Instability of Blasius boundary layer in the presence of steady streaks

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1. Motivation and objectives

It is well known that the instability of boundary layers is sensitive to the mean velocity profile, so that a small distortion to the basic flow may have a detrimental effect on its stability. The main interest of the present paper is in investigating the mechanisms by which a relatively weak distortion can significantly affect the instability of an otherwise-uniform Blasius flow. Specifically, we shall address two issues: (a) how the Tollmien-Schlichting instability, which operates in the absence of any distortion, is modified by a weak distortion, and (b) whether or not a weak distortion is able to cause inviscid instability.

Many factors can cause three-dimensional steady or unsteady distortions in the form of streamwise or longitudinal vortices. These include small steady or unsteady perturbations superimposed on the oncoming flow, imperfections at the leading edge, crossflow instability, and Görtler vortices induced by surface curvature, as well as certain excitation devices. Distortion of this kind also arises due to the nonlinear interaction between pairs of Tollmien-Schlichting waves. The resulting perturbed flows are spanwise-dependent but essentially unidirectional, i.e. the transverse velocity components are much smaller than the streamwise component. The instability of such transversely sheared flows has attracted a great deal of interest because it appears to be related to various aspects of the transition process, such as secondary instabilities and by-pass transition.

A distortion of particular interest occurs when the boundary layer is subject to relatively high free-stream turbulence level. As was first observed by Dryden (1936) and Taylor (1939), small low-frequency three-dimensional perturbations in the free stream produce significant distortion within the boundary layer, leading to alternating thickening and thinning of the layer in the spanwise direction. Steady disturbances also cause a similar type of variation (Bradshaw 1965). Recent experimental studies show that the distortions are in the form of elongated streaks (see e.g. Kendall 1985, Westin *et al.* 1994, Alfredsson & Matsubara 2001, and references therein), now commonly referred to as the Klebanoff mode, as a tribute to the contribution of Klebanoff (1971). These experiments have provided fairly complete quantitative data about the characteristics of Klebanoff modes themselves. However, the instability of the streaks and its role in the transition process remain poorly understood. The main obstacle of course is the random nature (in both time and space) of free-stream disturbances and the associated Klebanoff motion, which make a quantitative study extremely hard. Numerous researchers instead investigated steady distortions, induced in a controlled manner.

Hamilton & Abernathy (1994) used surface roughness elements to create single or multiple streamwise vortices. These vortices cause the distorted flow profile to have an

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inflection point. An inflection point, however, does not always lead to the inviscid instability. Only when the distortion exceeds a certain critical magnitude do localized inviscid-instability waves start to appear. These waves may decay, or occasionally develop into turbulent spots if the distortion strength is just above the critical value. As the distortion is increased further, the local instability leads to persistent self-sustaining turbulent spots.

Asai, Fukuoka & Nishioka (2000) and Asai (2001) investigated in detail the instability of an isolated streak, which was produced by a small screen set normal to the wall. The low-speed streak was shown to support both symmetric (varicose) and antisymmetric (sinuous) modes. These investigators mapped out the amplitude development as well as the spatial structure of each mode.

Bakchinov *et al.* (1995) generated periodically-distributed longitudinal vortices by arranging roughness-element arrays in a regular spacing along the spanwise direction. For strong modulation, inviscid-instability modes were found to develop out of the background disturbances, and their frequencies were well above those of unstable TS waves in the Blasius flow. At moderate modulation, instability waves with typical frequencies of T-S waves can be observed, but they amplify more rapidly than in the Blasius flow.

Obviously, the instability of boundary layers subject to finite-amplitude steady distortions is an interesting and important problem in its own right, and an attack on it requires a major numerical study. The main concern of this paper is with the sensitivity of the boundary-layer instability to a steady distortion. For this purpose, it is appropriate to consider the case where the distortion is relatively weak that it represents a sort of 'imperfection'. A weak distortion has the advantage of being more amenable to analytical treatment, yet as we shall argue, the resulting simple model may well offer relevant insights to the case of stronger distortion.

There have been numerous theoretical studies of the instability of shear flows (boundary layers or channel flows) perturbed by distortions in the form of streaks. The interested reader is referred to Anderson *et al.* (2001) for relevant references. Often the streaks were modeled in a rather *ad hoc* fashion. In the present work, we insist that the distortions must be realizable, at least in principle, i.e. they may be generated by either by a specific excitation device or by external disturbances. At any rate, they must be appropriate (approximate) solutions to the Navier-Stokes equations. To fix the idea, we consider the instability of the steady distortion that has been considered by Goldstein & Wundrow (1995), The basic observation is that the Blasius profile has small curvature near the wall, so that even a small distortion may lead to an inflection point and possibly to essentially-inviscid instabilities.

The essential physical and analytical insights can be gained by an asymptotic approach based on a high-Reynolds-number assumption, which stands as the only means for providing a self-consistent mathematical description of the key process involved.

2. Theoretical considerations

2.1. *Formulation*

We consider the two-dimensional incompressible boundary layer due to a uniform flow U_∞ past a semi-infinite flat plate. As in Goldstein & Wundrow (1995), a small-amplitude spanwise-dependent motion is assumed to be imposed at a distance L downstream from the leading edge. The Reynolds number is defined as

$$R = U_\infty L/\nu \ , \tag{2.1}$$

where ν is the kinematic viscosity. We shall assume that $R \gg 1$.

The flow is to be described in the Cartesian coordinate system (x, y, z), with its origin at the location where the crossflow is introduced, where x, y and z denote distances in the streamwise, normal and spanwise directions respectively, and they are all non-dimensionalized by $\delta = LR^{-1/2}$, the boundary-layer thickness at $x = 0$. The time variable t is normalized by δ/U_∞. The velocity (u, v, w) is non-dimensionalized by U_∞, while the non-dimensional pressure p is introduced by writing the dimensional pressure as $(p_\infty + \rho U_\infty^2 p)$, where p_∞ is a constant and ρ is the fluid density.

The profile of the Blasius boundary layer, $U_B(y)$, has the behaviour that as $y \to 0$,

$$U_B(y) \to \lambda y - \frac{\lambda^2}{48} y^4 + \dots$$

where the skin friction

$$\lambda = \lambda_0 (1 + x R^{-\frac{1}{2}})^{-1/2} \quad \text{with} \quad \lambda_0 \approx 0.332 \ . \tag{2.2}$$

Let Λ denote the characteristic length scale over which the spanwise variation of the imposed flow occurs. We assume that Λ is much larger than the local boundary-layer thickness δ, i.e.

$$\sigma \equiv \frac{\delta}{\Lambda} \ll 1 \ ,$$

so that the variation of the distortion can be described by the slow variable

$$Z = \sigma z \ . \tag{2.3}$$

A crossflow $\epsilon_M W_0(y, Z)$ is imposed at $x = 0$ by some excitation device. In the laboratory this may be achieved by inserting a thin wire with non-uniform cross-section into the main part of the boundary layer. A small screen set normal to the wall, as in the experiments of Asai *et al.* (2000), probably produces a similar effect.

The mean-flow distortion so generated is analyzed in detail by Goldstein & Wundrow (1995), who show that the flow in the region $x = O(\sigma^{-1})$ is fully interactive, but the distortion is too weak to affect the instability. The important location is at $x = O(\sigma^3 R^{\frac{1}{2}})$, where the perturbed streamwise velocity profile develops an inflection point in the wall layer $y = O(\sigma)$, if the imposed crossflow has a magnitude $\epsilon_M \sim R^{-\frac{1}{2}} (\ln \sigma)^{-1}$. For *periodic distortion*, a pair of oblique modes is in resonance with the distortion if the spanwise wavelength of the former is twice that of the latter. The characteristic streamwise wavelength of the instability modes is found to be comparable with that of the mean-flow distortion. The growth rate induced by the resonance has the same order of magnitude as that due to viscosity if $\sigma = O(R^{-\frac{1}{20}})$ but is larger if $\sigma \gg R^{-\frac{1}{20}}$. In the latter case, the instability is inviscid.

The main interest of the present paper will be *localized distortion* since distortions of this form were produced and studied in number of experiments. It will be shown that an inviscid instability may occur in a region farther downstream than that considered by Goldstein & Wundrow (1995).

The region in which this instability operates, as well as its characteristic time and length scales, can be determined by a scaling argument based on three considerations. First, suppose that at a typical streamwise location $x \sim l \gg O(\sigma^3 R^{\frac{1}{2}})$, the wall layer has a width $y \sim \hat{\sigma}$. Then the balance between the advection term $U_B \frac{\partial}{\partial x}$ and the diffusion term $R^{-\frac{1}{2}} \frac{\partial^2}{\partial y^2}$ in the streamwise momentum equation requires that

$$\frac{\hat{\sigma}}{l} \sim \frac{R^{-\frac{1}{2}}}{\hat{\sigma}^2} \ . \tag{2.4}$$

Secondly, for the distortion to be able to induce an essentially-inviscid instability, its curvature must be comparable to the $O(\hat{\sigma}^2)$ curvature of the Blasius flow in the wall layer, that is

$$\frac{\epsilon_D}{\hat{\sigma}^2} \sim \hat{\sigma}^2 \ , \tag{2.5}$$

where ϵ_D stands for the magnitude of the streamwise velocity of the distortion. Thirdly, if we seek instability modes with $O(\hat{\sigma})$ streamwise wavenumbers, then their growth rate would be of $O(\hat{\sigma}^4)$. It turns out that such modes exist if

$$\hat{\sigma}^5 \sim \sigma^2 \ . \tag{2.6}$$

This relation ensures that the spanwise modulation appears at the same order as the streamwise evolution in the final amplitude equation; this point will become clear later.

It follows from Eqs. (2.4) and (2.6) that the instability will operate in the region where $x \sim l = O(R^{\frac{1}{2}}\sigma^{\frac{6}{5}})$, and so we introduce the variable

$$\hat{x} = x/(\sigma^{\frac{6}{5}}R^{\frac{1}{2}}) \ . \tag{2.7}$$

For $\hat{x} = O(1)$, the instability modes that the perturbed mean flow can support have streamwise wavelength of $O(\sigma^{-\frac{2}{5}})$, much shorter than the $O(\sigma^{-1})$ spanwise length scale of the distortion. The phase speed is $O(\hat{\sigma})$ so that the frequency is $O(\sigma^{\frac{4}{5}})$. Without losing generality, in the rest of the paper we put

$$\hat{\sigma} = \sigma^{\frac{2}{5}} \ .$$

2.2. *Solution for the mean-flow distortion*

The solution for the distortion was considered in detail by Goldstein & Wundrow (1995). The required solution corresponds to the downstream limit of theirs, and also it is only necessary to present the solution in the viscous wall region, which has a width of $O(\sigma^{\frac{2}{5}})$. The appropriate transverse variable is

$$Y = \frac{y}{\sigma^{\frac{2}{5}}} \ . \tag{2.8}$$

The mean flow expands as

$$U = \hat{\sigma}\lambda_0 Y + \hat{\sigma}^4(\tilde{U} - \frac{\lambda_0^2}{48}Y^4 - \frac{1}{2}\lambda_0\hat{x}Y) + \dots \ , \tag{2.9}$$

$$W = R^{-\frac{1}{2}}\sigma^{-\frac{3}{5}}(\tilde{W} + \dots) \ . \tag{2.10}$$

The solution is simply the first terms of (3.43) in Goldstein & Wundrow (1995), namely

$$\tilde{U} = \bar{\sigma}\hat{x}B'(Z)F(\eta) \ , \quad \tilde{W} = \bar{\sigma}B(Z)G(\eta) \ , \tag{2.11}$$

where the similarity variable is defined as

$$\eta = (\lambda_0/\hat{x})^{\frac{1}{3}}Y \ . \tag{2.12}$$

The functions F and G satisfy the equations

$$F''' + \frac{1}{3}\eta^2 F'' - \frac{2}{3}\eta F' = -G, \quad G'' + \frac{1}{3}\eta^2 G' = 0 \ . \tag{2.13}$$

They are subject to the boundary conditions

$$F = F'' = G = 0 \quad \text{at} \quad \eta = 0 \ , \tag{2.14}$$

$$F \to \ln \eta \,, \quad G \to 1 \quad \text{as} \quad \eta \to \infty. \tag{2.15}$$

The boundary-value problem Eqs. (2.13)–(2.15) was solved numerically.

It should be pointed out that the curvature alteration in the wall layer is much larger than that in the main region despite the fact the streamwise velocities in both layers have the the same order of magnitude. This feature turns out to be important for the instability of the perturbed flow.

3. Results

3.1. *Linear instability*

When $\hat{x} = O(1)$, the distortion to the mean flow is still small in the whole flow field. An important point to note is that in the viscous wall region the curvature of the mean-flow distortion is comparable with that of the original Blasius flow, that is, the curvature of the total mean flow is altered by $O(1)$ in relative terms, and moreover becomes spanwise-dependent. This leads to a fundamental change of the instability property.

As was indicated by the scaling argument in Section 2, the admissible modes have streamwise wavenumbers of $O(\hat{\sigma})$, frequencies of $O(\hat{\sigma}^2)$ and growth rates of $O(\hat{\sigma}^4)$; so we introduce

$$\zeta = \hat{\sigma}\alpha x - \hat{\sigma}^2 \omega t \,, \quad X = \hat{\sigma}^4 x \,, \tag{3.1}$$

to describe the rapid oscillation and the relatively slow amplification of the modes respectively, where α and ω are the scaled wavenumber and frequency. We expand α and the phase speed $c \equiv \omega/\alpha$ as

$$\alpha = \alpha_0 + \hat{\sigma}\alpha_1 + \dots \,, \quad c = \frac{\omega}{\alpha} = c_0 + \hat{\sigma} c_1 + \dots \,.$$

The most unstable modes that the perturbed flow can support must have a spanwise length scale comparable with that of the distortion. Modes with a shorter spanwise length scale may be treated in a quasi-planar manner, but they have smaller growth rates, and moreover their phase speeds would be a function of Z, contradicting the experimental observation of Asai (2001) and Bakchinov *et al.* (1995) that the phase speed is constant along the spanwise direction. Such modes will be discarded. Therefore the spanwise variation of relevant instability waves is described by the variable Z. In the main part of boundary layer, the modes take the form, to leading order,

$$u = \epsilon A(X, Z)\bar{u}_1(y)\,\mathrm{e}^{\mathrm{i}\,\zeta} + c.c. + \dots, \tag{3.2}$$

where ϵ represents the magnitude of the modes, and A is the amplitude function.

Since the wavelength of the instability modes is long compared with the boundary layer thickness, the linear instability problem is governed by a five-zoned asymptotic structure that is akin to that for the upper-branch instability of the unperturbed Blasius boundary layer (cf. Bodonyi & Smith 1981, Goldstein & Durbin 1986). It consists of the upper layer, the main layer, the Tollmien layer, the viscous Stokes layer as well as the critical layer centered at the position where the basic flow velocity equals the phase velocity c.

The solution in each of these regions can be obtained by following what is now a fairly routine procedure (see e.g. Wu, Stewart & Cowley 1996). Matching these solutions gives the leading-order dispersion relation

$$c_0 = \frac{\alpha_0}{\lambda_0} \,, \tag{3.3}$$

and the relation for the growth rate

$$A_X - \frac{\mathrm{i}}{4\alpha_0} A_{ZZ} = (c^+ - c^-) + \Big[\frac{\lambda_0^2}{2R^{\frac{1}{4}}\hat{\sigma}^5(2\alpha_0 c_0)^{\frac{1}{2}}} + \mathrm{i}\,\chi_0\Big]A \ , \tag{3.4}$$

where $(c^+ - c^-)$ is the jump across the critical layer. In the linear regime ($\epsilon \ll 1$),

$$c^+ - c^- = \pi c_0 Y_c \Big\{-\frac{c_0^2}{4} + \tilde{U}_{YY}(Y_c, Z)\Big\} A \ , \tag{3.5}$$

where $Y_c = c_0/\lambda_0$ is the scaled critical level. Inserting Eq. (3.5) into Eq. (3.4) gives

$$A_X - \frac{\mathrm{i}}{4\alpha_0} A_{ZZ} = \Big(\gamma_0 + \gamma(Z)\Big) A \ , \tag{3.6}$$

where

$$\gamma_0 = -\frac{\pi c_0^4}{4\lambda_0} + \frac{\lambda_0^2}{2R^{\frac{1}{4}}\hat{\sigma}^5(2\alpha_0 c_0)^{\frac{1}{2}}} \ , \tag{3.7}$$

$$\gamma(Z;\hat{x}) = \pi c_0^2 (\frac{\hat{x}}{\lambda_0})^{\frac{1}{3}} F''(\eta_c)\bar{\sigma} B'(Z) \equiv \tilde{\gamma}(\hat{x})\bar{\sigma} B'(Z) \ , \tag{3.8}$$

with $\eta_c = \frac{c_0}{\lambda_0}(\frac{\lambda_0}{\hat{x}})^{\frac{1}{3}}$. Here use has been made of Eqs. (2.11) and (2.12), and the logarithmic factor $\bar{\sigma}$ has been absorbed into the definition of B. Obviously γ_0 is the growth rate in the absence of the distortion, with the second term in γ_0 representing the contribution from the viscous Stokes layer, which is the sole instability mechanism when the distortion is absent.

The derivation of Eq. (3.6) is based on the fact that the major curvature alteration occurs in a wall layer. As is indicated by Eq. (3.6), in this case the curvature alteration in the wall region is the sole quality that characterizes the instability of the perturbed flow; the distortion in the main part of the boundary layer turns out to be largely irrelevant.

Equation (3.6) can be viewed as a Schrödinger equation with a purely imaginary potential $\mathrm{i}\,\gamma(Z)$. It admits solution of the form

$$A = \Phi(Z)\,\mathrm{e}^{(a+\gamma_0)X} \ , \tag{3.9}$$

where a is a complex constant, and $\Phi(Z)$ satisfy

$$\Phi_{ZZ} = 4\,\mathrm{i}\,\alpha_0 \Big(\gamma(Z) - a\Big)\Phi \ . \tag{3.10}$$

For localized $\gamma(Z)$, the boundary conditions are

$$\Phi(Z) \to \mathrm{e}^{\mp(-4\,\mathrm{i}\,\alpha_0 a)^{\frac{1}{2}} Z} \quad \text{as} \quad Z \to \pm\infty \ , \tag{3.11}$$

where the square root is taken to be the one with a positive real part so that Φ decays to zero $\pm\infty$. We can derive a general result similar to the familiar 'semicircle theorem':

$$\min \gamma(Z) < \Re(a) < \max \gamma(Z) \ . \tag{3.12}$$

Equation (3.10) with Eq. (3.11) forms an eigenvalue problem to determine a. The real part of a represents the *excess growth rate* induced by the mean-flow distortion or streak. Depending on the size of $\hat{\sigma}$, the instability may be of quite different nature. Equations (3.7) and (3.8) indicate that the growth rate due to the viscosity is negligible if $\hat{\sigma} \gg R^{-\frac{1}{20}}$ or equivalently if the magnitude of the streamwise velocity of the distortion satisfies

$$\epsilon_D \gg R^{-\frac{1}{5}} \ .$$

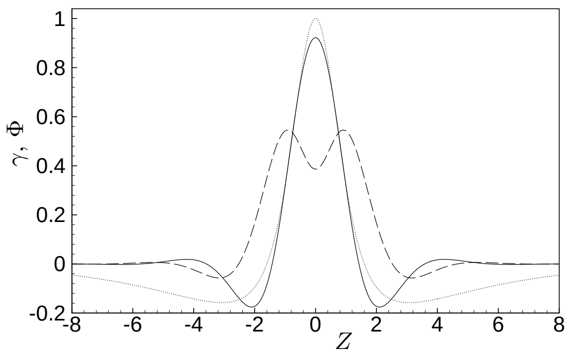

FIGURE 1. Distribution of distortion and eigenfunction, ········ distortion, ——— Φ_r – – – – Φ_i.

This implies that, when the distortion exceeds some threshold, the instability becomes essentially inviscid, with the growth rate

$$\kappa \equiv a_r - \frac{\pi c_0^4}{4\lambda_0} \ , \tag{3.13}$$

although it may be argued that inclusion of the viscous growth in this case would give a more general result. When $\epsilon_D \sim R^{-\frac{1}{5}}$, the distortion provides a modification to the viscous growth rate, and the modes may be viewed as a kind of *modified T-S waves*, even though they do not necessarily reduce to the usual T-S waves at the zero-distortion limit.

The eigenvalue problem Eqs. (3.10)–(3.11) is solved numerically for a spanwise distribution of the form

$$B'(Z) = B_0 \operatorname{sech}(\frac{Z}{d}) \tanh Z \ , \tag{3.14}$$

where d is a measure of the spanwise length scale of the distortion. Figure 1 depicts a typical shape, which is quite similar to that in the experiments of Asai *et al.* (2000) and Asai (2001). Also shown in the Figure is a typical distribution of the eigenfunction Φ. Clearly, the instability mode is confined to the region of the mean-flow distortion, and decays rapidly away from it. The mode is symmetric, i.e. varicose, in nature. No anti-symmetric (or sinuous) mode has been found. Asai *et al.* (2000) attributed the varicose modes to the inflection point in the normal direction $U_{YY} = 0$, and the sinuous modes to the inflection in the spanwise profile, $U_{ZZ} = 0$. The former is present in our theory, but the latter is absent. That sinuous modes are absent from our results is consistent with the conclusion of Asai *et al.*

In order to understand the general properties of the problem, calculations were first performed for the artificial case where $\tilde{\gamma} = 1$. The variation of a_r with B_0 is plotted in Fig. 2a for two fixed values of d. It shows that $a_r > 0$ when the distortion exceeds a *threshold magnitude* B_c. Below B_c, the localized mode does not exist. Instead there exists a continuous spectrum for which a is purely imaginary so that Φ is only bounded at $Z = \pm\infty$. The continuous spectrum can be viewed as the usual T-S waves, whose shape is deformed by $\gamma(z)$ but whose growth rates are not affected. The existence of a threshold means that the localized modes do not reduce to the usual T-S waves as the distortion is reduced; instead they merge with the continuous spectrum.

Figure 2b shows the variation of a_r with d. For each fixed B_0, there exists a threshold

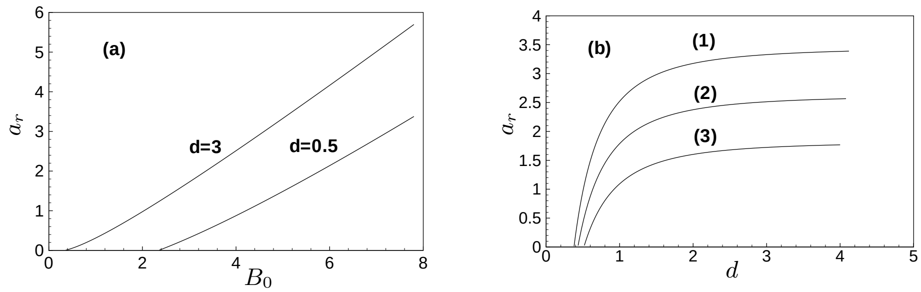

FIGURE 2. Variation of a_r with B_0 and d. Curves (1)-(3) correspond to $B_0 = 5$, 4 and 3.

d_c above which a localized mode comes into existence. The growth rate increases with the spanwise length scale d, and quickly saturates at the 'two-dimensional limit' when d is large enough. This conclusion is in agreement with the experimental finding of Asai (2001), although in his experiments the distortion has a rather large magnitude. It seems reasonable to suggest that the present simple theory captures some generic feature of the instability.

Next we present the instability results for the particular distortion considered in Section 2, for which $\tilde{\gamma}$ depends on $\hat{x}$ and is evaluated by solving Eqs. (2.13)–(2.15) numerically. It is found that $F > 0$ but $F'' < 0$, and as a result $\tilde{\gamma} < 0$ (see Eq. (3.8)). Therefore according to the result shown in Fig. 2, inviscid instability is possible only for $B_0 < 0$, i.e. when the distortion is characteristic of a low-speed streak.

For a given B_0, the inviscid growth rate $\kappa(\omega, \hat{x})$ as defined by Eq. (3.8) will be a function of $\hat{x}$ and $\omega = \alpha_0 c_0$, the frequency of the instability mode. As shown in Fig. 3a, in the streamwise region in which the distortion is significant, the perturbed flow supports a band of instability modes. The instability will manifest itself as an oscillation of the streak. In Fig. 3b, we plot the variation of the growth rate κ with $\hat{x}$ for three typical values of ω. As is illustrated, a mode with a suitable frequency experiences amplification in a finite streamwise region, beyond which it decays. The spatial extent and the frequency range of the unstable modes can best be demonstrated by plotting the contours of the growth rate $\kappa(\omega, \hat{x})$ in the $\hat{x} - \omega$ plane; see figure 4. For $B_0 = -7$, the perturbed flow is unstable in the streamwise window between $\hat{x} \approx 1.2$ and $\hat{x} \approx 19$. The unstable frequency band varies with $\hat{x}$, but roughly speaking the overall range is between $\omega = 0.2$ and 1.2. The frequency of the most unstable modes in the upstream end is fairly small, but increases with the downstream distance, implying that the oscillation of the streak will become progressively more rapid.

3.2. *Nonlinear instability*

As an instability mode amplifies, nonlinear effects may become important. For an instability wave with an asymptotically small growth rate, it is now well recognized that the dominant nonlinear interactions will first take place within the critical layer to produce a velocity jump across this layer. For reviews, see e.g. Goldstein (1994) and Cowley & Wu (1994). For the present problem, the nonlinear jump becomes comparable with the linear jump when

$$\epsilon = \hat{\sigma}^{\frac{17}{2}} \ . \tag{3.15}$$

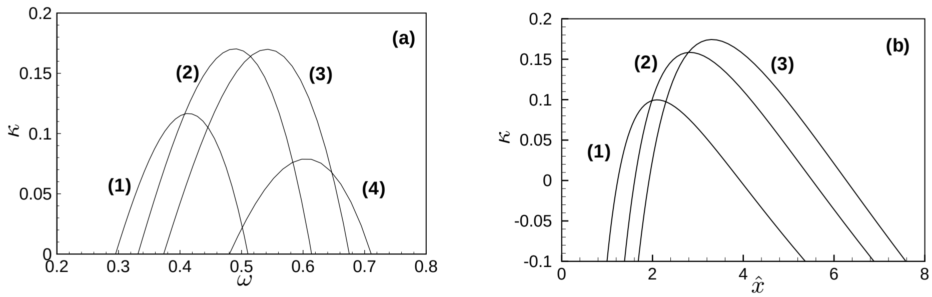

FIGURE 3. Inviscid growth rates κ of localized modes for $B_0 = -6$. (a) κ v.s. ω at $\hat{x} = 2$, 3,4 and 6, represented by curves (1)-(4) respectively. (b) κ v.s $\hat{x}$ for $\omega = 0.37$, 0.45 and 0.50 represented by curves (1)-(3) respectively.

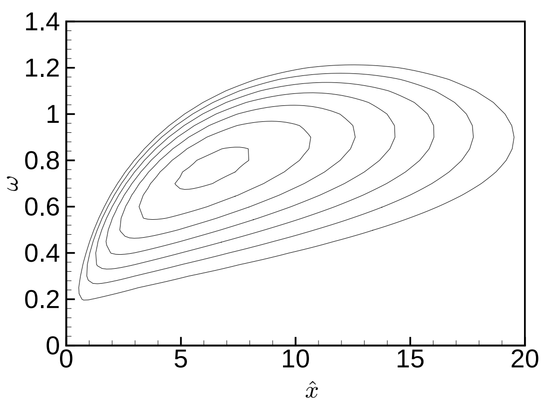

FIGURE 4. Contours of growth rates ($B_0 = -7$). The outermost contour corresponds to the neutral curve $\kappa(\omega, \hat{x}) = 0$.

We also assume that the Reynolds number scales with $\hat{\sigma}$ as follows

$$R^{-\frac{1}{2}} = r\hat{\sigma}^{13} \tag{3.16}$$

so that the viscous diffusion appears as a leading-order effect in the critical layer, where r is a parameter of order one, reflecting the effect of viscosity. In passing we note that the experiments of Bakchinov *et al.* (1995), in which the instability of a Blasius boundary layer subject to a spanwise-dependent distortion was investigated, point to the existence of a well defined critical layer, in which the mode attains its largest magnitude.

The nonlinear jump is the same as that calculated by Wu (1993) and Wu *et al.* (1996). Inserting that jump into Eq. (3.4), we obtain the amplitude equation that describes the nonlinear instability of the perturbed flow

$$A_X - \frac{\mathrm{i}}{4\alpha_0} A_{ZZ} = \Big(\gamma_0 + \gamma(Z)\Big) A + \mathrm{i}\, N(X, Z) \ , \tag{3.17}$$

where the nonlinear term

$$N = \int_0^\infty \int_0^\infty K(\xi, \eta | s) \Bigg\{ \xi^3 A(X - \xi) A(X - \xi - \eta) A^*_{ZZ}(X - 2\xi - \eta)$$

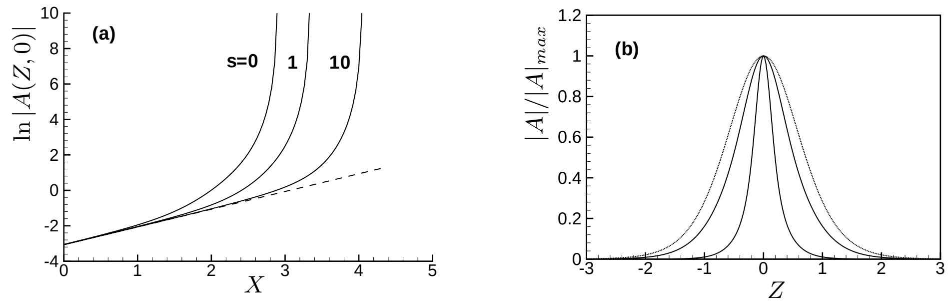

FIGURE 5. Nonlinear evolution of a localized mode ($\omega = 0.8$, $\hat{x} = 6.5$). (a) The amplitude on the symmetry plane $\ln|A(X,0)|$ *v.s.* X for $s = 0$ (purely inviscid case), 1 and 10. ········ the linear limit. (b) Spanwise distribution of $|A|$ at $X = 3.02$, 3.54 for $s = 1$, showing self-focusing as X increases. The least 'compact' of the three curves corresponds to the linear limit.

$$+\xi^2\eta A(X-\xi)\Big[A(X-\xi-\eta)A_Z^*(X-2\xi-\eta)\Big]_Z$$
$$+\xi^3\Big[A(X-\xi)A(X-\xi-\eta)A_Z^*(X-2\xi-\eta)\Big]_Z\Bigg\}\,\mathrm{d}\,\xi\,\mathrm{d}\,\eta\ , \quad (3.18)$$

with

$$K(\xi,\eta|s) = \mathrm{e}^{-s(2\xi^3+3\xi^2\eta)}\ , \quad s = \frac{1}{3}\alpha_0^2\lambda_0^2 r\ .$$

In Eq. (3.17), the amplitude function has been suitably renormalized so that the coefficient multiplying $\mathrm{i}\,N$ is unity. The amplitude A should match with the linear solution upstream, and so we have

$$A \to \Phi(Z)\,\mathrm{e}^{(a+\gamma_0)X} \quad \text{as} \quad X \to -\infty\ . \quad (3.19)$$

The nonlinear amplitude equation Eq. (3.17) was solved for the localized distortion. The mode was chosen to be the most unstable one in Fig. 4, which exists at $\hat{x} \approx 6.5$ and has frequency $\omega = 0.8$. The nonlinear evolution of $|A(X,0)|$, the amplitude on the symmetry plane, is shown in Fig. 5a for three viscous parameter values. Nonlinearity enhances the amplification, and apparently leads to a singularity at a finite distance downstream. Viscosity delays the formation of the singularity but cannot eliminate it. Figure 5b shows that the nonlinear effect deforms the shape of the mode, and a singularity of self-focusing type appears to be forming at $Z = 0$. A plausible structure for this singularity was proposed in Wu (1993). In the vicinity of the singularity, the present theory breaks down and strong three-dimensionality may act to 'regularize' the solution.

4. Conclusions and discussions

In this paper, we have shown that the instability of Blasius boundary layer can be significantly modified, and even fundamentally altered, by certain small-amplitude distortions which feature low-speed streaks. This occurs when the curvature of the distortion becomes comparable to that of the Blasius profile in a suitable vicinity of the wall. A self-consistent asymptotic theory is presented for distortions whose spanwise length scale is larger than the boundary-layer thickness. The instability of the perturbed flow is shown to be governed by a remarkably simple system, a Schrödinger equation with a purely imaginary potential.

A moderate distortion induces an excess growth rate comparable to that due to viscosity, and the instability modes can be viewed as a kind of modified T-S wave. This modification however is non-trivial because the spanwise shape is dictated by the distortion. The modes do not reduce to the usual T-S waves in the zero-distortion limit.

When the strength of the distortion exceeds a certain threshold, essentially-inviscid localized instability arises. The characteristic streamwise wavelength of the instability modes is much shorter than the spanwise length scale of the distortion, and their characteristic frequencies are higher than those of typical T-S waves on Blasius flow. Also the instability occurs in a limited streamwise window, and hence on the purely linear basis, the instability modes will die out. However they can enter a nonlinear regime if a significant magnitude is attained. The continued nonlinear development of these modes is governed by a modified form of the evolution equation derived by Wu (1993), and the nonlinear effect is found to be strongly destabilizing, causing the amplitude to break down rapidly in the form of a finite-distance singularity. The instability may lead to patches of streak oscillation, which may well breakdown into turbulent spots.

While the theory is built upon a set of rather restricted asymptotic relations, it does appear to be capable of reproducing the major laboratory observations qualitatively. For instance, the existence of a threshold magnitude and the occurrence of oscillation patches are in agreement with the conclusions of Hamilton & Abernathy (1994). As mentioned above, the theoretical prediction that the growth rate increases with the spanwise length scale of the distortion is consistent with the measurements of Asai (2001). The predicted frequency range of the inviscid unstable modes, as well as the excess growth exhibited by the T-S waves confirm the findings of Bakchinov *et al.* (1995).

It should be noted that the distortions in the experiments are actually comparable with the basic Blasius flow so that they must be governed by nonlinear equations as opposed to the linear equations employed in our theory. Based on the above broad agreement, it seems reasonable to argue that the nonlinear structure of the distortion should not affect the qualitative feature of the instability, and that the simple model captures the key physics of the instability. From the qualitative point of view, the failure to describe the sinuous instability mode seems to be the only obvious shortcoming of the model.

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