On mass conservation and desingularization of the Level Set/Vortex Sheet method

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1. Motivation and objectives

The Level Set/Vortex Sheet (LSVS) method has been introduced by Herrmann (2002, 2003a, 2004a,b). Its objective is to provide a framework for the derivation of the so-called Large Surface Structure (LSS) model (Herrmann 2003b) that describes the primary breakup of turbulent liquid jets and sheets. The advantage of the LSVS method as compared to other, more traditional approaches is the fact that it contains explicit local source terms for each individual physical process that occurs at the phase interface, thus making these directly accessible to modeling attempts of the LSS subgrid terms.

As has been argued by Yecko et al. (2002) and Li et al. (2004), viscous effects might play an important role during primary breakup. The LSVS method, on the other hand, has been derived theoretically for the limit of inviscid fluids. However, its numerical implementation necessitates the introduction of a desingularization of the governing equations by introducing shear layers of finite width (Herrmann 2004a), thus imitating some effects of viscous fluids to a certain extend. This effect shall be analyzed in this paper.

Furthermore, as with any level set based method, the problem of mass conservation has to be addressed. To this end, different level set correction methods have already been proposed (Bourlioux 1995; Sussman & Fatemi 1999; Enright et al. 2002). Unfortunately, within the scope of the LSVS method, these correction methods lead to unacceptable fluctuations in the surface tension term due to the fact that all corrections are performed locally (Herrmann 2004a). On this account, a method to de-localize the correction methods has been proposed by Coyajee et al. (2004), resulting in significant, but unfortunately insufficient improvements with respect to the LSVS method. Hence, an alternative and rather simple and straightforward approach is proposed here, named refined level set grid (RLSG) method.

This paper is divided into three parts. First, the governing equations of the LSVS method for three-dimensional two-phase interface dynamics are summarized. Also, the numerical methods employed to solve the LSVS equations and the RLSG method and its implications within the LSVS method are presented. Second, numerical results are presented addressing both the performance of the RLSG method and the effect of desingularization inherent in the LSVS method. Finally, conclusions are drawn and an outlook to future work is given.

2. The Level Set/Vortex Sheet method

The LSVS method describes the dynamics of the phase interface $\Gamma$ between two inviscid, incompressible fluids 1 and 2, as shown in Fig. 1. Defining the iso-surface of a level set scalar $G = 0$ to be the location of the phase interface, the motion of $\Gamma$ can be tracked...
by the so called level set equation (Osher & Sethian 1988),
\[
}\frac{\partial G}{\partial t} + \mathbf{u} \cdot \nabla G = 0,
\]
where \( \mathbf{u} \) is the velocity vector.

The velocity field generated by the phase interface can be described by a vortex sheet strength \( \eta \), which is equal to the jump in tangential velocity at the phase interface. The transport equation for \( \eta \) (Pozrikidis 2000; Herrmann 2003b) is given by
\[
\frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta = -\mathbf{n} \times [(\eta \times \mathbf{n}) \cdot \nabla \mathbf{u}] + \mathbf{n} \left[ (\nabla \mathbf{u} \cdot \mathbf{n}) \cdot \eta \right]
\]
\[
+ \frac{2(A+1)}{\text{We}} (\mathbf{n} \times \nabla \kappa) + 2A \mathbf{n} \times \mathbf{a}.
\]

Here, \( \mathbf{n} \) is the interface normal vector, \( A \) the Atwood number, \( \text{We} \) the Weber number, \( \kappa \) the interface curvature, and \( \mathbf{a} \) is the average acceleration of fluid 1 and fluid 2 at the interface. Note that Eq. (2.2) contains on the right hand side individual, local source terms describing the physical processes at the phase interface explicitly, namely, from left to right, two stretching terms, a surface tension term, and a density difference term.

Strictly speaking, Eqs. (2.1) and (2.2) are valid only at the location of the interface itself. However, to facilitate the numerical solution of both equations throughout the whole computational domain, \( \eta \) is set constant in the interface normal direction,
\[
\nabla \eta \cdot \nabla G = 0,
\]
and \( G \) is chosen to be a distance function away from the interface,
\[
\left| \nabla G \right|_{G \neq G_0} = 1.
\]

Equations (2.1) and (2.2) are coupled by the self-induced velocity \( \mathbf{u} \) of the vortex sheet. To calculate \( \mathbf{u} \), the vector potential \( \psi \) is introduced,
\[
\Delta \psi = \omega.
\]

Here, the vorticity vector \( \omega \) is calculated following a vortex-in-cell type approach (Christiansen 1973; Cottet & Koumoutsakos 2000) by either
\[
\omega(x) = \int_{\Gamma} \eta(x_{\Gamma}) \delta(x - x_{\Gamma}) dx_{\Gamma},
\]
named method $\mathcal{M}_2$ in the following, or
\[
\omega(x) = \int_V \eta(x') \delta(x - x') \delta(G(x') - G_0) |\nabla G(x')| dx',
\] (2.7)
named method $\mathcal{M}_3$. In general, method $\mathcal{M}_2$ is preferable, because Eq. (2.6) ensures vorticity conservation (Herrmann 2004a). However, $\mathcal{M}_2$ necessitates the geometric reconstruction of the interface location and is thus very tedious in three dimensions. Therefore, method $\mathcal{M}_2$ is used in two-dimensions, whereas $\mathcal{M}_3$ is the method of choice in three-dimensions.

In Eqs. (2.6) and (2.7), $\delta$ is the delta-function that is approximated by the following smoothed version (Peskin 1977),
\[
\delta_{\varepsilon}(x) = \begin{cases} 
\frac{1}{2\varepsilon} & \left[ 1 + \cos \left( \frac{\pi x}{\varepsilon} \right) \right] : |x| \leq \varepsilon \\
0 & : |x| > \varepsilon 
\end{cases}
\] (2.8)

Note that this in fact changes the tangential velocity boundary condition at the phase interface from a jump, as appropriate for inviscid fluids, to a smoothed, constant shear layer thickness type boundary condition, reminiscent of viscous fluids. It is in principle possible to recapture the shear layer thickness and profile of viscous fluids by modifying $\varepsilon$ and the functional form of Eq. (2.8), however, in the present work, Eq. (2.8) is used as shown and only the effect of varying the shear layer thickness is analyzed.

Finally, $u$ can be calculated from
\[
u(x) = \int_V \delta(x - x') (\nabla \times \psi) dx'.
\] (2.9)

In summary, Eqs. (2.1), (2.2), and (2.5) - (2.9) constitute the LSVS method and describe the three-dimensional two-phase interface dynamics.

2.1. Numerical methods

Numerically, Eqs. (2.1) and (2.2) are solved in a narrow band (Peng et al. 1999) by a 5th-order WENO scheme (Jiang & Peng 2000) using a 3rd-order TVD Runge-Kutta time discretization (Shu & Osher 1989). The reinitialization of $G$ (2.4) is solved by the iterative procedure outlined in Sussman et al. (1994) and Peng et al. (1999). The redistribution of $\eta$, Eq. (2.3), is solved by a Fast Marching Method (Sethian 1996; Adalsteinsson & Sethian 1999; Herrmann 2003a). The interested reader is referred to Herrmann (2002, 2003a, 2004a) for a detailed description of the numerical methods employed in the level set/vortex sheet method and a summary of the domain decomposition parallelization approach used.

2.1.1. Refined Level Set Grid (RLSG) method

Tracking interfaces by a standard level set approach (Osher & Sethian 1988) unavoidably introduces volume, respectively mass errors that are proportional in size to the employed numerical grid size. To avoid these errors, two different approaches can be followed in principle. One can correct the level set solution using an interface tracking method that either inherently preserves the volume, as for example the volume of fluid method (Noh & Woodward 1976; Kothe & Rider 1995; Gueyffier et al. 1999), or at least preserves the volume with higher accuracy than the level set method alone, like for example marker particles (Brackbill et al. 1988; Rider & Kothe 1995; Unverdi & Tryggvason 1992). Correction methods along these lines have been proposed by Bourlioux (1995),
Sussman & Fatemi (1999), Enright et al. (2002), and van der Pijl et al. (2004). Alternatively, one can reduce the mass error by refining the underlying numerical grid. This can be done adaptively (Ham & Young 2003). However, this introduces an additional level of complexity that is not necessary in certain cases. Since the goal of the LSVS method is to perform three-dimensional DNS of the primary breakup process, it can be assumed that a refined grid is necessary in large portions of the phase interface. The grid on which the level set equation is solved shall thus be refined as a whole. Note that due to the narrow band implementation, this does not constitute a prohibitive numerical cost, since the $G$-grid needs only be stored in a small neighborhood of the $G = 0$ iso-surface, see Fig. 2. This refined level set grid (RLSG) method is described in more detail in the following.

Let $\Delta x$ be the cell size of the equidistant Cartesian grid on which the $\eta$-equation, Eq. (2.2), is solved. This grid is termed the $\eta$-grid in the following. The level set equation, Eq. (2.1), is then solved on a narrow band consisting of equidistant Cartesian grid cells of size $\Delta_G x$ width

$$\Delta_G x = \Delta x / n_G, \quad (2.10)$$

where $n_G$ is the grid refinement factor. The width of the narrow band $\alpha_{T,G}$ is chosen in such a way that enough cells are present to allow for the evaluation of the 5th-order WNENO stencil during a single CFL-limited time step on the $\eta$-grid. To fulfill the CFL-criterion on the $G$-grid, subcycling typically has to be employed. This results in a width of the narrow band of

$$\alpha_{T,G} = \begin{cases} 
9\Delta_G x & : n_G \leq 4 \\
(1.5n_G + 3)\Delta_G x & : n_G > 4 
\end{cases}, \quad (2.11)$$

see Fig. 2. All other narrow band widths described in Herrmann (2003a, 2004a) are defined accordingly.

The coupling of the $\eta$-grid and the $G$-grid is two-fold: for one, the level set scalar field solved on the finer $G$-grid has to be transferred to the $\eta$-grid. Let $G_G$ be the level set scalar defined on the $G$-grid and $G_\eta$ be the level set scalar defined on the $\eta$-grid. Then, remembering that any $G$ is defined as the distance function away from the interface, the embedded interface $G_G = 0$ can be viewed as a higher order approximation of the interface as defined by $G_\eta = 0$. This implies that at the same node location, the value of $G_\eta$ should be exactly equal to the value of $G_G$, since both values describe the distance to the same interface geometry. It is important to note that this coupling does not constitute a filtering operation of $G_G$ to $G_\eta$. To do this, a more complex marker particle based scheme would have to be employed (Oberlack et al. 2001; Pitsch 2002). Here, the goal is rather to make use of a higher order approximation of the interface to eliminate numerical errors on the coarser grid.

In practice, $G_\eta$ is determined from $G_G$ on all $\eta$-grid nodes that are directly adjacent to the $G_\eta = 0$ interface. All other $G_\eta$ values up to a certain distance away from the interface are then reconstructed using the Fast Marching Method.

Secondly, the velocity $u$ is initially only defined on the $\eta$-grid. To solve Eq. (2.1), $u$ has to be transferred to the $G$-grid. This is done by simple trilinear interpolation.

Additionally, by solving the level set equation separate from the $\eta$-equation on a refined grid, the RLSG method allows for a different approach in calculating the source terms in Eq. (2.2). These can still be evaluated using $G_\eta$ (Herrmann 2004a). However, to make full use of the available geometry information on the $G$-grid, these source terms $S(x_\eta)$ defined on the $\eta$-grid can now also be evaluated using $G_G$ on the $G$-grid and then surface averaged onto the $\eta$-grid. This process is a three-step procedure: first, all source
terms on the right-hand side of Eq. (2.2) are evaluated on the $G$-grid. Then, these terms are redistributed in the interface normal direction by solving Eq. (2.3) using the Fast Marching Method. Finally, the surface integration is performed by evaluating

$$
S(x_\eta) = \frac{\int_{\Delta \Gamma_\eta} S(x_{\Gamma_G}) d\Gamma_G}{\int_{\Delta \Gamma_\eta} d\Gamma_G} = \frac{\int_{\Delta V_\eta} S(x_G) \delta(G_G) |\nabla G_G(x_G)| dx_G}{\int_{\Delta V_\eta} \delta(G_G) |\nabla G_G(x_G)| dx_G},
$$

(2.12)

where $\Delta \Gamma_\eta$ is the part of the $G_G = 0$ interface that lies within the $\eta$-cell located at $x_\eta$ and $\Delta V_\eta$ is the volume of that $\eta$-grid cell. The integration above is performed on the $G$-grid.

Using this procedure also avoids one potential pitfall when evaluating the surface tension term. Using finite differences, this term involves a stencil that extends at least three cells in the front normal direction. Thus, as soon as two fronts approach each other closer than six grid cells, the results for the surface tension term will be incorrectly influenced due to the single valued nature of $G$. Ideally the surface tension term should depend only on the position of the interface, requiring a geometric reconstruction of the interface location which is tedious in three dimensions.

To alleviate the stencil problem, the calculation of the surface tension term is split into three steps. First, the curvature $\kappa$ is calculated on the $G$-grid using a standard 3x3 stencil. Then, $\kappa$ is redistributed on the $G$-grid using the Fast Marching Method. Finally, $\nabla \times \kappa$ is evaluated using central differences. The intermediate FMM step effectively limits the stencil size of the surface tension term to just two cells in the interface normal direction, thereby improving the results considerably.

Note that using a $G$-grid of $n_G \geq 4$, interfaces can now approach each other up to $\Delta x$, before the stencil problem occurs. Modifying the surface averaging step in such a way, that $G$-nodes with two fronts closer than $4\Delta_G x$ are rejected in the averaging procedure, will then avoid the stencil problem altogether. The derivation of such an averaging procedure will be addressed in future work.

### 3. Results

#### 3.1. Zalesak's disk

The solid body rotation of a notched circle, also known as Zalesak's disk (Zalesak 1979), is one of the standard test problems for evaluating the accuracy of level set methods. A disk of radius 0.15, notch width 0.05, and notch height 0.25 is placed in a $1 \times 1$ box at $(0.5, 0.75)$. The velocity field is given by

$$
u(x,t) = (0.5 - y, x - 0.5).$$

(3.1)

Figure 3 shows the shape of the interface at $t = 2\pi$ after one full rotation of the disk using no correction scheme, the particle correction method, and the RLSG method with varying $n_G$. Obviously, using no correction method at all causes the notch height to decrease substantially and the lower sharp corners to become significantly rounded. This in turn increases the area $A/A_0$ of the lower of the disk and decreases the length of the interface $s/s_0$ considerably, as shown in Fig. 4.

Using the particle correction method improves the results significantly. However a slight asymmetry occurs. The area of the disk decreases slightly, see Tab. 1, while showing noticeable fluctuations over time, Fig. 4. These are due to the local, non-continuous correction step of the particle correction method.
Figure 3. Interface shape after one full rotation of Zalesak’s disk. Solid line denotes numerical solution and dash-dotted line is exact solution. From left to right: no correction method, particle correction method, RLSG method \( n_G = 2 \), \( n_G = 4 \), and \( n_G = 8 \).

Figure 4. Normalized area \( A/A_0 \) (left) and interface length \( s/s_0 \) (right) during one full rotation of Zalesak’s disk. Particle correction method (solid line), no correction method (open box), RLSG method \( n_G = 2 \) (open circle), \( n_G = 4 \) (full box).

<table>
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<th>RLSG ( n_G = 4 )</th>
<th>RLSG ( n_G = 8 )</th>
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</table>

Table 1. Normalize disk area \( A/A_0 \) and interface length \( s/s_0 \) after one full rotation of Zalesak’s disk

Employing the RLSG method with \( n_G = 2 \) results in markedly improved results as compared to using no correction method at all. The area of the disk is preserved better than in the case of the particle correction method. However, this is due to two errors canceling each other: one at the sharp corners leading to an area decrease and the other at the top of the notch leading to an area increase, see Fig. 3. This cancelation of errors results in an area decrease of only 0.28%. The total length of the interface, on the other hand decreases by about 2.7%.

Successively refining the \( G \)-grid continues to improve these results. The interface shape obtained with \( n_G = 4 \) is already superior to that of the particle correction method. For \( n_G = 8 \) almost no difference between the exact solution and the numerical result can be
discerned, see Fig. 3, and 99.99% of the disk’s area and 99.4% of the interface length is preserved, see Tab. 1.

The results of this test case indicate, that the RLSG method with $n_G \geq 4$ performs comparable, if not superior, to the particle correction method with respect to area preservation, while maintaining symmetry and avoiding any fluctuations introduced by the correction step of the particle correction method.

3.2. Vortex sheet roll-up

Pure vortex sheets represent a special class of phase interfaces in that no surface tension forces exist. Thus, they constitute an ill posed problem, since linear theory predicts that the growth rate of a sinusoidal disturbance of wave number $k$ is proportional to $k$. Consequently, as pointed out by Moore (1979), a vortex sheet develops a singularity at some critical time $t_c$, typically a discontinuity in curvature. In order to calculate the behavior of the vortex sheet beyond $t_c$, some form of desingularization has to be introduced, either by replacing the exact equations by desingularized versions, see for example the vortex blob method (Krasny 1986), or by adding physical effects like surface tension forces (Pullin 1982) or viscosity (Tryggvason et al. 1991).

In the case of the LSVS method, desingularization is two-fold. For one, all transport equations are solved by a finite difference scheme on an underlying numerical grid. Thus, all derivatives of finite quantities are inherently limited by the employed grid spacing. Secondly, the delta functions in Eqs. (2.6) and (2.7) are approximated by the numerical delta function $\delta_\varepsilon$ of finite width $\varepsilon$, Eq. (2.8). Thus, vorticity is not solely located at the interface, but spread out to neighboring grid nodes. To some extent, this crudely mimics the effect of viscosity (Tryggvason et al. 1991), since the shear layer is not of zero thickness as in the inviscid case, but rather has a finite, constant thickness proportional to $\varepsilon$. However, if the extend of the desingularization is reduced, i.e. the grid is refined and $\varepsilon$ reduced, the LSVS method should be able to reproduce the Moore singularity at $t_c$.

Furthermore, after the critical time, the vortex sheet rolls up in the inner core region in such a way, that adjacent interfaces are tightly packed, see Fig. 5, requiring high fidelity level set solution algorithms, i.e. level set correction methods.

In the following, first, the performance of the different level set correction methods beyond the critical time is analyzed. Then, the LSVS method in the non-desingularized limit is verified. Finally, the influence of varying the shear layer thickness is ascertained.
3.2.1. Influence of level set correction methods

The vortex sheet roll-up is initiated by placing a sinusoidal interface

\[ G(x, t = 0) = y - A_0 \sin \left( \frac{2\pi}{B} \left( x - A_0 \sin \frac{2\pi}{B} \right) \right), \]

with amplitude \( A_0 = 0.01 \), \( B = 1 \), and vortex sheet strength

\[ \eta_{VS}(x, t = 0) = \frac{\eta^*}{\sqrt{1 + \frac{4\pi A_0}{B} \cos \frac{2\pi}{B} x + 2 \left[ \frac{2\pi A_0}{B} \cos \frac{2\pi}{B} x \right]^2}}, \]

with \( \eta^* = -1 \), into a box of size \( B \times B \), resolved by an \( \eta \)-grid of 256 \( \times \) 256 equidistant Cartesian grid cells. Periodic boundary conditions are employed at the left and right boundaries, and no-slip walls are used at the bottom and top boundaries, with the tangential wall velocity set to \( u_w = \mp 0.5 \). In all cases, \( \varepsilon = 16/256 \). Simulations were performed using no correction method at all, the particle correction method, and the RLSG method with \( n_G = 2, 4 \), and 8.

Figure 5 shows the interface shape at \( t = 3 \) obtained by using no correction method and the RLSG method with \( n_G = 8 \). As discussed in Herrmann (2004a), numerical diffusion and the incorrect merging of characteristics leads to significantly less turns in the inner core region where these errors are dominant, when using no correction method as compared to both the results by Krasny (1986) and those obtained using a correction method.

To evaluate the performance of the RLSG method as compared to the particle correction method, Fig. 6 shows a zoom of the inner core region of the vortex sheet at \( t = 3 \). Figure 7 depicts the interface curvature \( \kappa \) along the normalized interface arc length \( s/L \), and Fig. 8 shows the tangential derivative of the interface curvature, \( \partial_\theta \kappa \), i.e. the term proportional to the surface tension term in Eq. (2.2). Looking first at the interface shape, using any of the two correction methods yields very similar good results as compared to using no correction method at all. On close inspection, however, it can be seen that the RLSG method with \( n_G = 2 \) and \( n_G = 4 \) generates slightly less respectively more turns than the particle correction method. Using the RLSG method with \( n_G = 8 \) leads to results that are virtually indistinguishable from those obtained with \( n_G = 4 \), demonstrating grid convergence with respect to the \( G \)-grid.

The analysis of the interface curvature distribution reveals the drawbacks of the particle correction method. As discussed in Herrmann (2004a) and Coyajee et al. (2004), a correction method that corrects the level set scalar locally, like the particle correction method, will introduce fluctuations in the higher derivatives of \( G \). As can be seen in Fig.
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Figure 7. Interface curvature $\kappa$ for vortex sheet roll-up at $t = 3$, particle correction method (top left) (Herrmann 2004a), RLSG method with $n_G = 2$ (top right), $n_G = 4$ (bottom left), and $n_G = 8$ (bottom right).

Figure 8. Tangential curvature derivative $\partial_r \kappa$ for vortex sheet roll-up at $t = 3$, particle correction method (top left), RLSG method with $n_G = 2$ (top right), $n_G = 4$ (bottom left), and $n_G = 8$ (bottom right).

8 this would lead to unacceptable fluctuations in the surface tension term, making any local correction method inapplicable within the context of the LSVS method. The de-localization method developed in Coyajee et al. (2004) could help alleviate this problem and will thus be further analyzed in the future. The RLSG method, on the other hand, delivers smooth, virtually non-fluctuating distributions of both the curvature, and more importantly, the tangential curvature derivative, making it ideal within the context of the LSVS method. The variations in both $\kappa$ and $\partial_r \kappa$ are due to the fact that the vortex sheet is slightly elongated, see Fig. 5, resulting in the shown distributions. Also, the results for $n_G = 4$ and $n_G = 8$ show virtually identical distributions of both $\kappa$ and $\partial_r \kappa$, demonstrating again, grid convergence, even for these higher derivatives of $G$.

3.2.2. Moore singularity

The initial conditions used to recover the Moore singularity are those proposed by Meiron et al. (1982). Initially, the vortex sheet is flat and located at $y = 0$ inside a box of size $[0, 2\pi] \times [-\pi, \pi]$ with a sinusoidal disturbance of the vortex sheet strength, $\eta = 1 + a \cos x$ and $a = 0.125$. With these initial conditions, the predicted critical time is $t_c \approx 2.84$ (Shelley 1992). Three calculations were performed, successively reducing the desingularization, i.e. the grid spacing and the spreading parameter, from $\Delta x = 2\pi/256$ with $\varepsilon = 16/256$, to $\Delta x = 2\pi/512$ with $\varepsilon = 8/256$, and to $\Delta x = 2\pi/1536$ with $\varepsilon = 6/256$. Since the amplitude of the disturbance throughout the simulations remains relatively small, no level set correction method has been used.
Figure 9. Vortex sheet curvature (left) and curvature derivative (right) at times close to the critical time, $t = 3.0$ (left) and $t = 3.2$ (right) for $\varepsilon = 16/256$ (dashed), $\varepsilon = 8/256$ (dotted), and $\varepsilon = 6/256$ (solid).

Figure 10. Estimation of the critical time $t_c$.

Figure 9 shows the distribution of curvature and curvature derivative, $\partial_s \kappa$, along the normalized interface arc length $s/L$ at two different times close to the critical time $t_c$. Both the curvature and the curvature derivative distributions clearly show the onset of the Moore singularity at $s/L = 0.5$. Decreasing the desingularization obviously increases the curvature derivative at any given time. However, the $\varepsilon = 6/256$ case shows some small fluctuations of $\partial_s \kappa$ not present in the other two cases. This is due to the fact that a decrease of $\varepsilon$ increases the growth rate of higher wave number fluctuations (Krasny 1986). The initial amplitudes of these higher wave number disturbances are due to the numerical error associated with the employed numerical grid. Obviously, the $1536 \times 1536$ grid is barely sufficient to delay the growth of these higher wave number disturbances. A further reduction of $\varepsilon$ would thus require a substantially finer grid. Furthermore, at $t = 3.2$ the $\partial_s \kappa$ distribution for both $\varepsilon = 8/256$ and $\varepsilon = 6/256$ exhibits two local minima directly adjacent to the central maximum. This behavior is consistent with that reported by Shelley (1992).

To determine $t_c$ in the non-desingularized limit, the critical time for each of the three calculations is estimated using a procedure proposed by Shelley (1992). The values of $1/\partial_s \kappa$ at $s/L = 0.5$ between $t = 1.0$ and $t = 3.2$ are extrapolated to zero using a third order polynomial that provides an excellent fit to the data points. The resulting critical times are shown in Fig. 10 as a function of the spreading parameter $\varepsilon$. A linear extrapolation is then used to estimate $t_c$ for $\varepsilon \to 0$ and $\Delta x \to 0$, yielding a critical time of $t_c = 3.001$. Although this time is slightly larger than the critical time predicted by theory, it is in excellent agreement with the results of Shelley (Shelley 1992), who estimates $t_c = 3.015$ using a high accuracy point-vortex method.
3.3. Kelvin-Helmholtz instability in the linear regime

A velocity shear between two fluids can give rise to the so-called Kelvin-Helmholtz instability, if the velocity profile between the two fluids exhibits an inflection point (Rayleigh 1880). This phenomenon was first studied by Helmholtz (1868) and Kelvin (1871) in the inviscid limit for a shear layer of zero thickness, i.e. a velocity discontinuity. For fluids of equal density, the growth rate $w$ in the linear regime for the unbounded case is

$$w(\text{We}) = \frac{k}{2} \sqrt{1 - \frac{2k}{\text{We}}} \quad (3.4)$$

The case of a shear layer of finite size and constant shear was subsequently analyzed by Rayleigh (1880). In the inviscid and $\text{We} = \infty$ limit, the growth rate $w$ in the linear regime becomes

$$w(d) = \frac{k}{2} \sqrt{\frac{\exp(-2kd) + 2kd - 1}{(kd)^2}} - 1 \quad (3.5)$$

with $d$ the shear layer thickness defined as

$$d = \frac{\eta}{(\partial u/\partial y)_{\text{max}}} \quad (3.6)$$

Although numerical solutions are possible for arbitrary shear layer velocity profiles (Michalke 1964; Yecko et al. 2002), comparisons will be limited to the above analytical solutions for validation purposes.

The initial conditions for the level set scalar $G$ are given by Eq. (3.2) with $A_0 = 1.10^{-5}$ and $B = 1$. The initial vortex sheet strength distribution is calculated from

$$\eta(x, t = 0) = \frac{w(\text{We})}{w(\text{We} = \infty)} (\eta_{\text{VS}}(x, t = 0) - \eta^*) + \eta^* \quad (3.7)$$

with $\eta^* = -1$ and $\eta_{\text{VS}}$ given by Eq. (3.3). Periodic boundary conditions are employed at the left and right domain boundary, whereas slip conditions are used at the lower and upper boundary. This in theory constitutes a bounded shear layer flow. However, the upper and lower boundaries are placed sufficiently far away from the interface, as to have no further influence on the presented results, thus allowing comparisons with the theoretical results of the unbounded case. All simulations are performed in a $B \times 2B$ box employing an $\eta$-grid of $256 \times 512$ equidistant Cartesian grid cells. Due to the small amplitude of the disturbance, no level set correction method is required.

Figure 11 compares the growth rates $w$,

$$w = \frac{1}{t_1} \int_0^{t_1} w(t) dt, \quad t_1 = 0.5 \quad (3.8)$$

to the results obtained by linear theory for varying We, Eq. (3.4). Table 2 lists the respective numerical values. For the lowest value of the spreading parameter, $\varepsilon = 4/256$, method $M_2$, Eq. (2.6), slightly under-predicts the linear growth rate by about 3%. This is due to the introduction of the numerical spreading function in Eq. (2.6) and the consequently theoretically reduced linear growth rate, cp. Eq. (3.5). Method $M_3$, Eq. (2.7), on the other hand marginally over-predicts the linear growth rate for $\varepsilon = 4/256$, but gives slightly lower $w$ than $M_2$ for larger $\varepsilon$. The reason for this behavior is not directly apparent, but it is most likely due to the lack of a consistent level set based interpolation step to calculate the vortex sheet induced velocity $u$, Eq. (2.9), (Herrmann 2004a).
Figure 11. Growth rate $w$ of the Kelvin-Helmholtz instability in the linear regime, method $M_2$, $\varepsilon = 4/256$ (circle, left), $M_3$, $\varepsilon = 4/256$ (box, left), $M_2$, $\varepsilon = 8/256$ (circle, right), $M_3$, $\varepsilon = 8/256$ (box, right), $M_2$, $\varepsilon = 16/256$ (full triangle, right), $M_3$, $\varepsilon = 16/256$ (open triangle, right), and linear theory (lines).

<table>
<thead>
<tr>
<th>Method / We</th>
<th>13</th>
<th>20</th>
<th>50</th>
<th>100</th>
<th>1000</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory</td>
<td>0.574</td>
<td>1.915</td>
<td>2.718</td>
<td>2.938</td>
<td>3.122</td>
<td>3.142</td>
</tr>
<tr>
<td>$M_2$, $\varepsilon = 4/256$</td>
<td>0.552</td>
<td>1.856</td>
<td>2.638</td>
<td>2.852</td>
<td>3.021</td>
<td>3.034</td>
</tr>
<tr>
<td>$M_3$, $\varepsilon = 4/256$</td>
<td>0.726</td>
<td>1.991</td>
<td>2.756</td>
<td>2.965</td>
<td>3.129</td>
<td>3.140</td>
</tr>
<tr>
<td>$M_2$, $\varepsilon = 8/256$</td>
<td>0.426</td>
<td>1.744</td>
<td>2.505</td>
<td>2.717</td>
<td>2.886</td>
<td>2.899</td>
</tr>
<tr>
<td>$M_3$, $\varepsilon = 8/256$</td>
<td>0.429</td>
<td>1.703</td>
<td>2.472</td>
<td>2.682</td>
<td>2.849</td>
<td>2.863</td>
</tr>
<tr>
<td>$M_2$, $\varepsilon = 16/256$</td>
<td>0.329</td>
<td>1.568</td>
<td>2.316</td>
<td>2.521</td>
<td>2.685</td>
<td>2.698</td>
</tr>
<tr>
<td>$M_3$, $\varepsilon = 16/256$</td>
<td>0.267</td>
<td>1.569</td>
<td>2.332</td>
<td>2.539</td>
<td>2.703</td>
<td>2.717</td>
</tr>
</tbody>
</table>

Table 2. Linear growth rates $w$ of the Kelvin-Helmholtz instability for varying $\varepsilon$ and $We$, 256 x 512 grid, methods $M_2$ and $M_3$.

Altogether, increasing $\varepsilon$ results in a consistent reduction of the growth rate for both $M_2$ and $M_3$.

In order to further analyze this behavior, Fig. 12 depicts the growth rate $w$ as a function of the shear layer thickness $d = \varepsilon$ in the limit of $We = \infty$ as compared to the analytical solutions, Eqs. (3.4) and (3.5). The corresponding numerical values are shown in Tab. 3. While the qualitative behavior of the simulations and the theory for a linear velocity profile is similar, both $M_2$ and $M_3$ exhibit overall larger growth rates and hence reach their neutrally stable solution at a larger shear layer thickness of $d = \varepsilon \approx 0.8$. This quantitative deviation is due to the different velocity profile of Eq. (3.5) and the LSVS method. The former assumes a linear profile, whereas the latter employs a rather complex profile due to the schemes outlined in section 2. In principle, it seems possible to change the velocity profile used in the LSVS method by modifying the numerical delta function $\delta_x$, Eq. (2.8). A detailed comparison with the viscous linear theory by Yecko et al. (2002) and Li et al. (2004) will reveal the necessary modifications to Eq. (2.8).
4. Conclusions and future work

A method has been presented that addresses the problem of mass conservation inherent in the level set approach. This so-called Refined Level Set Grid (RLSG) method achieves mass conservation results comparable to other level set correction methods, like for example the particle correction method, but demonstrates clearly superior performance with respect to the convergence of higher derivatives of the level set scalar. In particular the convergence of the interface curvature derivative is crucial within the context of the LSVS method and has been demonstrated in the case of vortex sheet roll-up.

Although the LSVS method has been derived for the limit of inviscid fluids, the numerical implementation necessitates the introduction of a finite shear layer thickness, representing an additional level of desingularization. The non-desingularized, inviscid limit of the Moore singularity can however be recovered, if the the extend of the desingularization is successively reduced.

The influence of the desingularization introduced by the finite shear layer thickness has been further analyzed in the case of the Kelvin-Helmholtz instability in the linear regime. It was found that the finite, constant shear layer thickness qualitatively mimicks certain features of viscous fluids. However, only a detailed comparison to the fully viscous theory by Yecko et al. (2002) and Li et al. (2004), planned for the future, will help quantify this effect and point to possible enhancements of the LSVS method.
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REFERENCES


SUSSMAN, M. & FATEMI, E. 1999 An efficient, interface-preserving level set redistancing


