Stability for the Wall-Pierce-Moin implicit scheme

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1. Motivation and Objectives

Explicit schemes are not well suited for simulation of low Mach number flow, due to the severe restriction on the time-step. Implicit schemes on the other hand, give rise to large non-linear systems coupling all dependent variables. Iterative methods have to be used for solving these systems, but iteration to full convergence in each time-step may be time-consuming. One way to reduce the work, is to limit the number of iterations, such that the system is not fully solved, but still has a reasonable accuracy. Sometimes such a method is called semi-implicit.

In Wall *et al.* (2002) a new method for the Navier-Stokes equations was introduced. It uses a staggered grid as suggested in Pierce (2001), and it is semi-implicit in the sense mentioned above. There is a stability analysis in Wall *et al.* (2002) for the simplified case where the system has been reduced to the acoustic wave equation with constant speed of sound and periodic solutions on a uniform staggered grid. In this analysis, it is assumed that the system of equations is solved exactly in each step, i.e., the fully implicit scheme is analyzed. It is shown that the scheme is not only unconditionally stable, but also energy conserving in the sense that the Fourier modes are propagated without any change in magnitude.

The purpose with this paper is to generalize the stability analysis for the fully implicit scheme in two ways. The first generalization is that we allow for nonzero advective velocity. The second one is that we start from the full non-linear difference scheme, and linearize from there. We will assume constant coefficients, and use Fourier analysis to prove unconditional stability and energy conservation even in this case.

2. The model equation and the linearization

The simplified model problem is obtained by disregarding the viscous terms, and substitute the enthalpy equation by an equation of state $p = p(\rho)$ connecting the density ρ and the pressure p. With u denoting the advective velocity, the system of differential equations is

$$\begin{aligned} g_t + (\rho u^2)_x + p_x &= 0\,, \\ \rho_t + g_x &= 0\,, \end{aligned}$$

where $g = \rho u$ and $p = p(\rho)$. The speed of sound a is defined by $a^2 = dp/d\rho$. The system can be expressed in terms of the two variables g and ρ. With

$$U = \begin{bmatrix} g \\ \rho \end{bmatrix}, \qquad F(U) = \begin{bmatrix} g^2/\rho + p(\rho) \\ g \end{bmatrix},$$

we have

$$U_t + (F(U))_x = 0\,.$$

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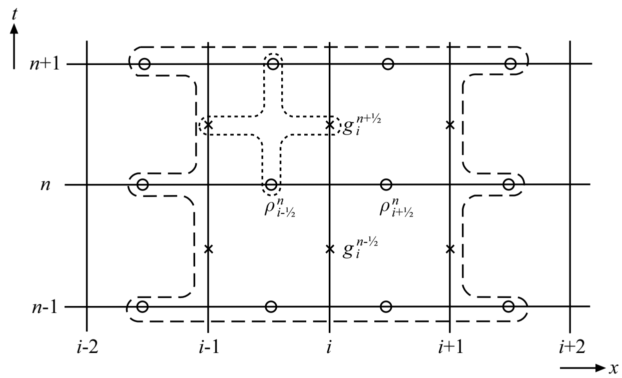

FIGURE 1. The staggered grid and the computational stencil

Let $U' = (g',\, \rho')^T$ be a perturbation around the state $U = [g,\, \rho]^T$. The linearized system is obtained as

$$U'_t + \frac{\partial F}{\partial U} U'_x = 0\,, \tag{2.1}$$

where the Jacobian matrix is given by

$$\frac{\partial F}{\partial U} = \begin{bmatrix} 2g/\rho & -g^2/\rho^2 + a^2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2u & a^2 - u^2 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $\partial F / \partial U$ are $u \pm a$.

For the discretization on the staggered grid, we use here a slightly different notation compared to Wall *et al.* (2002), such that subscripts and superscripts denote the location of the variables, see Fig 1. The notation is

$$\begin{aligned} g_i^{n+1/2} &= g(x_i,\, t_{n+1/2}) = g(i\Delta x,\, (n+1/2)\Delta t)\,, \\ \rho_{i+1/2}^n &= \rho(x_{i+1/2},\, t_n) = \rho((i+1/2)\Delta x,\, n\Delta t)\,. \end{aligned}$$

For averages, the sub/super-scripts denote the location where the average is centered. For example,

$$\overline{\rho}_i^{n+1/2} = \frac{1}{4}(\rho_{i+1/2}^{n+1} + \rho_{i-1/2}^{n+1} + \rho_{i+1/2}^n + \rho_{i-1/2}^n)$$

denotes the average centered at $(x_i,\, t_{n+1/2})$, see Fig 2.

With $\lambda = \Delta t / \Delta x$, the scheme is

$$\begin{aligned} g_i^{n+1/2} - g_i^{n-1/2} &+ \lambda(\overline{g}_{i+1/2}^n \overline{u}_{i+1/2}^n - \overline{g}_{i-1/2}^n \overline{u}_{i-1/2}^n) \\ &+ \frac{\lambda}{4}(p_{i+1/2}^{n+1} + 2p_{i+1/2}^n + p_{i+1/2}^{n-1} - p_{i-1/2}^{n+1} - 2p_{i-1/2}^n - p_{i-1/2}^{n-1}) = 0\,, \\ \rho_{i-1/2}^n - \rho_{i-1/2}^{n-1} &+ \lambda(g_i^{n-1/2} - g_{i-1}^{n-1/2}) = 0\,. \end{aligned} \tag{2.2}$$

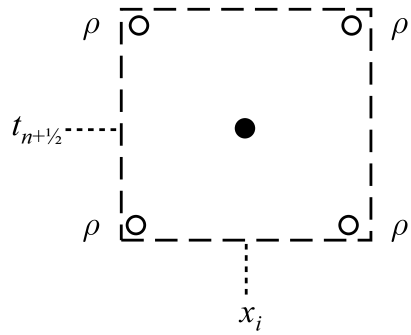

FIGURE 2. The average for ρ

Here

$$\overline{u}^n_{i+1/2} = \frac{g^{n+1/2}_{i+1}}{\rho^{n+1}_{i+3/2} + \rho^{n+1}_{i+1/2} + \rho^n_{i+3/2} + \rho^n_{i+1/2}} + \frac{g^{n+1/2}_{i}}{\rho^{n+1}_{i+1/2} + \rho^{n+1}_{i-1/2} + \rho^n_{i+1/2} + \rho^n_{i-1/2}}$$
$$+ \frac{g^{n-1/2}_{i+1}}{\rho^{n}_{i+3/2} + \rho^{n}_{i+1/2} + \rho^{n-1}_{i+3/2} + \rho^{n-1}_{i+1/2}} + \frac{g^{n-1/2}_{i}}{\rho^{n}_{i+1/2} + \rho^{n}_{i-1/2} + \rho^{n-1}_{i+1/2} + \rho^{n-1}_{i-1/2}},$$
$$\overline{g}^n_{i+1/2} = \frac{1}{4}(g^{n+1/2}_{i+1} + g^{n+1/2}_{i} + g^{n-1/2}_{i+1} + g^{n-1/2}_{i}),$$

and $p^k_j = p(\rho^k_j)$ for all j, k.

We make a few remarks concerning the method (2.2).

We first note that the second equation by itself is explicit. However, the first equation involves not only the $g^{n+1/2}$-values, but also the ρ^{n+1}-values via the p^{n+1}- and $\overline{u}^n$-expressions. This means that the scheme is fully implicit in the sense that the solution $\{g^{n+1/2}, \rho^{n+1}\}$ at the highest time level is coupled across several points in space, and depends on all the values behind.

Secondly we note that (2.2) is effectively a one-step scheme, even if the first equation couples three time levels for ρ. It is easily seen by the fact that only one level for each of the variables ρ and g are required to start the scheme. Assume that ρ^0 and $g^{1/2}$ are known. The second equation is first advanced one step providing the ρ^1-values. After that we let $n = 2$ in the second equation and couple it to the first equation with $n = 1$. This provides the ρ^2- and $g^{3/2}$-values, and the scheme is then advanced by stepping up n and repeating the last part of the algorithm.

We next turn to the linearization. As seen in Fig 1, there are 18 different variables occuring in the discrete equations, and by lining them up in a vector V, the scheme can be written in the form $G(V) = 0$. Then the linearized equations are

$$\frac{\partial G}{\partial V}(V)V' = 0\,.$$

In the evaluation of the Jacobian $\partial G/\partial V$, it is assumed that all values are independent of x and t, such that

$$g^n_i = g\;,\quad u^n_i = u\;,\quad dp/d\rho|_{i,n} = a^2$$

for all i, n. As an example, we get

$$\frac{\partial}{\partial g_{i+1}^{n+1/2}}(\overline{g}_{i+1/2}^n \overline{u}_{i+1/2}^n) = \frac{\partial \overline{g}_{i+1/2}^n}{\partial g_{i+1}^{n+1/2}} \overline{u}_{i+1/2}^n + \overline{g}_{i+1/2}^n \frac{\partial \overline{u}_{i+1/2}^n}{\partial g_{i+1}^{n+1/2}} = \frac{1}{4}u + g\frac{1}{4\rho} = \frac{u}{2}.$$

For completeness we list all the derivatives for the convection term, the other ones are simpler.

	$\overline{g}_{i+1/2}^n \overline{u}_{i+1/2}^n$	$\overline{g}_{i-1/2}^n \overline{u}_{i-1/2}^n$
$\partial/\partial g_{i+1}^{n+1/2}$	$u/2$	0
$\partial/\partial g_{i}^{n+1/2}$	$u/2$	$u/2$
$\partial/\partial g_{i-1}^{n+1/2}$	0	$u/2$
$\partial/\partial g_{i+1}^{n-1/2}$	$u/2$	0
$\partial/\partial g_{i}^{n-1/2}$	$u/2$	$u/2$
$\partial/\partial g_{i-1}^{n-1/2}$	0	$u/2$
$\partial/\partial \rho_{i+3/2}^{n+1}$	$-u^2/16$	0
$\partial/\partial \rho_{i+1/2}^{n+1}$	$-u^2/8$	$-u^2/16$
$\partial/\partial \rho_{i-1/2}^{n+1}$	$-u^2/16$	$-u^2/8$
$\partial/\partial \rho_{i-3/2}^{n+1}$	0	$-u^2/16$
$\partial/\partial \rho_{i+3/2}^{n}$	$-u^2/8$	0
$\partial/\partial \rho_{i+1/2}^{n}$	$-u^2/4$	$-u^2/8$
$\partial/\partial \rho_{i-1/2}^{n}$	$-u^2/8$	$-u^2/4$
$\partial/\partial \rho_{i-3/2}^{n}$	0	$-u^2/8$
$\partial/\partial \rho_{i+3/2}^{n-1}$	$-u^2/16$	0
$\partial/\partial \rho_{i+1/2}^{n-1}$	$-u^2/8$	$-u^2/16$
$\partial/\partial \rho_{i-1/2}^{n-1}$	$-u^2/16$	$-u^2/8$
$\partial/\partial \rho_{i-3/2}^{n-1}$	0	$-u^2/16$

We drop the $'$ notation for the new variables, and arrive at the complete linearized system

$$
\begin{aligned}
&g_i^{n+1/2} + \frac{\lambda u}{2}(g_{i+1}^{n+1/2} - g_{i-1}^{n+1/2}) \\
&\quad - \frac{\lambda u^2}{16}(\rho_{i+3/2}^{n+1} + \rho_{i+1/2}^{n+1} - \rho_{i-1/2}^{n+1} - \rho_{i-3/2}^{n+1}) + \frac{\lambda a^2}{4}(\rho_{i+1/2}^{n+1} - \rho_{i-1/2}^{n+1}) \\
&-g_i^{n-1/2} + \frac{\lambda u}{2}(g_{i+1}^{n-1/2} - g_{i-1}^{n-1/2}) \\
&\quad - \frac{\lambda u^2}{8}(\rho_{i+3/2}^{n} + \rho_{i+1/2}^{n} - \rho_{i-1/2}^{n} - \rho_{i-3/2}^{n}) + \frac{\lambda a^2}{2}(\rho_{i+1/2}^{n} - \rho_{i-1/2}^{n}) \\
&\quad - \frac{\lambda u^2}{16}(\rho_{i+3/2}^{n-1} + \rho_{i+1/2}^{n-1} - \rho_{i-1/2}^{n-1} - \rho_{i-3/2}^{n-1}) + \frac{\lambda a^2}{4}(\rho_{i+1/2}^{n-1} - \rho_{i-1/2}^{n-1}) = 0\,, \\
&\lambda(g_i^{n-1/2} - g_{i-1}^{n-1/2}) + \rho_{i-1/2}^{n} - \rho_{i-1/2}^{n-1} = 0\,.
\end{aligned}
$$

Here λ, u, a are all constants. Note that the system is consistent with the linear system (2.1).

3. Stability analysis

For the stability analysis, we Fourier transform the system by substituting $g_j^{n+1/2} = \hat{g}^{n+1/2} e^{i\omega x_j}$ etc.. With $\xi = \omega \Delta x$, we get

$$
\begin{aligned}
&(1 + \lambda u i \sin\xi)\hat{g}^{n+1/2} - \frac{\lambda u^2}{8} i(\sin\frac{\xi}{2} + \sin\frac{3\xi}{2})\hat{\rho}^{n+1} + \frac{\lambda}{2} a^2 i \sin\frac{\xi}{2}\hat{\rho}^{n+1} \\
&\quad + (-1 + \lambda u i \sin\xi)\hat{g}^{n-1/2} - \frac{\lambda u^2}{4} i(\sin\frac{\xi}{2} + \sin\frac{3\xi}{2})\hat{\rho}^{n} + \lambda a^2 i \sin\frac{\xi}{2}\hat{\rho}^{n} \\
&\quad - \frac{\lambda u^2}{8} i(\sin\frac{\xi}{2} + \sin\frac{3\xi}{2})\hat{\rho}^{n-1} + \frac{\lambda}{2} a^2 i \sin\frac{\xi}{2}\hat{\rho}^{n-1} = 0\,, \\
&2\lambda i \sin\frac{\xi}{2}\hat{g}^{n-1/2} + \hat{\rho}^{n} - \hat{\rho}^{n-1} = 0\,.
\end{aligned} \tag{3.1}
$$

Let

$$
\hat{V}^n = \begin{bmatrix} \hat{g}^{n-1/2} \\ \hat{\rho}^n \end{bmatrix}.
$$

With the notation

$$
s_1 = \sin\frac{\xi}{2}\,, \quad s_2 = \sin\xi\,, \quad s_3 = \sin\frac{3\xi}{2}\,,
$$

the system can be written in two-step form

$$
Q_2 \hat{V}^{n+1} + Q_1 \hat{V}^n + Q_0 \hat{V}^{n-1} = 0\,, \tag{3.2}
$$

where

$$
\begin{aligned}
Q_2 &= \begin{bmatrix} 1 + \lambda u i s_2 & -\frac{\lambda}{8} i[(s_1 + s_3)u^2 - 4 s_1 a^2] \\ 0 & 0 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} -1 + \lambda u i s_2 & -\frac{\lambda}{4} i[(s_1 + s_3)u^2 - 4 s_1 a^2] \\ 2\lambda i s_1 & 1 \end{bmatrix}, \\
Q_0 &= \begin{bmatrix} 0 & -\frac{\lambda}{8} i[(s_1 + s_3)u^2 - 4 s_1 a^2] \\ 0 & -1 \end{bmatrix}.
\end{aligned}
$$

Looking for the amplification factor z of this two-step scheme, we substitute $\hat{V}^n = z^n\hat{V}$, and put the determinant equal to zero. We get

$$\begin{aligned}
&\mathrm{Det}(Q_2 z^2 + Q_1 z + Q_0) \\
&\quad = \mathrm{Det}\begin{bmatrix} (1+\lambda u i s_2)z^2 + (-1+\lambda u i s_2)z & -\frac{\lambda}{8} i[(s_1+s_3)u^2 - 4s_1a^2](z^2+2z+1) \\ 2\lambda i s_1 z & z-1 \end{bmatrix} \\
&\quad = [z^2 - z + (z^2+z)\lambda u i s_2](z-1) - \frac{\lambda^2}{4} s_1 z[(s_1+s_3)u^2 - 4s_1a^2](z+1)^2 = 0\,.
\end{aligned}$$

One root is zero. The reason for this is that for convenience we have included $\hat{g}^{n-3/2}$ in the two-step form (3.2), but it is not present in the equations (3.1). (If $\hat{g}^{-3/2}$ would be given a non-zero value at the start, it would never show up in the computation.) Furthermore, since we have a one-step scheme for two variables ρ and g as explained above, there are only two remaining roots z.

In order to find these remaining roots, we substitute

$$\alpha = \lambda u s_2\,, \quad \beta = \frac{\lambda^2}{4} s_1[(s_1+s_3)u^2 - 4s_1a^2]\,,$$

and obtain

$$z^2 - 2\frac{1+\beta}{1+i\alpha-\beta}z + \frac{1-i\alpha-\beta}{1+i\alpha-\beta} = 0\,,$$

with the solutions

$$z = \frac{1+\beta \pm \sqrt{4\beta-\alpha^2}}{1+i\alpha-\beta}\,,$$

where

$$4\beta - \alpha^2 = \lambda^2[(s_1^2 + s_1s_3 - s_2^2)u^2 - 4s_1^2a^2]\,.$$

The coefficient multiplying u^2 is identically zero, which is seen by introducing the notation $\theta = \xi/2$ and expanding:

$$\begin{aligned}
s_1^2 + s_1s_3 - s_2^2 &= -\frac{1}{4}[(e^{i\theta}-e^{-i\theta})^2 + (e^{i\theta}-e^{-i\theta})(e^{3i\theta}-e^{-3i\theta}) - (e^{2i\theta}-e^{-2i\theta})^2] \\
&= -\frac{1}{4}(e^{2i\theta} - 2 + e^{-2i\theta} + e^{4i\theta} - e^{-2i\theta} - e^{2i\theta} + e^{-4i\theta} - e^{4i\theta} + 2 - e^{-4i\theta}) = 0\,.
\end{aligned}$$

Hence, $4\beta - \alpha^2 = -4\lambda^2 s_1^2 a^2 \le 0$, and

$$|z|^2 = \left|\frac{1+\beta \pm i\sqrt{\alpha^2-4\beta}}{1-\beta+i\alpha}\right|^2 = \frac{1+2\beta+\beta^2+\alpha^2-4\beta}{1-2\beta+\beta^2+\alpha^2} = 1\,.$$

The only possibility for a multiple root z is $4\beta-\alpha^2 = 0$, i.e., $\xi = 0$, but this is the trivial case with a constant solution being forwarded without any change. Hence, we have shown unconditional stability. Furthermore, we note that both roots of the characteristic equation have modulus one, which means that there is energy conservation for the periodic case.

4. Future plans

The analysis above is done for a quite simplified case, and for generalization there are several directions to go:

- Analyze the effect of not solving the equations to full convergence in each time-step (making the scheme "semi-implicit").
- Introduce non-periodic boundary conditions.
- Include also the enthalpy equation in the analysis.
- Treat the multidimensional case.

The first item is important, but very difficult if applied to the full system. An analysis of a simpler model problem could give some indications.

Regarding the non-periodic boundary conditions, there is first of all a question of how to impose the boundary conditions for the full system of differential equations. For solid walls, the conditions are given on physical grounds, but for open boundaries there are still many difficulties to overcome. Furthermore, the method analyzed here has a quite wide computational stencil as illustrated by Fig 1. Therefore, extra numerical boundary conditions must be constructed in order to get the algebraic system well defined at each time level. Stability has to be secured, but in addition, a special challenge here would be to keep energy conservation in the case this holds for the system of differential equations.

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