Simple stochastic model for laminar-to-turbulent subcritical transition

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1. Motivation and objectives

The main purpose of this paper is to study the effects of stochastic perturbations on a non-normal dynamical system mimicking laminar-to-turbulent subcritical transition. The mechanism of non-normal transient linear growth has received much attention, both experimentally and theoretically, during the past decade, especially after the seminal work of Trefethen *et al.* (1993). The main reason is that this explains the onset of turbulence when the laminar flow passes to a turbulent regime without linear instability. Non-normality of the linearized Navier-Stokes evolution operator leads to the transient growth of velocity disturbances, even though the steady mean flow is linearly stable. A typical example of such transient growth is the function $t\exp(-t)$. Let us remind that the matrix A is normal, if $AA^* = A^*A$, where $*$ denotes the Hermitian transpose, otherwise it is non-normal. The nonlinear interactions lead to a further amplification of the initially small but finite disturbances. Nonlinear terms play a vital role in the redistribution of energy to those disturbances which exhibit a linear transient growth. Thus the transition to turbulence is not a consequence of the linear instability of the stationary laminar flow; rather, it is the result of the interaction of the non-normality-producing transient amplification of velocity perturbations and energy-conserving nonlinearities driving the system into the basin of attraction of the turbulent regime. A comprehensive review of the up-to-date results on such interactions and the resulting onset of shear-flow turbulence can be found in the review by Grossmann (2000) and the book by Schmid and Henningson (2001).

Several theoretical studies have been devoted to stochastically-forced dynamical systems involving a non-normal operator (Farrel & Ioannou 1993; Bassam & Dahlem 2001). It has been found that these systems have an extraordinary sensitivity to random perturbations, which leads to a great amplification of the variances. However, this research has focused only on linear non-normal systems.

The objective of this paper is to study the interaction between the following three factors: non-linearity, non-normality, and stochastics. In order to gain some insight into this problem, we shall examine the role of external noise in a simple non-normal dynamical system mimicking laminar-to-turbulent subcritical transition

$$\frac{du}{dt} = -2\varepsilon u + \left(u^2 + v^2\right)^{\frac{1}{2}} v,$$

$$\frac{dv}{dt} = -\varepsilon v + u - \left(u^2 + v^2\right)^{\frac{1}{2}} u, \tag{1.1}$$

where u and v mimic streamwise vortices and streamwise streaks respectively; ε is a small parameter, chosen in analogy with the inverse Reynolds number. This dynamical system

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has been suggested by Trefethen *et al.* (1993) as a simple model explaining the subcritical transition of a flow obeying the Navier-Stokes equations. It should be noted that several other low-dimensional models have been proposed to explain the onset of a turbulent regime for high Reynolds numbers (e.g. Gebhardt & Grossmann 1994, Baggett *et al.* 1995, 1997). The dynamical system (1.1) has three stable equilibrium points including $(0,0)$. The main feature of the system (1.1) is that for $\varepsilon << 1$ the linearized evolution operator for the fixed point $(0,0)$ is a highly-non-normal matrix that leads to a large transient growth of $v(t)$ prior to an eventual exponential decay. It can easily be found that for the non-zero initial conditions $u(0) = \varepsilon u_0$ and $v(0) = 0$, the solution of the linearized equations is of the form $v(t) = u_0(e^{-\varepsilon t} - e^{-2\varepsilon t})$, $u(t) = \varepsilon u_0 e^{-2\varepsilon t}$. The function $v(t)$ achieves a maximum of order one, on a time scale of order ε^{-1}. Furthermore, although both eigenvalues are negative $(\lambda_1 = -\varepsilon\, ; \lambda_2 = -2\varepsilon)$, finite fluctuations with exceedingly low amplitude can excite the transition from the fixed point $(0,0)$. The main problem here is to find the minimum amplitude of all fluctuations capable to excite this transition and its dependence on the parameter ε of the form ε^{α}. The threshold exponent α is found to be 3 . This tells us that the basin of attraction of $(0,0)$ shrinks very rapidly as $\varepsilon \to 0$ (Chapman 2002).

2. Non-normal dynamical system with noise

One of the purposes of this paper is to understand how random perturbations can affect the dynamics of the non-normal system (1.1). We simply add two generic uncorrelated Gaussian white-noise terms to the right-hand side of (1.1). The dynamical system (1.1) can then be written in the form of the stochastic differential equations (Gardiner 1996)

$$du = (-2\varepsilon u + \left(u^2+v^2\right)^{\frac{1}{2}} v)dt + (2\delta)^{\frac{1}{2}}\, dW_1(t)\,,$$

$$dv = (-\varepsilon v + u - \left(u^2+v^2\right)^{\frac{1}{2}} u)dt + (2\delta)^{\frac{1}{2}}\, dW_2(t)\,, \qquad (2.1)$$

where $W_1(t)$ and $W_2(t)$ are the uncorrelated standard Wiener processes. Here we assume for simplicity that the intensity of the noise parameter, δ, is the same for both stochastic terms.

For the deterministic system (1.1) only small but finite initial perturbations can escape from the basin of attraction for a fixed point at the origin. In this case the main problem is to answer the question “What are the minimum amplitude of the form ε^{α} and the threshold exponent α for transition to turbulence?”. In the stochastic case the key question is “What is the long-time effect of adding noise terms to the nonlinear non-normal dynamical system?”. Due to the highly sensitive way that non-normal systems are affected by random perturbations, we can expect that the presence of noise on the right-hand side of (1.1) may lead to a transition, even for zero initial conditions

$$u(0) = 0, \quad v(0) = 0. \qquad (2.2)$$

We believe that this is physically significant since in practical situations random fluctuations may often be what induce the subcritical transition in fluid flow. To illustrate the stochastic sensitivity of the non-normal system (2.1), consider its linear approximation

$$du = -2\varepsilon u dt + (2\delta)^{\frac{1}{2}} dW_1(t),$$

$$dv = (-\varepsilon v + u)dt + (2\delta)^{\frac{1}{2}} dW_2(t) \qquad (2.3)$$

with zero initial conditions (2.2). This is a relatively simple stochastic dynamical system in which the variable $u\,(t)$ is the Ornstein-Uhlenbeck process, with well-known statistical properties, while $v(t)$ is the non-Markov random process whose properties can be easily found (Gardiner 1996). The second moments are very important statistical characteristics of the system (2.3), since they mimic the kinetic energy of fluid flow. One can find the following explicit representations for them:

$$m_1(t) \equiv Eu^2(t) = \frac{\delta}{2\varepsilon}(1 - e^{-4\varepsilon t}),$$

$$m_2(t) \equiv Eu(t)v(t) = -\frac{2\delta}{3\varepsilon^2}e^{-3\varepsilon t} + \frac{\delta}{2\varepsilon^2}e^{-4\varepsilon t} + \frac{\delta}{6\varepsilon^2},$$

$$m_3(t) \equiv Ev^2(t) = (-\frac{\delta}{\varepsilon^3} - \frac{\delta}{\varepsilon})e^{-2\varepsilon t} + \frac{4\delta}{3\varepsilon^3}e^{-3\varepsilon t} - \frac{\delta}{2\varepsilon^3}e^{-4\varepsilon t} + \frac{\delta}{\varepsilon} + \frac{\delta}{6\varepsilon^3},$$

where E denotes the expectation operator. The limiting values $\bar{m}_i = \lim_{t\to\infty} m_i(t)$ are

$$\bar{m}_1 = \frac{\delta}{2\varepsilon}, \qquad \bar{m}_2 = \frac{\delta}{6\varepsilon^2}, \qquad \bar{m}_3 = \frac{\delta}{\varepsilon} + \frac{\delta}{6\varepsilon^3}. \tag{2.4}$$

From (2.4) we can see that owing to the non-normality of the system (2.3) as $\varepsilon \to 0$ for constant δ, all second moments tend to infinity. The stationary second moment $\bar{m}_3$ exhibits the highest degree of sensitivity. Even for very weak noise, say $\delta \sim \varepsilon^2$ then $\bar{m}_3 \to \infty$ as $\varepsilon \to 0$.

It is instructive to investigate the effect that non-normality has on the probability of exit from the zero-attraction point. This problem is closely related to the famous 'Kramer's exit problem' which concerns the escape of random trajectories of a stochastic dynamical system from the domain of attraction of the underlying deterministic dynamical system (Gardiner 1996). We have calculated numerically the empirical exit probabilities of random trajectories from the neighborhood of the zero point $U = \{(u, v) : u^2 + v^2 \leq 0.01\}$ up to $t = 10$. The results in figure 1 demonstrate that even for a very small intensity of noise ($\delta = 10^{-3}$) the exit probability p_e is close to unity. In particular, for $\varepsilon = 2^{-5} \approx 0.03$ and $\delta = 5 \times 10^{-5}$ the exit probability is greater than 0.6. For $\delta = 2 \times 10^{-4}$ the probability p_e is greater than 0.8. For $\delta = 10^{-3}$ this probability is very close to one.

An analytical treatment of the stochastic dynamical system (2.1) is rather difficult, although some approximations are possible, and indeed useful (see (4.3) for the slowly-varying energy of the non-normal system). We have performed simulations of random trajectories of (2.1) for different values of ε and δ. Our numerical results show that, either by increasing the intensity of noise δ or by decreasing the non-normality parameter ε, the stochastic system (2.1) undergoes a series of phase transitions. We have found three qualitatively different regimes. For a fixed value of ε, this phenomenon can be interpreted as a noise-induced transition. A detailed discussion can be found in the excellent book by Horsthemke & Lefever (1984).

Figures 2 and 3 illustrate these transitions in terms of stochastic trajectories of the non-normal dynamical system for $\varepsilon = 10^{-2}$ and $\delta = 10^{-4}$ (figure 2), $\delta = 10^{-2}$ (figure 3). For very small values of δ ($\delta < 10^{-12}$), we have observed that the random trajectory is concentrated around the equilibrium point at $(0, 0)$. As δ increases, the trajectory then begins to become more concentrated in the vicinity of one of the non-trivial fixed points (Fig. 2). Further increase of the noise intensity parameter δ leads to the stochastic orbits containing all three fixed points (Fig. 3). It should also be noted that these noise-induced

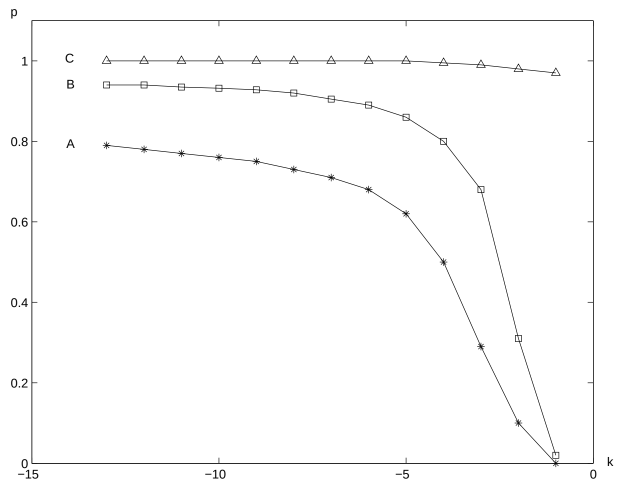

FIGURE 1. The exit probability p_e as a function of non-normal parameter $\varepsilon = 2^{-k}$. Curves A, B and C correspond to $\delta = 5 \cdot 10^{-5},\ 2 \cdot 10^{-4},\ 10^{-3}$.

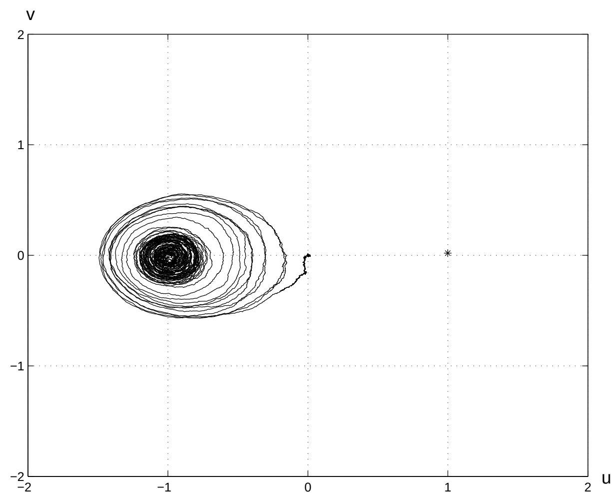

FIGURE 2. The stochastic trajectory for $\varepsilon = 10^{-2}, \delta = 10^{-4}$; initial conditions: $u(0) = 0$ and $v(0) = 0$.

transitions can also be analyzed in terms of the extrema of the stationary probability density $p_{st}(u, v)$ (Horsthemke & Lefever 1984).

3. Underlying Hamiltonian structure

The behavior of the trajectories when the values of ε and δ are small can be explained by the existence of a Hamiltonian structure in (2.1). If we introduce the Hamiltonian function

$$H\left(u,v\right) = \frac{1}{3}\left(u^2 + v^2\right)^{\frac{3}{2}} - \frac{1}{2}u^2, \tag{3.1}$$

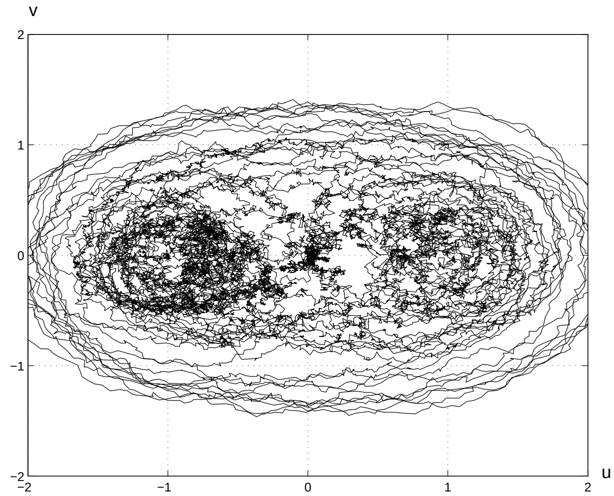

FIGURE 3. The stochastic trajectory for $\varepsilon = 10^{-2}, \delta = 10^{-2}$.

then the dynamical system (2.1) can be rewritten as

$$du = -2\varepsilon u dt + \frac{\partial H}{\partial v} dt + (2\delta)^{\frac{1}{2}}\, dW_1(t),$$

$$dv = -\varepsilon v dt - \frac{\partial H}{\partial u} dt + (2\delta)^{\frac{1}{2}}\, dW_2(t). \tag{3.2}$$

In the limits $\varepsilon \to 0$ and $\delta \to 0$, the system (3.2) becomes conservative, so

$$\frac{du}{dt} = \frac{\partial H}{\partial v}, \qquad \frac{dv}{dt} = -\frac{\partial H}{\partial u}, \tag{3.3}$$

and, therefore, $H(u,v) = E = \text{const}$. The phase trajectories $u(t)$ and $v(t)$ of (3.3) move along the level set

$$C(E) = \{(u,v):\ H(u,v) = \frac{1}{3}\left(u^2+v^2\right)^{\frac{3}{2}} - \frac{1}{2}u^2 = E\}. \tag{3.4}$$

with the speed

$$V(u,v) = (\frac{\partial H}{\partial v}, -\frac{\partial H}{\partial u}). \tag{3.5}$$

It follows from the existence of the Hamiltonian (3.1) that the trajectories are periodic, and that the period of the oscillations $T(E)$ can be found to be

$$T(E) = \int_{C(E)} |V(u,v)|^{-1} ds, \tag{3.6}$$

where the integral is taken along the level curves $C(E)$.

In figure 4 we plot the one-parameter family of curves generated by (3.4) that gives us the full phase portrait of the conservative system (3.3). There are three equilibrium points, at $(0,0)$ and $(\pm 1, 0)$. One can see that the phase portrait is similar to that of the Duffing equation without dissipation. Linearization of (3.3) at $(1,0)$ and $(-1,0)$ gives us the period 2π. While moving out, the periodic trajectories have longer periods and tend to infinity as we approach the saddle connection. The situation is more complicated in the presence of dissipative terms. An addition of the two terms $-2\varepsilon u$ and $-\varepsilon v$ changes

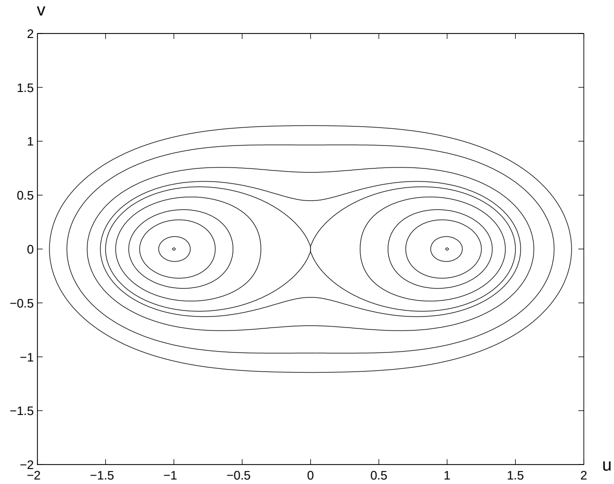

FIGURE 4. The phase portrait of the conservative system (3.3).

the direction of the vector field in an alternative way, to that of the dissipative Duffing equation. Of course, the global effect is to destroy the closed orbits. In particular, the fixed point $(0,0)$ becomes linearly stable, but the width of its basin of attraction decreases as ε^3.

4. Stochastic differential equation for the energy

In a further analysis of the effect of randomness and dissipation, it is of interest to consider the reduced equation for energy. In the general case ($\varepsilon \neq 0,\ \delta \neq 0$) the energy of the system $E = H(u,v)$ is not a constant, but rather a random function of time. If we apply the Itô formula for $E = H(v,u)$ (Gardiner 1996) we can obtain the governing equation for the energy,

$$dE = (-2\varepsilon \frac{\partial H}{\partial u} u - \varepsilon \frac{\partial H}{\partial v} v + \delta \frac{\partial^2 H}{\partial u^2} + \delta \frac{\partial^2 H}{\partial v^2})dt$$
$$+ (2\delta)^{\frac{1}{2}} \frac{\partial H}{\partial u} dW_1(t) + (2\delta)^{\frac{1}{2}} \frac{\partial H}{\partial v} dW_2(t). \tag{4.1}$$

It is clear that for small values of both the dissipation parameter ε, and the noise parameter δ, after some transient period of time, the phase trajectories of (3.2) will be very close to the level curves $C(E)$. There are three different families of periodic orbits, separated by the saddle connection (see Fig. 4). Let us denote those components of the level set by $C_i(E)$ $(i = 1,2,3)$. The overall dynamics of (2.1) can be viewed as a composition of a fast motion along the level curve $C_i(E)$ and of a slow motion normal to the energy levels with the possible transitions, for example, from $C_1(E)$ to $C_3(E)$. In this case one can eliminate the fast motion to derive an equation for the slowly-varying energy $E(t)$. It is well known (Gardiner 1996) that the fast variables can be eliminated when there exists a stationary distribution function, independent of small parameters. Let us introduce the following normalized measure corresponding to the fast motion (Freidlin 1996) along the energy-level curve $C_i(E)$

$$\rho_i(u,v) = \frac{1}{T_i(E)\,|V(u,v)|}. \tag{4.2}$$

The equation for the energy $E(t)$ can be derived as follows. Let us multiply the equation (4.1) by the measure (4.2), and integrate along the level curve $C_i(E)$ (Freidlin 1996). The equation for $E(t)$ then takes the form of a one-dimensional stochastic differential equation

$$\frac{dE}{dt} = S_i(E) - D_i(E) + \sigma_i(E)\frac{dW}{dt}, \tag{4.3}$$

where the rate of energy supply due to the noise is

$$S_i(E) = \frac{\delta}{T_i(E)} \int_{C_i(E)} \left(\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2}\right) |V(u,v)|^{-1} ds, \tag{4.4}$$

while the rate of removal of energy by dissipation can be written as

$$D_i(E) = \frac{\varepsilon}{T_i(E)} \int_{C_i(E)} \left(2\frac{\partial H}{\partial u}u + \frac{\partial H}{\partial v}v\right) |V(u,v)|^{-1} ds. \tag{4.5}$$

The intensity of noise is

$$\sigma_i^2(E) = \frac{\delta}{T_i(E)} \int_{C_i(E)} |V(u,v)| ds. \tag{4.6}$$

The details of the derivation of the above formula can be found in the book by Freidlin (1996). For very small values of ε and δ, most of the probability is concentrated on the level curves $C_i(E)$. We have in essence a deterministic motion with speed V along the level curves. In general, we have stochastically-sustained oscillations for which the energy generation $S_i(E)$ due to the noise, and the dissipation $D_i(E)$, are in balance with the stochastic term, whose intensity $\sigma_i(E)$ is a function of energy itself.

5. Conclusions and future work

In summary, we have investigated the effects of the additive Gaussian perturbations on a non-normal dynamical system mimicking laminar-to-turbulent subcritical transition both analytically and numerically. We have derived explicit representations for the second moments and found that the dynamical system with a non-normal transient linear growth is highly sensitive to the presence of weak random perturbations. We have calculated numerically the empirical exit probabilities of random trajectories from the neighborhood of a zero fixed point. We have found that even for very small values of the intensity of noise parameter ($\delta = 10^{-3}$) the exit probability is close to unity. We have also found that an increase of the intensity of noise parameter, or a decrease of the non-normality parameter, will lead to certain qualitative changes in the behavior of the trajectories. This can be interpreted as noise-induced phase transitions. By using the Itô formula and the adiabatic elimination procedure, we have derived a stochastic equation governing the slow evolution of the energy of the system.

We believe that the study of the impact of noise on non-normal dynamical systems is physically significant, since, in practical situations, random fluctuations may often be what induce the subcritical transition in fluid flow. The transition appears to become an essentially random event. The generic feature of laminar-to-turbulent transition in shear flow is that it does not have a critical, reproducible Reynolds number (see Grossman 2000). Regarding the model (2.1), it should be noted that its nonlinearity is quite different from that of the Navier-Stokes equations: therefore, it does not really describe the laminar-to-turbulent transition in fluid flow. However, it gives the general features of such transition involving transient growth, and the interaction between non-linear and stochastic modes. The stochastic dynamic system (2.1) is fundamentally different from the deterministic one (1.1) that has only two degrees of freedom. We can regard (2.1) as an *effective* dynamical system with many degrees of freedom in which two variables u and v play the role of order parameters, while the stochastic noise terms approximate other degrees of freedom and their influence on u and v. Further research is needed to identify the statistical characteristic of the noise terms in (2.1).

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