text{ and } \ p = 9 + i, \\
& G^* = G, && (2.22) \\
T_{13} :\ & t^* = \frac{(s_l^2 t^2 - x_k x_k)\left[\varepsilon\omega_{13}(G)(s_l^2 t^2 - x_k x_k) - t\right]}{x_k x_k - s_l^2\left[\varepsilon\omega_{13}(G)(s_l^2 t^2 - x_k x_k) - t\right]^2}, \\
& x_i^* = \frac{x_i(s_l^2 t^2 - x_k x_k)}{s_l^2\left[\varepsilon\omega_{13}(G)(s_l^2 t^2 - x_k x_k) - t\right]^2 - x_k x_k}, \\
& G^* = G, && (2.23) \\
T_{13+[i]} :\ & t^* = \frac{t(s_l^2 t^2 - x_k x_k)}{-\left[\varepsilon\omega_p(G)(s_l^2 t^2 - x_k x_k) + x_{[i]}\right]^2 + s_l^2 t^2 - x_k x_k + x_{[i]}^2}, \\
& x_i^* = \frac{(s_l^2 t^2 - x_k x_k)\left[\varepsilon\omega_p(G)(s_l^2 t^2 - x_k x_k) + x_{[i]}\right]}{s_l^2 t^2 - x_k x_k + x_{[i]}^2 - \left[\varepsilon\omega_p(G)(s_l^2 t^2 - x_k x_k) + x_{[i]}\right]^2}, \\
& x_j^* = \frac{x_{[j]}(s_l^2 t^2 - x_k x_k)}{-\left[\varepsilon\omega_p(G)(s_l^2 t^2 - x_k x_k) + x_{[i]}\right]^2 + s_l^2 t^2 - x_k x_k + x_{[i]}^2}, \\
& G^* = G, \\
& i = 1,2,3\,,\ \ j = 1,2,3/i \ \text{ and } \ p = 13 + i, && (2.24)
\end{aligned}
$$

Unless stated otherwise, the indices i, j and k denote $1, 2, 3$. The notation for the indices i and j in Eq. (2.22) and Eq. (2.24) denote that i can be any of 1, 2 and 3, and j refers to the remaining two.

2.2. *Generalized symmetries*

In mathematical physics it is known that many fundamental equations, such as the Burgers equation or the Korteweg-de Vries equation, admit a much wider class of symmetries which go beyond the classical point symmetries called generalized symmetries (see e.g. Bluman & Kumei 1989). Some authors call them Lie-Bäcklund or Noether symmetries. Generalized symmetries are defined such that the infinitesimals ξ^t, ξ^i and η^G in Eq. (2.5) not only depend on all dependent and independent variables, but may also comprise derivatives of G up to a given order n.

The actual derivation of the generalized symmetries is almost identical to that of the classical point symmetries. However the necessary algebra becomes increasingly more tedious for large orders of derivatives in the infinitesimals. For mathematical convenience, we here adopt Boyer's formulation of the generalized symmetries. He proved that all infinitesimals of the independent variables, here ξ^t and ξ^i, may be set to zero if at least all first-order derivatives of G are included in η: see Bluman & Kumei (1989). Hence without loss of generality we search for the generalized symmetry

$$\tilde{\mathrm{X}} = \tilde{\eta}(t, \boldsymbol{x}, G, \underset{1}{G}, \underset{2}{G}, \dots)\frac{\partial}{\partial G} \tag{2.25}$$

where $\underset{n}{G}$ denotes the set of all n^{th} order spatial derivatives of G. In the present context we may exclude any time derivative of G from $\underset{n}{G}$ since it can immediately be replaced by the right-hand side of Eq. (2.2). In the following, we indicate any derivative with respect to G by index notation. The time-derivative of G is defined as $\dfrac{\partial G}{\partial t} \equiv G_t$ while the spatial derivatives are abbreviated by $\dfrac{\partial G}{\partial x_i} \equiv G_{,i}$, $\dfrac{\partial^2 G}{\partial x_i \partial x_j} \equiv G_{,ij}$, etc.. Any derivative with respect to $\tilde{\eta}$ is given in the usual ∂-notation.

Boyer's theorem also states that once a point symmetry such as any of Eqs. (2.9)–(2.16) is known it may readily be written in the form of Eq. (2.25) where $\tilde{\eta}$ is given – see e.g. Bluman & Kumei (1989) – by

$$\tilde{\eta} = \eta^G - G_t\,\xi^t - G_{,i}\,\xi^i = \eta^G - s_l\sqrt{G_{,m}G_{,m}}\,\xi^t - G_{,i}\,\xi^i. \tag{2.26}$$

In the latter equality, G_t has been replaced using Eq. (2.2). For example X_{10-12} in Eq. (2.14) may be written as

$$\tilde{\mathrm{X}}_{10-12} = \omega_{9+[i]}(G)\left(-s_l\sqrt{G_{,m}G_{,m}}\,x_i - G_{,i}s_l^2 t\right)\frac{\partial}{\partial G}, \quad i = 1, 2, 3. \tag{2.27}$$

From Eq. (2.26) it is apparent that any point symmetry is linear in G_t and $G_{,i}$. However, the converse may not always be true.

Keeping $\tilde{\eta}$ completely general and applying Eq. (2.25) and any necessary prolongation of $\tilde{\mathrm{X}}$ – see e.g. Bluman & Kumei (1989) – to Eq. (2.2) we obtain

$$\left[\left(\tilde{\eta}\frac{\partial}{\partial G} + \frac{\mathcal{D}\tilde{\eta}}{\mathcal{D}t}\frac{\partial}{\partial G_t} + \frac{\mathcal{D}\tilde{\eta}}{\mathcal{D}x_m}\frac{\partial}{\partial G_{,m}}\right)\left(G_t - s_l\sqrt{G_{,n}G_{,n}}\right)\right]\Bigg|_{\text{eqn (2.2)}} = 0, \tag{2.28}$$

where $\mathcal{D}/\mathcal{D}t$ and $\mathcal{D}/\mathcal{D}x_m$ are defined as

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + G_t\frac{\partial}{\partial G} + G_{t,i}\frac{\partial}{\partial G_{,i}} + G_{t,ij}\frac{\partial}{\partial G_{,ij}} + \dots \tag{2.29}$$

and

$$\frac{\mathcal{D}}{\mathcal{D}x_m} = \frac{\partial}{\partial x_m} + G_{,m}\frac{\partial}{\partial G} + G_{,mi}\frac{\partial}{\partial G_{,i}} + G_{,mij}\frac{\partial}{\partial G_{,ij}} \dots \tag{2.30}$$

Expanding Eq. (2.28) we obtain

$$\begin{aligned}&\left[\frac{\partial\tilde\eta}{\partial t} + G_t\frac{\partial\tilde\eta}{\partial G} + G_{t,i}\frac{\partial\tilde\eta}{\partial G_{,i}} + G_{t,ij}\frac{\partial\tilde\eta}{\partial G_{,ij}} + \dots\right.\\ &\left.\left.- s_l\frac{G_{,m}}{\sqrt{G_{,n}G_{,n}}}\left(\frac{\partial\tilde\eta}{\partial x_m} + G_{,m}\frac{\partial\tilde\eta}{\partial G} + G_{,mi}\frac{\partial\tilde\eta}{\partial G_{,i}} + G_{,mij}\frac{\partial\tilde\eta}{\partial G_{,ij}}\dots\right)\right]\right|_{\text{eqn (2.2)}} = 0.\end{aligned} \tag{2.31}$$

As denoted by $|_{\text{eqn (2.2)}}$, when solving Eq. (2.31) the reduced G-equation, Eq. (2.2), may be introduced to replace any term of the form $G_{t,ij\dots}$ by the differential consequences of G_t. This finally leads to a single determining equation for $\tilde\eta$ of the form

$$\begin{aligned}&\frac{\partial\tilde\eta}{\partial t} - s_l\frac{G_{,m}}{\sqrt{G_{,n}G_{,n}}}\frac{\partial\tilde\eta}{\partial x_m} + s_l\left[\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}}\right]\frac{\partial\tilde\eta}{\partial G_{,ij}}\\ &+ \dots + s_l\Delta_{i_1i_2\dots i_n}(\underset{1}{G},\underset{2}{G},\dots\underset{n}{G})\frac{\partial\tilde\eta}{\partial G_{,i_1i_2\dots i_n}} = 0,\end{aligned} \tag{2.32}$$

where $\Delta_{i_1i_2\dots i_n}$ comprises all the terms emerging from the differential consequences of Eq. (2.2). Several things are important to note about Eq. (2.32). No derivative with respect to G and $G_{,i}$ appears, so any solution for $\tilde\eta$ can depend arbitrarily on G and $G_{,i}$.

Most importantly, Eq. (2.32) is closed. We may readily verify this by choosing $\tilde\eta$ to depend only on derivatives of G up to the order n indicated by $\underset{n}{G}$. Computing all differential consequences of Eq. (2.2), i.e. determining all $\Delta_{i_1i_2\dots i_n}$ up to order n, we find that they contain derivatives of G only up to $\underset{n}{G}$. Hence, Eq. (2.32) constitutes a linear hyperbolic equation in $\tilde\eta$ depending on the set of variables: t, $\boldsymbol{x}$, G, $G_{,i}$, $G_{,ij}$, $\dots$, $G_{,i_1i_2\dots i_n}$, where G and $G_{,i}$ appear only as parameters.

Solutions for $\tilde\eta$ with increasing derivative order $\underset{n}{G}$ may be successively obtained, beginning with the lowest derivative order. First, we consider $\tilde\eta$ solely depending on G-derivatives up to order one. Hence, we limit $\tilde\eta$ to be a function only of t, $\boldsymbol{x}$, G and $\underset{1}{G}$. As a consequence equation Eq. (2.32) reduces to

$$\frac{\partial\tilde\eta}{\partial t} - s_l\frac{G_{,m}}{\sqrt{G_{,n}G_{,n}}}\frac{\partial\tilde\eta}{\partial x_m} = 0. \tag{2.33}$$

The characteristic equations of Eq. (2.33) are

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1 \quad \text{and} \quad \frac{\mathrm{d}x_m}{\mathrm{d}\epsilon} = -s_l\frac{G_{,m}}{\sqrt{G_{,n}G_{,n}}}. \tag{2.34}$$

Equation (2.34) may readily be integrated to yield the complete solution of Eq. (2.33),

$$\tilde\eta = \mathcal{G}\left(\boldsymbol{C}, G, \underset{1}{G}\right) \quad \text{where} \quad C_i = x_i + s_l\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}}\,t. \tag{2.35}$$

$\mathcal{G}$ is an arbitrary function of its arguments and should be once-differentiable with respect to $\boldsymbol{C}$. Note that in determining $\boldsymbol{C}$, $G_{,i}$ appeared only as a parameter.

It is important to note that $\boldsymbol{C}$ may be considered the fundamental characteristic. It nicely exemplifies the solution structure of Eq. (2.2) which may be interpreted, and also constructed, geometrically. Given an initial condition for the G field we may propagate it in time with the speed s_l along rays normal to each instantaneous G field, as described by the family of curves $\boldsymbol{C}$.

The generalized symmetry Eq. (2.35) appears to be considerably simpler in form than the classical point symmetries in Subsection 2.1. However we can show that all point symmetries, Eqs. (2.9)–(2.16), are included in the solution Eq. (2.35) by virtue of Boyer's relation Eq. (2.26). E.g. the symmetries Eq. (2.14), which are rewritten in Eq. (2.27) using Boyer's formulation, can be derived from Eq. (2.35) by restricting $\tilde{\eta}$ to $\mathcal{G} = -\omega_{9+[i]} s_l \sqrt{G_{,n}G_{,n}} C_i,\ \ i = 1,2,3$. However, any solution to Eq. (2.33) which is not among Eqs. (2.9)–(2.16) using Eq. (2.26) cannot be written as a point symmetry.

Enlarging the dependence of $\tilde{\eta}$ in the next step in Eq. (2.32) by the set of variables $\underset{2}{G}$, Eq. (2.33) extends to

$$\frac{\partial \tilde{\eta}}{\partial t} - s_l \frac{G_{,m}}{\sqrt{G_{,n}G_{,n}}} \frac{\partial \tilde{\eta}}{\partial x_m} + s_l \left[\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}} \right] \frac{\partial \tilde{\eta}}{\partial G_{,ij}} = 0. \tag{2.36}$$

The corresponding set of characteristic equations, Eq. (2.34) is expanded by

$$\frac{\mathrm{d}G_{,ij}}{\mathrm{d}\epsilon} = s_l \left[\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}} \right]. \tag{2.37}$$

In order to solve Eq. (2.34) and Eq. (2.37) we combine the characteristic ODE's for t and $G_{,ij}$ to obtain

$$\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t} = s_l \left[\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}} \right] \tag{2.38}$$

Since $G_{,ij}$ is a symmetric second-order tensor, Eq. (2.38) constitutes a quadratic tensor equation in which $G_{,k}$ is a vector-valued parameter. For the purpose of solving Eq. (2.38) we derive the identity

$$\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t} = \frac{\mathrm{d}G^{-1}_{,ki}G_{,ij}G^{-1}_{,jl}}{\mathrm{d}t} = 2\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t} + G^{-1}_{,ki}\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t}G^{-1}_{,jl} \tag{2.39}$$

which may be rewritten as

$$G^{-1}_{,ki}\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t}G^{-1}_{,jl} = -\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t}. \tag{2.40}$$

$G^{-1}_{,ij}$ is the matrix inverse of $G_{,ij}$ and $G_{,ik}G^{-1}_{,kj} = G^{-1}_{,ik}G_{,kj} = \delta_{ij}$. Multiplying Eq. (2.38) with $G^{-1}_{,ki}$ and $G^{-1}_{,jl}$ we find, using Eq. (2.40), that

$$-\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t} = s_l \left[\frac{\delta_{kl}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,k}G_{,l}}{(G_{,n}G_{,n})^{3/2}} \right]. \tag{2.41}$$

Equation (2.41) may immediately be integrated with respect to t since the right-hand side does not depend on $G_{,kl}$. We introduce an additional identity from the Caley-Hamilton theorem (see Appendix A)

$$G^{-1}_{,ij} = \frac{3}{2G^3_{,kk} - 3G^2_{,kk}G_{,kk} + (G_{,kk})^3} \left[((G_{,kk})^2 - G^2_{,kk})\delta_{ij} - 2G_{,kk}G_{,ij} + 2G^2_{,ij} \right] \tag{2.42}$$

where $G^n_{,ij} = \underbrace{G_{,ik_1}G_{,k_1k_2}\cdot\dots\cdot G_{,k_{n-1}k_n}G_{,k_nj}}_{n}$. As the final solution of Eq. (2.38), we obtain the characteristic tensor

$$D_{ij} = s_l\left[\frac{\delta_{ij}}{\sqrt{G_{,m}G_{,m}}} - \frac{G_{,i}G_{,j}}{(G_{,m}G_{,m})^{3/2}}\right]t + \frac{3}{2\lambda_3 - 3\lambda_2\lambda_1 + \lambda_1^3}\left[(\lambda_1^2 - \lambda_2)\delta_{ij} - 2\lambda_1 G_{,ij} + 2G_{,ik}G_{,kj}\right]. \qquad (2.43)$$

In the latter characteristic the abbreviations λ_i are defined according to $\lambda_1 = G_{,kk}$, $\lambda_2 = G_{,kl}G_{,lk}$ and $\lambda_3 = G_{,kl}G_{,lm}G_{,mk}$. Hence the complete solution to Eq. (2.32) is derived, where the derivative order $\underset{n}{G}$ has been limited to $n = 2$, as

$$\tilde{\eta} = \mathcal{H}(\boldsymbol{C}, G, \underset{1}{G}, \mathbf{D}). \qquad (2.44)$$

$\boldsymbol{C}$ and $\mathbf{D}$ are respectively defined by Eq. (2.35) and Eq. (2.43) and $\mathcal{H}$ is an arbitrary function of its arguments, being at least once-differentiable with respect to $\boldsymbol{C}$ and $\mathbf{D}$.

In principle the next step to obtain further generalized symmetries would be to include $\underset{3}{G}$ in $\tilde{\eta}$ in Eq. (2.32). The mathematical complexity of the characteristic equations rapidly increases when higher-order derivatives of G are introduced into $\tilde{\eta}$. However it is important to note that this can be done in principle, and leads to an infinite sequence of generalized symmetries of Eq. (2.2).

3. Summary

It is demonstrated that the G-equation for premixed combustion admits a very broad variety of symmetry properties, including those from classical mechanics. It is particularly interesting that the number of symmetries depends strongly on whether the underlying flow velocity is zero or non-zero. For zero flow, sixteen distinct symmetries have been established. For this case also, an infinite series of generalized symmetries has been established.

In Oberlack *et al.* (2001) it was shown that the symmetries of the G-equation with non-zero velocity are very useful in aiding the modeling process of the G-equation for turbulent premixed combustion. It is expected that the present findings may also be used to help improve turbulent combustion models.

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Appendix A. Derivation of $\mathbf{A}^{-1}$ in terms of matrix-polynomials of $\mathbf{A}$ in $\mathbb{R}^3$

In Eq. (2.41) we need to express the inverse of the symmetric tensor $G_{,ij}$, denoted in the following by $\mathbf{A}$, in terms of polynomials of $G_{,ij}$ itself. From tensor-invariant theory it is known that $\mathbf{A}^{-1}$ may be expressed in terms of $\mathbf{A}$ in the form

$$A_{ij}^{-1} = a_1\delta_{ij} + a_2 A_{ij} + a_3 A_{ij}^2, \tag{A 1}$$

where the a_i may depend on the three scalar tensor invariants of $\mathbf{A}$ denoted as

$$\lambda_1 = A_{kk}, \;\; \lambda_2 = A_{kk}^2 \;\text{ and }\; \lambda_3 = A_{kk}^3. \tag{A 2}$$

Multiplying Eq. (A 1) with $\mathbf{A}$ and expressing $\mathbf{A}^3$ in terms of lower order polynomials and scalar invariants Eq. (A 2) using the Caley-Hamilton theorem – e.g. Spencer (1971) – we obtain

$$\delta_{ij} = a_1 A_{ij} + a_2 A_{ij}^2 + a_3\left[\lambda_1 A_{ij}^2 - \frac{1}{2}A_{ij}\left(\lambda_1^2 - \lambda_2\right) + \frac{1}{3}\delta_{ij}\left(\lambda_3 - \frac{3}{2}\lambda_2\lambda_1 + \frac{1}{2}\lambda_1^3\right)\right]. \tag{A 3}$$

Ordering the scalar coefficients of δ_{ij}, A_{ij} and A_{ij}^2 we obtain a linear set of equations for the a_i, $\; i = 1, 2, 3$. The result for the a_i may be inserted into Eq. (A 1) to yield the final solution

$$A_{ij}^{-1} = \frac{3}{2\lambda_3 - 3\lambda_2\lambda_1 + \lambda_1^3}\left[(\lambda_1^2 - \lambda_2)\delta_{ij} - 2\lambda_1 A_{ij} + 2A_{ij}^2\right]. \qquad \text{(A 4)}$$