A space-time correlation theory for turbulent shear flows

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1. Motivations and objectives

The objective of this research is to investigate the large-eddy simulation (LES) prediction of velocity space-time correlations in turbulent shear flows. In the application of LES methodology to aeroacoustics (e. g. Wang & Moin 2000; Colonius & Lele 2005), Lighthill's acoustic analogy (Lighthill 1952) is often used to compute the far-field noise. According to Lighthill's theory, the acoustic intensity radiated by a turbulent flow depends on the space-time correlations of the Lighthill stresses, which are related to the space-time correlations of velocity fluctuations based on Proudman's (1952) analysis. Our previous research (He, Rubinstein & Wang 2002; He, Wang & Lele 2004) shows that the space-time correlations in isotropic turbulence are determined by the instantaneous wavenumber energy spectrum and the sweeping velocity; the latter is the root mean square of total energy. Therefore, if LES with an appropriate SGS model captures the time evolution of the energy spectrum, the space-time correlation, and hence noise, is expected to be computed correctly. However, realistic flows are not isotropic and homogeneous. In turbulent shear flows, the space-time correlations are no longer dominated by the sweeping velocity. Thus, an accurate prediction of the energy spectrum is not the only requirement for SGS modeling. The convection and shear may cause additional effects on space-time correlations, which must be accounted for by the SGS models.

In this study, we investigate the space-time correlations in turbulent shear flows. An elliptic hypothesis is proposed to model the space-time correlations in the streamwise direction. It is shown that Taylor's hypothesis is a linear approximation to the correlation contours, whereas the elliptic hypothesis is a second order approximation to the contours. An analytical expression for space-time correlations is formulated from the elliptic hypothesis, which relates the space-time correlations to two-point spatial correlations and propagation velocities. The propagation velocities are neither the sweeping velocity nor the mean velocity in general. The results will be used to guide a subsequent computation aimed at elucidating the SGS modeling effect on space-time correlations in turbulent shear flows.

2. Main results

The space-time correlation in a statistically stationary and homogeneous flow is defined as

$$R(r,\tau) = \langle u_i(\mathbf{x},t)u_i(\mathbf{x}+\mathbf{r},t+\tau)\rangle, \tag{2.1}$$

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where u_i is a component of velocity fluctuation $\mathbf{u}$ and τ is the temporal separation. No summation over repeated indices is implied unless otherwise indicated. For notational simplicity, the index on the correlation function R has been dropped. In the present analysis, we limit spatial separation to the streamwise direction: $\mathbf{r} = (r, 0, 0)$, in which turbulence is statistically homogeneous. In previous studies of the space-time correlations, velocity fluctuations are frequently considered to be convected by a propagation velocity V such that

$$R(r, \tau) = R(r - V\tau, 0). \tag{2.2}$$

If the fluctuations are convected in frozen patterns, the propagation velocity is the mean velocity. This is the result from Taylor's well-known hypothesis. Eq. (2.2) also implies that

$$R(r, \tau) = \int E(k) \cos[k(r - V\tau)]dk, \tag{2.3}$$

which indicates that the space-time correlation is determined by the wavenumber energy spectrum $E(k)$ and propagation velocity. Therefore, if LES can predict both the wavenumber energy spectrum and propagation velocity, it can predict the space-time correlations. However, Taylor's hypothesis is a rather crude approximation and may fail in flows with a strong shear (Lumley 1965). It is desirable to derive a more general and accurate model for the space-time correlations in turbulent shear flows.

Consider an iso-correlation contour $R(r, \tau) = C$, where C is constant. If we can find a point $(r_c, 0)$ on the contour, it implies that $R(r, \tau) = R(r_c, 0)$. Therefore, the space-time correlations are determined by the spatial correlation $R(r, 0)$ and the solution of the equation $R(r, \tau) = R(r_c, 0)$. Both numerical simulations (Kim & Hussain 1993) and experiments (Wills 1964) have shown that in turbulent shear flows the correlation contours form closed curves with a single peak at the origin $(r, \tau) = (0, 0)$. The contours decrease in value with increasing r or τ, and decay to zero as r or τ goes to infinite. This ensures the existence of a solution to the equation $R(r, \tau) = R(r_c, 0)$. The most prominent feature of the contours is that they have a preferred direction.

The correlation contours can be reasonably approximated by second order algebraic curves. A higher-order approximation can be introduced if necessary. Based on the experimental and numerical observations, we propose an elliptic approximation, i.e. the contours of space-time correlations are ellipses. The ellipses can be obtained by Taylor's expansion of the space-time contours $R(r, \tau) = C$

$$R_{rr}r^2 + 2R_{r\tau}r\tau + R_{\tau\tau}\tau^2 = C, \tag{2.4}$$

where the subscripts denote derivatives evaluated at $r = 0$ and $\tau = 0$. Note that $R_\tau(0, 0) = R_r(0, 0) = 0$ because the flow is statistically stationary and homogeneous in the direction of separation. $R(0, 0)$ has been absorbed into C. Suppose that (r, τ) and $(r_c, 0)$ are the two points on the ellipse, that is $R(r, \tau) = R(r_c, 0)$, we obtain

$$r_c = \sqrt{r(r - 2V_l\tau) + V_t^2\tau^2}, \tag{2.5}$$

where

$$V_l = -R_{r\tau}(R_{rr})^{-1}, \tag{2.6}$$
$$V_t^2 = R_{\tau\tau}(R_{rr})^{-1}. \tag{2.7}$$

Here V_l is the same propagation velocity defined by Wills (1964). V_t also has the dimension of velocity and may thus be considered as a propagation velocities. As a result, we have

$$R(r,\tau) = R\left(\sqrt{r(r-2V_l\tau)+V_t^2\tau^2},0\right) \tag{2.8}$$

Eq. (2.8) implies that the space-time correlations in a turbulent shear flow can be determined from the spatial correlations and appropriately defined propagation velocities.

In the degenerate case of $R_{rr}R_{\tau\tau} - R_{r\tau}^2 = 0$, (2.5) can be simplified to

$$r_c = r - V_l\tau, \tag{2.9}$$

which is Taylor's hypothesis. It is now clear that Taylor's hypothesis is a linear approximation to the correlation contours, which is the degenerate case of the elliptic hypothesis. If there exists a dominant propagation velocity, the ellipse is very flat in the direction normal to the propagation direction. This means that the ratio of two principal axes of the ellipse, $R_{rr}R_{\tau\tau} - R_{r\tau}^2$, is negligibly small. Thus, Taylor's hypothesis is a good approximation.

The two velocities in (2.5) can be analytically calculated from the Navier-Stokes equations using the Taylor expansion technique (Gotoh & Kaneda 1991; Kaneda 1993). In the following, we derive the first propagation velocity as an example. We start with the incompressible Navier-stokes equations

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v},\\
&\nabla\cdot\mathbf{v} = 0,
\end{aligned} \tag{2.10}$$

where ν is the kinematic viscosity and ρ is the density, hencefore absorbed into pressure. The velocity $\mathbf{v}$ and pressure p are decomposed into their mean parts, $\mathbf{U}$ and P, and their fluctuation parts, $\mathbf{u}$ and p', such that

$$\begin{aligned}
\mathbf{v} &= \mathbf{U} + \mathbf{u},\\
p &= P + p'.
\end{aligned} \tag{2.11}$$

For simplicity, we consider parallel flows whose mean velocity has the form $\mathbf{U} = (U_1(x_2), 0, 0)$. The velocity and pressure fluctuations must then satisfy

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= -U_1(x_2)\frac{\partial u_i}{\partial x_1} - u_2\frac{\partial U_1(x_2)}{\partial x_2}\delta_{i1} - u_j\frac{\partial u_i}{\partial x_j} + \left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle - \frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j\partial x_j},\\
p' &= -\nabla^{-2}\frac{\partial^2}{\partial x_i\partial x_j}\left[U_1\delta_{i1}u_j + U_1\delta_{j1}u_i + u_iu_j - \langle u_iu_j\rangle\right].
\end{aligned} \tag{2.12}$$

Note that in (2.12), the summation convention over repeated indices applies. From the definition of the space-time correlation, we can calculate

$$R_{rr} = \left\langle u_i(\mathbf{x},t)\frac{\partial^2 u_i(\mathbf{x}+\mathbf{r},t+\tau)}{\partial r^2}\right\rangle_{r=\tau=0} = -\left\langle \frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\right\rangle \tag{2.13}$$

$$R_{r\tau} = \left\langle u_i(\mathbf{x},t)\frac{\partial^2 u_i(\mathbf{x}+\mathbf{r},t+\tau)}{\partial r\partial \tau}\right\rangle_{r=\tau=0} = -\left\langle \frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\frac{\partial u_i(\mathbf{x},t)}{\partial t}\right\rangle \tag{2.14}$$

Substituting (2.12) into (2.14) leads to

$$R_{r\tau} = U_1(x_2)\left\langle \frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\right\rangle + \frac{\partial U_1(x_2)}{\partial x_2}\delta_{i1}\left\langle u_2(\mathbf{x},t)\frac{\partial u_1(\mathbf{x},t)}{\partial x_1}\right\rangle$$

$$-\frac{1}{2\pi}\iiint \frac{\partial U_1(y_2)}{\partial y_2}\left\langle \frac{\partial u_2(\mathbf{y},t)}{\partial y_1}\frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\right\rangle \frac{y_i-x_i}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} \tag{2.15}$$

where the velocity fluctuations are assumed to be homogeneous and their triple correlations and the viscous terms are ignored.

Therefore, the first propagation velocity is

$$V_l = U_1(x_2) + \frac{\partial U_1(x_2)}{\partial x_2}\delta_{i1}\left\langle u_2\frac{\partial u_1}{\partial x_1}\right\rangle\left\langle \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_1}\right\rangle^{-1}$$
$$-\frac{1}{2\pi}\iiint \frac{\partial U_1(y_2)}{\partial y_2}\left\langle \frac{\partial u_2(\mathbf{y},t)}{\partial y_1}\frac{\partial u_i(\mathbf{x},t)}{\partial x_1}\right\rangle \frac{y_i-x_i}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\left\langle \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_1}\right\rangle^{-1} \tag{2.16}$$

Expression (2.16) shows that the propagation velocity V_l is dominated by the mean velocity if the shear rate is relatively small. However, when the shear rate is large, the contribution from the second term can be significant. This observation is in agreement with the numerical results in channel flows obtained by Kim and Hussain (1993): the propagation velocity for most of the outer layer is essentially identical to the local mean velocity, whereas the propagation velocity in the near-wall region is about one half of the mean velocity at the center of the channel, which is larger than the local mean velocity. Therefore, the space-time correlations in turbulent shear flows present additional requirements to LES beyond the wavenumber energy spectra.

3. Future work

An elliptic approximation for space-time correlations in turbulent shear flows has been proposed. According to this approximation, the space-time correlations of velocity fluctuations are determined by the two-point spatial correlations and the propagation velocities. It is also shown that Taylor's hypothesis is a linear approximation to the space-time correlations. To accurately predict the space-time correlations, one must accurately predict the spatial correlations and the propagation velocities.

Future work will include the numerical verification of the elliptic hypothesis in turbulent shear flows. The space-time correlations will be evaluated in a channel flow using DNS and LES with different SGS models. The results obtained will be analyzed in terms of propagation velocities. These fluctuating quantities to be analyzed include velocity, pressure and the Lighthill stress tensor.

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