Modeling the "rapid" part of the velocity/pressure-gradient correlation in inhomogeneous turbulence

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1. Motivation and objectives

The Reynolds-stress transport equation (RSTE) includes the velocity/pressure-gradient correlation $< p_{,j}u_i >$. Following Rotta (1951), it is common practice to split this correlation into two parts: the pressure-strain correlation $< pu_{i,j} >$ and the pressure-diffusion part $< pu_i >_{,j}$. This approach has advantages if one simulates homogeneous turbulence with a two-equation turbulence model. In homogeneous turbulence, both the pressure diffusion term $< pu_i >_{,i}$ and the pressure-strain term $< pu_{i,i} >$ do not contribute to the transport of the turbulent kinetic energy (k).

In inhomogeneous turbulent flows, however, the modeling of the pressure-diffusion term in the k-equation is challenging. Direct numerical simulation (DNS) data from free shear flows (Rogers & Moser 1994; Moser *et al.* 1998) show that the contribution of the pressure diffusion in the turbulent kinetic energy balance is not negligible, especially in the central core of the flow. The current practice of using a single model for both the pressure diffusion and the turbulent diffusion is not likely to be successful because the two have qualitatively different profiles. It was shown by Lumley (1978), that in homogeneous turbulence the "slow" part of the pressure diffusion term can be modeled as

$$-\frac{1}{\rho} < pu_i >^{(s)}_{,i} = \frac{1}{5} < u_m u_m u_i >_{,i} . \tag{1.1}$$

Thus, a model for the turbulent diffusion can absorb only this "slow" term. The contribution of the "rapid" part and the surface integral in the exact integral expression for the pressure diffusion term (Chou 1945) should also appear in the modeled turbulent kinetic energy transport equation.

Judging by DNS data, the term $< pu_i >_{,j}$ cannot be ignored in the transport equations for Reynolds stresses and should be modeled as well as the pressure-strain term $< pu_{i,j} >$. Thus, two models for the pressure-containing correlations are necessary, and splitting the correlation $< p_{,j}u_i >$ into two parts becomes a disadvantage. The direct modeling of the correlation $< p_{,j}u_i >$ is a natural choice. Here, models for the "rapid" part of the velocity/pressure-gradient correlation in the Reynolds-stress transport equation (RSTE) and of the pressure-diffusion term in the turbulent kinetic energy transport equation (Poroseva 2000) are considered and tested against post-processed DNS data from a wake and a mixing layer (Rogers & Moser 1994; Moser *et al.* 1998).

2. Model expressions

To derive the model for the "rapid" part of $< p_{,j}u_i >$ in inhomogeneous turbulence, it was suggested by Poroseva (2000) to go back to the original idea of Chou (1945) and consider the exact integral expression for the velocity/pressure-gradient correlation $< p_{,j}u_i >$. This expression holds rigorously in an incompressible flow. For a compressible flow, it works as an approximation. The "rapid" part of that expression can be written as following:

$$-\frac{1}{\rho} < p_{,j}u_i >^{(r)} = -\frac{1}{2\pi}\int\int\int \left[U'_{m,n} < u'_n u_i >'_{,m}\right]'_{,j} \frac{1}{r} dV'. \tag{2.1}$$

The prime "′" indicates that the quantities should be evaluated at a point P', with coordinates x'_i, which ranges over the volume V', and r is the distance from point P' to point P with coordinates x_i. The correlation on the left-hand side of Eq. (2.1) is evaluated at point P, whereas all derivatives on the right side are taken at P'. The integrand in Eq. (2.1) is non-zero only over the volume where the two-point correlation $< u'_n u_i >$ (or more precisely $< u'_{n,m} u_i >$) does not vanish. In other words, for a fixed point P, only those points P' which lie within the length scale of the two-point correlation measured from P, contribute to the integral in Eq. (2.1). If one assumes that the function $U'_{m,n}$ varies more slowly than the two-point correlation within the volume determined by the length scale of the two-point correlation, then, to the first approximation, we can rewrite Eq. (2.1) as

$$-\frac{1}{\rho} < p_{,j}u_i >^{(r)} = -\frac{1}{2\pi} U_{m,n} \int\int\int < u'_n u_i >'_{,mj} \frac{1}{r} dV' \tag{2.2}$$

(Chou 1945). Notice that this is not an assumption of turbulence homogeneity: all functions are regarded as functions of space coordinates.

Chou (1945) suggested modeling the sum of the two pressure-containing correlations as

$$\Pi_{ij}^{(r)} = -\frac{1}{\rho}\left(< p_{,j}u_i > + < p_{,j}u_j >\right)^{(r)} = a_{nmji}U_{m,n},$$

though no specific form for the tensor function a_{nmji} was derived in his work. The idea to model not the sum, but each correlation separately, was applied for the first time to the pressure-strain correlation $< pu_{i,j} >$ by Rotta (1951). This idea has been found more fruitful for the correlation $< p_{,j}u_i >$ also (Poroseva 2000) and it yields

$$-\frac{1}{\rho} < p_{,j}u_i >^{(r)} = a_{nmji}U_{m,n}$$

and

$$\Pi_{ij}^{(r)} = (a_{nmji} + a_{nmij})U_{m,n},$$

where

$$a_{nmji} = \frac{1}{2\pi}\int_{V'} < u'_n u_i >'_{,mj} \frac{1}{r} dV'.$$

In this way, more specific conditions can be imposed on a_{nmji}:

(i) symmetry in permutation of indices m and j;

(ii) from continuity: $a_{mmji} = 0$;

(iii) from Green's theorem: $a_{njji} = 2 < u_n u_i >$,

and a number of model coefficients can be eliminated. The final model expression for a_{nmji} includes two coefficients:

$$a_{nmji} = -\frac{1}{5}(< u_i u_j > \delta_{mn} + < u_i u_m > \delta_{jn}) + \frac{4}{5} < u_i u_n > \delta_{jm} + \\ C_1[\frac{1}{2}(< u_i u_j > \delta_{mn} + < u_i u_m > \delta_{jn}) + < u_i u_n > \delta_{jm} + \\ k(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn}) - < u_j u_m > \delta_{in} - \\ 2(< u_j u_n > \delta_{im} + < u_m u_n > \delta_{ij})] + \\ C_2[\frac{1}{2}(< u_i u_j > \delta_{mn} + < u_i u_m > \delta_{jn} - < u_j u_n > \delta_{im} - \\ < u_m u_n > \delta_{ij}) + k\delta_{in}\delta_{jm} - \frac{3}{2} < u_j u_m > \delta_{in}]. \tag{2.3}$$

Then the model for the "rapid" part of the pressure-containing term in the RSTEs is

$$\Pi_{ij}^{(r)} = -\frac{1}{\rho}(< p_{,i} u_j > + < p_{,j} u_i >) = \\ -\left(\frac{1}{5} + \frac{1}{2}C_1 + C_2\right)(< u_i u_m > U_{m,j} + < u_j u_m > U_{m,i}) \\ +\left(\frac{4}{5} - C_1 - \frac{1}{2}C_2\right)(< u_i u_m > U_{j,m} + < u_j u_m > U_{i,m}) \\ +k(C_1 + C_2)(U_{i,j} + U_{j,i}) - (4C_1 + C_2) < u_m u_n > U_{m,n}\delta_{ij}. \tag{2.4}$$

Note that expression Eq. (2.4) is obtained with the assumption of flow incompressibility. Thus terms in Eq. (2.3) that involve δ_{mn} make no contribution in Eq. (2.4).

Coefficients C_1 and C_2 are, in the general case, unknown functions of several parameters (see discussions in Lumley 1978; Reynolds 1987; Ristorcelli 1995; Girimaji 2000). However, the behavior of these functions can be specified for some known limit states of turbulence.

Setting $< u_i u_j >= 2/3k\delta_{ij}$ it is easy to show that expression Eq. (2.3) satisfies, for any values of the coefficients C_1 and C_2, the exact solution for isotropic turbulence subjected to sudden distortion (Rotta 1951; Crow, 1968; Reynolds 1976)

$$a_{nmji} = k\left(\frac{8}{15}\delta_{ni}\delta_{mj} - \frac{2}{15}(\delta_{nm}\delta_{ji} + \delta_{nj}\delta_{mi})\right).$$

In the case of homogeneous turbulence, an additional condition based on permutation of the indices n and i can be imposed on Eq. (2.3). It results in the following relation between model coefficients:

$$\frac{1}{5} - \frac{5}{2}C_1 - C_2 = 0. \tag{2.5}$$

Under the condition given by Eq. (2.5), model Eq. (2.4) for $\Pi_{ij}^{(r)}$ transforms to the known model of Launder *et al.* (1975). Note that this connection between coefficients exists only in a homogeneous flow, and does not hold in the general case.

Model Eq. (2.4) for $\Pi_{ij}^{(r)}$ is realizable. If $< u_\alpha u_\alpha >$ $(\alpha = 1, 2, 3)$ are the eigenvalues of the Reynolds-stress tensor, then we obtain the following realizability constraint

$$\Pi_{11}^{(r)} = 0, \text{ if } < u_1 u_1 >= 0$$

corresponding to the limit state of two-component turbulence (Schumann 1977; Pope 1985; Shih *et al.* 1994). Index 1 is chosen arbitrarily. Writing Eq. (2.4) for $\Pi_{11}^{(r)}$, one gets

$$\Pi_{11}^{(r)} = 2k(C_1 + C_2)U_{1,1} + (-4C_1 - C_2) < u_m u_n > U_{m,n} = 0$$

or

$$C_1 = -C_2 \frac{2kU_{1,1} - < u_m u_n > U_{m,n}}{2kU_{1,1} - 4 < u_m u_n > U_{m,n}}. \tag{2.6}$$

Taking into account that among all eigenvalues only $< u_2 u_2 >$ and $< u_3 u_3 >$ are not equal to zero and $< u_2 u_2 > + < u_3 u_3 >= 2k$ as well as $U_{m,m} = 0$, then Eq. (2.6) can be rewritten as

$$C_1 = -C_2 \frac{U_{1,1}(4k - < u_\beta u_\beta >) + 2U_{\beta,\beta}(k - < u_\beta u_\beta >)}{U_{1,1}(10k - 4 < u_\beta u_\beta >) + 8U_{\beta,\beta}(k - < u_\beta u_\beta >)}. \tag{2.7}$$

where $\beta = 2$ or 3 (no summation on β). This condition is satisfied with $C_1 = C_2 = 0$, for instance, but it allows other solutions for specific turbulence states. Again, this connection between coefficients is valid only in the two-component limit. Consideration of two-component homogeneous turbulence is interesting itself and will be addressed in the future. Here, we simply notice that Eq. (2.5) and Eq. (2.7) do not contradict each other, but for specific turbulence states they fully define the set (C_1, C_2).

In the transport equation for the turbulent kinetic energy, model expression Eq. (2.4) for $\Pi_{ij}^{(r)}$ contracts to a model for the "rapid" part of the pressure diffusion term with the only model coefficient:

$$-\frac{1}{\rho} < pu_i >_{,i}^{(r)} = \left(-\frac{3}{5} + C_k\right) P \tag{2.8}$$

where

$$C_k = \frac{15}{2} C_1 + 3C_2. \tag{2.9}$$

Here, $P = - < u_i u_j > \frac{\partial U_i}{\partial x_j}$. The coefficient C_k is a function of the same parameters as coefficients C_1 and C_2. In homogeneous turbulence, substitution of Eq. (2.5) in Eq. (2.9) gives $C_k = 0.6$.

An important question in modeling the pressure diffusion term $< pu_i >_{,i}$ is whether a model expression for this term should be of the diffusive type also. However, this question addresses the complete model for $< pu_i >_{,i}$, which includes "rapid", 'low", and "surface" parts in inhomogeneous turbulence, and does not apply to each part separately. Only the "rapid" part is the concern of the present study. Moreover, the properties of a model for $< pu_i >_{,i}$ should not be considered separately from a model for the trace of the dissipative tensor ε_{ij}. On the contrary, the physical assumptions and mathematical approximations used in deriving the two models should be the same if the models are to be consistent. This is the only way one can hope for inaccuracy in a model for one term to be compensated for by a model for another term. In other words, the equality sign stands not between each exact term and a model expression for it, but between the sum of exact terms and the sum of models for them:

$$< pu_i >_{,i} - \nu < u_{i,j} u_{i,j} > = \frac{1}{2}(\Pi_{ii}^{(M)} - \varepsilon_{ii}^{(M)}).$$

With such an approach, the properties of the sum should be conserved in modeling rather than the properties of each term separately. The commonly-used model $\varepsilon_{ij} = 2/3\delta_{ij}\varepsilon$ is not consistent with any known model for Π_{ij} to the author's knowledge. More work should be done in this direction.

3. Results

Post-processed DNS data for the wake (Moser *et al.* 1998) were used to evaluate terms in the model expression Eq. (2.4). Eq. (2.8) was tested against DNS data in two flows: the plane mixing layer (Rogers & Moser 1994) and the plane wake (Moser *et al.* 1998). DNS data for the pressure diffusion in the turbulent kinetic energy balance (solid lines) are compared with model profiles (dashed lines) in Fig. 1. Because at present DNS data are not available separately for the "slow" and "rapid" parts of the pressure diffusion, only evaluation of the joint performance of Eq. (1.1) and Eq. (2.8) is possible. Therefore, inaccuracies of model Eq. (1.1), derived under the assumption of turbulence homogeneity (Lumley 1978), are absorbed in the value of C_k. To calculate model profiles, DNS data for the production and turbulent diffusion terms are used. The optimal value of the coefficient C_k in Eq. (2.8) is determined by adjusting the maximum of a calculated profile to match the maximum of the DNS profile. Its value was found to be equal to 0.52 in the mixing layer (Fig. 1a) and 0.5 in the wake (Fig. 1b).

DNS profiles of $\Pi_{12}^{(r)}$, $\Pi_{11}^{(r)}$, and $\Pi_{33}^{(r)}$ in the wake (solid lines) are compared to model ones (dashed lines) in Fig. 2. Model profiles are obtained by substituting the DNS data for the mean velocity and Reynolds stresses in the model expression Eq. (2.4) for $\Pi_{ij}^{(r)}$. The coefficient $C_1 = 0.4$ was chosen by matching the calculated profiles to the DNS ones. The value of C_2, $C_2 = -5/6$, is calculated from Eq. (2.9) using known C_k and C_1. The agreement between the DNS data and the calculated profiles is very good, except that the levels of $\Pi_{11}^{(r)}$ and $\Pi_{33}^{(r)}$ are slightly overpredicted. One of the possible reasons for this disagreement is the dependence of C_2 on the coefficient C_k, which is not exact, but absorbs the inaccuracies of the model for the "slow" part of the pressure diffusion.

It is important to clarify the question of the functional form of the coefficients C_k, C_1, and C_2. Conditions Eq. (2.5) and Eq. (2.7) provide relationships between the coefficients in particular limit states of turbulence, but do not indicate on what parameters the coefficients depend. This information can be partly drawn from DNS, experimental data, and results from RANS calculations of classical self-similar free shear flows, equilibrium boundary layers with and without pressure gradients, and separated flows. Such flows were calculated by Poroseva (2000) and Poroseva & Iaccarino (2001), using the k-ε turbulence model with the "rapid" part of the pressure diffusion term taken into account. Good predictions for mean velocity, shear stress, friction coefficient, and turbulent kinetic energy profiles were obtained in all test flows.

The value of C_k in Poroseva (2000) and Poroseva & Iaccarino (2001) is calibrated by matching the non-dimensional turbulent kinetic energy level in the flow. Consequently, its value depends on the conditions which determine that level. The turbulent kinetic energy level is not unique even in the same flow. Indeed, it does appear from both experiments and DNS (see discussion in Rogers & Moser 1994 and Moser *et al.* 1998) that, even in geometrically-equivalent flow situations at the same Reynolds number, multiple asymptotic states are observed. This difference between alternative states manifests itself in both the statistics and the flow structure. Under appropriate scaling, the mean-velocity and shear-stress profiles are universal or near universal, but normal stresses and turbulent

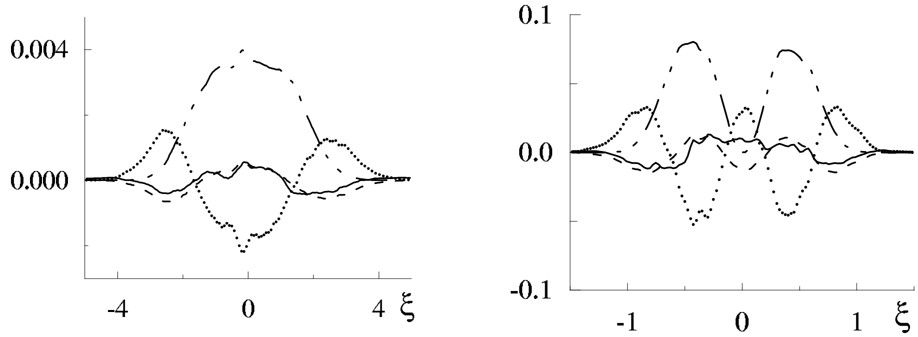

FIGURE 1. Partial turbulent kinetic energy balance: a) mixing layer, b) wake. (– – – –) calculated pressure diffusion profiles; (—··—) DNS production, (———) DNS pressure diffusion, (········) DNS turbulent diffusion.

kinetic energy profiles are non-unique. DNS confirms that statistical differences reflect the differences in the large-scale structure of turbulence, which depends strongly "on uncontrolled and possibly unknown properties of the initial or inlet conditions" (Moser *et al.* 1998). Also, large-scale structure is influenced by flow geometry, boundary conditions, external forces etc. (Tsinober 1998). Among other factors connected with large-scale structure are Reynolds number and flow geometry. The coefficient C_k seems to be influenced by both factors. However, the database used by Poroseva (2000) and Poroseva & Iaccarino (2001) does not facilitate distinguishing their influence on the coefficient value, as data for different test flows were obtained at different Reynolds numbers.

On the other hand, the coefficient C_k is an explicit function Eq. (2.9) of the two coefficients C_1 and C_2, which are linked to each other by Eq. (2.7) in the two-component turbulence limit. Because mean velocity gradients are involved in that expression, a functional dependence of C_k on velocity gradients could be expected. However, each test flow was successfully reproduced with the same value of C_k. Moreover, mean velocity gradients, like mean velocity components, appear not to be sensitive to initial or inlet conditions, whereas large-scale structure and, as a result, the value of C_k strongly depend on them. This suggests that although there may be some dependence of the coefficient on mean velocity gradients, it seems to be of secondary importance in inhomogeneous flow.

4. Conclusions

In the present research, model expressions for the "rapid" parts of the velocity/pressure-gradient correlation in the RSTE and the pressure diffusion in the turbulent kinetic energy transport equation (Poroseva 2000) were evaluated using DNS data. For the velocity/pressure-gradient correlation such data are available in a wake (Moser *et al.* 1998). For the pressure diffusion, there exist DNS data in two flows: a wake and a mixing layer (Rogers & Moser 1994; Moser *et al.* 1998). DNS data and calculated model profiles agree very well.

The contribution of the "rapid" part of the pressure diffusion appears in the standard k-ε equation through an additional model coefficient C_k. THere is probably some dependence of the coefficient on mean velocity gradients, but it is apparently of secondary importance. Each test flow was successfully reproduced with a constant value of C_k in Poroseva (2000) and Poroseva & Iaccarino (2001). The value $C_k = 0.6$ can be rigorously

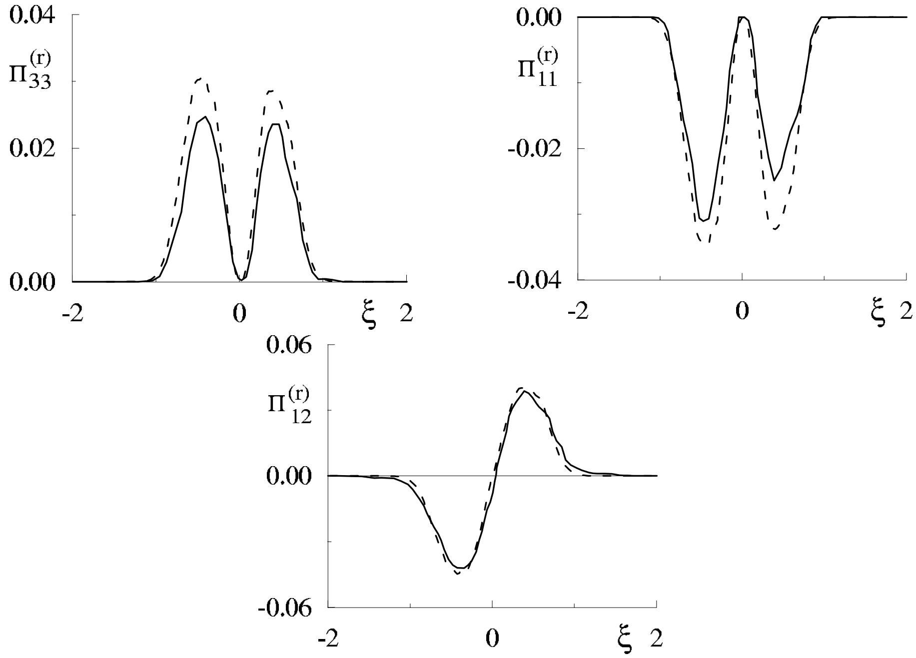

FIGURE 2. "Rapid" part of velocity/pressure-gradient correlations in the Reynolds-stress budget in the wake. (———) DNS profiles, (- - - -) calculated profiles

derived for homogeneous turbulence. However, the optimum value of the coefficient in inhomogeneous flows changes from flow to flow, depending on different factors controlling the large-scale turbulence structure. Among these factors are initial or inlet conditions, Reynolds number, flow geometry, and boundary conditions. These conditions make large-scale structure non-universal, even in geometrically equivalent flow situations at the same Reynolds number, as DNS and experimental data demonstrate. Moreover, it is not obvious that all conditions controlling the large scales have been defined and, even if they were known, the link between them and the large-scale structure they produce may not necessarily be evident and predictable.

Acknowledgments

The author would like to thank Dr. M. M. Rogers (NASA-Ames) for providing the DNS database and Mr. C. A. Langer (Mechanical Engineering Dept., Stanford University) for pointing out some references and for fruitful discussions.

REFERENCES

CHOU, P. Y. 1945 (Also Zhou) On velocity correlations and the solutions of the equations of turbulent fluctuation. *Quart. Appl. Math.* **3**, 38-54.

CROW, S. C. 1968 Viscoelastic properties of fine-grained incompressible turbulence. *J. Fluid Mech.* **33**, 1-20.

GIRIMAJI, S. S. 2000 Pressure-strain correlation modelling of complex turbulent flows. *J. Fluid Mech.* **422**, 91-123.

LAUNDER, B. E., REECE, G. J. & RODI, W. 1975 Progress in development of a Reynolds-stress turbulent closure. *J. Fluid Mech.*, **68**, 537-566.

LUMLEY, J. L. 1978 Computational modeling of turbulent flows. *Adv. Appl. Mech.* **18**, 123-177.

MOSER, R. D., ROGERS, M. M. & EWING, D. W. 1998 Self-similarity of time-evolving plane wakes. *J. Fluid Mech.* **367**, 255-288.

POPE, S. B. 1985 PDF methods for turbulent reactive flows. *Prog. Energy Combust. Sci.* **11**, 119-192.

POROSEVA, S. V. 2000 New approach to modeling the pressure-containing correlations. *Proc. 3rd Int. Symp. on Turbulence, Heat and Mass Transfer,* (Nagoya, Japan), 487-493.

POROSEVA, S. V. & BÉZARD, H. 2001 On ability of standard k-ε model to simulate aerodynamic turbulent flows. *CFD J.* **9**, 464-470.

POROSEVA, S. V. & IACCARINO, G. 2001 Simulating separated flows using the k-ε model. *Annual Research Briefs*, Center for Turbulence Research., NASA Ames/Stanford Univ., 375–383.

REYNOLDS, W. C. 1976 Computation of turbulent flows. *Ann. Rev. Fluid Mech.* **8**, 183-208.

REYNOLDS, W. C. 1987 Fundamentals of turbulence for turbulence modeling and simulation. *Lecture Notes for Von Karman Institute, AGARD* CP-93.

RISTORCELLI, J. R., LUMLEY, J. L. & ABID, R. 1995 A rapid-pressure covariance representation consistent with the Taylor-Proudman theorem materially-frame-indifferent in the 2D limit. *J. Fluid Mech.* **292**, 111-152.

ROGERS, M. M. & MOSER R. D. 1994 Direct simulation of a self-similar turbulent mixing layer. *Phys. Fluids* **6**, 903-923.

ROTTA, J. C. 1951 Statistische Theorie nichthomogener Turbulenz. 1. *Z. Phys.* **129**, 547-572.

SHIH, T.-H., SHABBIR, A. & LUMLEY, J. L. 1994 Realizability in second moment turbulence closures revisited. *NASA* TM 106469.

TSINOBER, A. 1998 Is concentrated vorticity that important? *Euro. J. Mech. - B/Fluids.* **17**(4), 421-449.