Curvature correction and application of the $v^2 - f$ turbulence model to tip vortex flows

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1. Motivation and objectives

Lifting surfaces such as airplane wings and helicopter rotors generate coherent trailing vortices. The persistence of these vortices is known to be a hazard to following aircraft (in airplane wakes) and a significant source of noise and vibration (in helicopters). The physics of the flow is extremely complex in the near-field region of a tip vortex, as the process is largely turbulent, highly three dimensional, and involves multiple cross flow separations as depicted in Fig.1. Streamwise vorticity, mainly in the form of a feeding sheet, is seen to separate from the wing surface and roll-up into the tip vortex along with a variety of minor structures. Downstream of the trailing edge, these structures rapidly evolve into the largely axisymmetric and coherent tip vortex. Many experimental studies on wing tip vortices (Chow et al. (1997)) have reported largely reduced turbulence levels in the vortical core, even in the near-field. This has been attributed to the near-solid body rotation that exists in the inner core. Analytical studies based on linear stability theory of isolated vortices (Jacquin & Pantano (2002) and references therein) have also supported this argument by showing the damping of imposed disturbances in the core. As a result, downstream of the trailing edge, the major diffusion mechanism appears to be laminar rather than turbulent (Zeman (1995)).

The high Reynolds numbers, typically $O(10^7)$, encountered in typical flight conditions renders the cost of Large Eddy Simulations prohibitive. Therefore, one has to resort to robust, high fidelity Reynolds Averaged Navier Stokes (RANS) simulations. Although RANS models cannot be expected to describe accurately the intricate details of the turbulent flow-field, it has been previously demonstrated by Duraisamy (2005) that reliable solutions of the mean flow-field of a tip vortex can be achieved within engineering accuracy.

In this work, the $v^2 - f$ turbulence model of Durbin (1991) is used to simulate vortex formation from a wing in a wind tunnel. The key phenomenon of streamline curvature and its effect on the turbulence levels is addressed by means of a correction to the eddy viscosity coefficient. The primary objective of this work is to evaluate the predictive capabilities of baseline and corrected $v^2 - f$ turbulence model as applied to tip vortex flows, and also to compare it with other widely used closures such as the Spalart Allmaras (SA) model and Menter’s SST model.

2. Background

In traditional $k - \varepsilon$ models, the Reynolds stress anisotropy tensor $b_{ij}$ is defined by:

$$b_{ij} = C_k \frac{k}{\varepsilon} S_{ij}. \tag{2.1}$$
In the \( v^2 - f \) model, the damping of the turbulent fluctuations in the wall-normal direction is accounted for, and the corresponding anisotropy tensor is \( b_{ij} = C_\mu \frac{v^2}{k} S_{ij} \). Since the mean strain rate tensor \( S_{ij} \) is frame indifferent, it becomes obvious that non-inertial and streamline curvature effects need to be explicitly incorporated. Representation of such effects (primarily non-inertial) have been attempted in the literature and generally fall into the following categories:

- Sensitize \( C_\mu \) to the invariants of strain \( (\eta_1) \) and vorticity \( (\eta_2) \).
- Modify the production / dissipation terms of the \( k \) or \( \epsilon \) equations.

This work extends the effort of Petterson Reif et al. (1999), in which the \( v^2 - f \) turbulence model was sensitized to frame-rotation effects. Specifically, a Galilean-invariant curvature correction is analyzed in the context of vortex flows and applied within the original framework.

3. Review of existing framework

In order to account for frame-rotation effects, a modification has been proposed to the \( v^2 - f \) model by Petterson Reif et al. (1999). This modification is based on the behavior of the equilibrium solution of homogeneous plane shear flow \( (du/dy = S) \) subject to orthogonal frame rotation (angular velocity \( \Omega \)). In essence, the coefficient \( C_\mu \) is replaced by \( C^*_\mu(\eta_1, \eta_2) \), where

\[
\eta_1 = \frac{k^2}{\epsilon^2} \left( \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right)^2 \quad \text{and} \quad \eta_2 = \frac{k^2}{\epsilon^2} \left( \frac{1}{2} \left[ \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right] + C_\omega \Omega_{ij} \right)^2.
\]

The functional form of \( C^*_\mu \) is guided by the bifurcation of the fixed points of the resulting system in the \( \epsilon/\text{Sk} - R \) phase space, where \( R = \sqrt{\eta_2/\eta_1} \). This is done such that the equilibrium solutions mimic the behavior of EARSMs.

The evolution equation for the time scale ratio \( (\epsilon/\text{Sk}) \) is given by

\[
\frac{d}{\sigma} \left( \frac{\epsilon}{\text{Sk}} \right) = \left[ \frac{P}{\epsilon - \frac{C_{\epsilon 2} - 1}{C_{\epsilon 1} - 1}} \left( \frac{\epsilon}{\text{Sk}} \right)^2 \left( C_{\epsilon 1} - 1 \right) \right],
\]

with \( \sigma = \text{St} \). The two branches of the fixed point solution are thus identified by:

\[
\left( \frac{\epsilon}{\text{Sk}} \right)_\infty = 0,
\]
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It is known that the equilibrium $\frac{\epsilon}{Sk} = 0$ is characterized by a power law solution for $k$ and that $\frac{\epsilon}{Sk} \neq 0$ is associated with an exponential growth rate. On the former branch,

$$\left( \frac{P}{\epsilon} \right)_{\infty} = \frac{C_{e2} - 1}{C_{e1} - 1}$$  \hspace{1cm} (3.4)

The final functional form was selected to be:

$$C_{\mu}^* (\eta_1, \eta_2) = C_{\mu} \left( \frac{1 + \alpha_2 |\eta_3| + \alpha_3 |\eta_3|}{1 + \alpha_4 |\eta_3|} \right) \left[ \frac{1 + \alpha_5 \eta_1}{1 + \alpha_5 \eta_2} + \alpha_1 \sqrt{\eta_1} \sqrt{|\eta_3| - \eta_3} \right]^{-1}$$  \hspace{1cm} (3.6)

with $\eta_3 = \eta_1 - \eta_2$ and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} = \{0.055, 0.5, 0.25, 0.2, 0.025\}$. The bifurcation diagram for the modified $v^2 - f$ model and the EARS of Gatski & Speziale (1993) is shown in Fig. 2. Note that at $R = 1$, the equilibrium solution of the original model is recovered: i.e.

$$\left( \frac{P}{\epsilon} \right)_{\infty} = \frac{C_{e2} - 1}{C_{e1} - 1}, \quad \left( \frac{v^2}{k} \right)_{\infty} = 0.367, \quad \left( \frac{\epsilon}{Sk} \right)_{\infty} = 0.416.$$  \hspace{1cm} (3.7)

4. Incorporation of streamline curvature effects

In order to incorporate curvature effects, the eddy viscosity coefficient $C_{\mu}^*$ is sensitized within the framework introduced in the previous section. As discussed in Gatski & Jongen (2000) and in Hellsten et al. (2002), curvature sensitivity can be introduced in an EARS by transforming the anisotropy evolution equation to a local coordinate system in which the weak equilibrium condition can be approached. However, the definition of such a coordinate system is non-trivial.

As shown in the Appendix, curvature sensitivity can be introduced into an EARS by obtaining a Galilean invariant measure of the local rotation rate and including it in the objective vorticity tensor $\Omega$ in a consistent manner. A similar approach is proposed...
for the $v^2 - f$ model in that the invariant $\eta_2$ is now defined as

$$\eta_2 = \frac{k^2}{\epsilon^2} |W_{ij}|^2 = \frac{k^2}{\epsilon^2} \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + C_\omega (\tilde{\Omega}_{ij} + \tilde{\Omega}_{ji}) |^2 \quad (4.1)$$

where $\tilde{\Omega}_{ij} = -\epsilon_{ijk} \tilde{\omega}_k$ is an antisymmetric tensor that results from the transformation to a local basis (refer Appendix for further discussion).

In the context of EARSMSs, the various methods of determining $\tilde{\Omega}$ can be classified into acceleration-based and strain rate-based methods (Hellsten et al. (2002)). Strain rate-based methods are generally found to be more robust (Hellsten et al. (2002), Wallin & Johansson (2002)) and will be pursued in the present approach. Specifically, two different methods (due to Spalart & Shur (1997) and Wallin & Johansson (2002) respectively) were explored:

$$\tilde{\omega}_i = \frac{S'_{jk} S'_{kl} \epsilon_{lij}}{2 \Pi_1}, \quad \text{and} \quad \tilde{\omega}_i = \frac{\Pi_1^2 \delta_{ij} + 12 \Pi_2 S_{ij} + 6 \Pi_1 S_{ik} S_{kj}}{2 \Pi_1^2 - 12 \Pi_2^2} S_{pl} S'_{ij} S'_{pqj}, \quad (4.2)$$

where $(\cdot)'$ denotes a material derivative, $\Pi_1 = \text{trace}(S^2)$, and $\Pi_2 = \text{trace}(S^3)$. While the two transformations are identical in two dimensions, the latter is known to yield the exact rotation vector in a 3D axisymmetric flow. In practice, the contribution of the curvature correction term is limited.

5. Analysis for Lamb-Oseen vortex

Consider a curved homogeneous shear flow with circular streamlines with no frame rotation. The curvature parameter is defined as (Holloway & Tavoularis (1992))

$$C_f = \frac{U}{r} \frac{dU}{dr}, \quad (5.1)$$

where $U$ is the tangential velocity, and $r$ is the radius. Another useful parameter is the Richardson number (Holzapfel (2004)):

$$Ri = \frac{2 U \frac{\partial U}{\partial r} (U r)}{(r \frac{\partial}{\partial r} (\frac{U}{r}))^2}, \quad (5.2)$$

which is normally used to explain the stabilization of rotating flow and its analogy with stable stratification. The curvature correction vector for this flow can be shown to be $\omega_z = U/r$. Hence, defining the non dimensional shear $S = (dU/dr)(k/\epsilon)$, the invariants are given by:

$$\eta_1 = \frac{S^2}{2} (1 - C_f)^2, \quad \eta_2 = \frac{S^2}{2} (1 + C_f + C_\omega C_f)^2. \quad (5.3)$$

In the absence of the curvature correction, the $C_\omega$ term is neglected.

For purposes of analysis, the Lamb-Oseen vortex, which is defined by

$$U(r) = \frac{1}{r} \left( 1 - e^{-\alpha r^2} \right), \quad (5.4)$$

will be considered. The value $\alpha = 1.25643$ is used so that the core-radius $r_c = 1$ (or in other words, the peak tangential velocity occurs at $r = r_c = 1$). For this flow, the parameters are illustrated in Fig. 3.

It has to be mentioned that for a given turbulence model, the previously discussed bifurcation behavior is the same for all classes of flows and depends only on the ratio
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$R = \pm \sqrt{\eta_2/\eta_1}$. However, certain values of $R$ might not be attainable in different classes of flows.

From (5.3) and the definition of $C_f$ and $R_i$, it is easy to see that

$$C_f = \frac{R - 1}{R + 1 + C_\omega} \quad \text{and} \quad R_i = \frac{2(R - 1)(2R + C_\omega)}{2 + C_\omega} \quad (5.5)$$

Hence, the bifurcation results can be represented in terms of $C_f$ and $R_i$, an example of which is shown in Fig. 4. In addition, the relationship between the $C_f$ and $r$, i.e.,

$$C_f = \frac{U/r}{dU/dr} = \frac{1 - e^{-\alpha r^2}}{2\alpha r^2 e^{-\alpha r^2} - (1 - e^{-\alpha r^2})} \quad (5.6)$$

could be inverted to give an explicit expression:

$$\bar{r}(C_f) = \sqrt{\frac{\gamma - W(-1,-ge^{-\gamma})}{\alpha}}, \quad g = \frac{C_f + 1}{2C_f^2} \quad (5.7)$$

and $W$ represents the Lambert function. This allows for a representation of the bifurcation diagram directly in terms of the radius as seen from Fig. 5. It is clearly seen that the curvature correction reduces the effective eddy viscosity coefficient, and the production to dissipation ratio increases much more gradually with increasing radius.

An important fact to be recognized is that for this particular flow, the condition $|C_f| < 1$ is not realized (Fig. 3a). Correspondingly, the region $|R| < \frac{C_f}{2}$ is not realized in the
Figure 5. Bifurcation diagram as a function of radius

(b)

Figure 6. Sample streamwise section of the mesh

bifurcation diagram 5b. Therefore, if rotational correction is used, the resulting eddy viscosity coefficient will be less than that in plane shear ($R = 1$).

Note that the current analysis assumes complete spatial homogeneity and should be interpreted on a point-to-point basis.

6. Validation of tip vortex formation

The experiment corresponds to a rounded tip NACA 0012 wing of 4 ft. chord and 3 ft. span in a 32 × 48in. wind tunnel section. The chord based Reynolds number is $4.6 \times 10^6$, the Mach number is approximately 0.1, and the angle of attack is $10^\circ$. The experiments were conducted by Chow et al. (1997) at the NASA Ames research center. This test case was chosen because of the availability of a comprehensive set of measurements in the wake region and on the wing surface. Static pressure, mean velocity and Reynolds stress data are available at select axial planes, starting at a distance of $x/c=0.591$ ahead of the trailing edge to $x/c=0.678$ from the trailing edge. The flow is tripped at the leading edge so that the flow can be considered fully turbulent.
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A multiblock structured grid consisting of 9.3 Million mesh points and 62 blocks was used for the computation. A sample streamwise section of the grid is shown in Fig. 6. As seen in the figure, the mesh points are very finely clustered in the region of vortex formation. Previous studies (Duraisamy (2005)) have shown that around 20 points are required per vortex core diameter to keep numerical diffusion to a minimum. Accordingly, in the vortex formation and evolution regions, a mesh spacing of 0.003c is maintained in the cross-stream directions.

Simulations were carried out with the following turbulence models:
1. Baseline Spalart Allmaras model
2. Spalart Allmaras model with empirical correction of production term (Dacles-Mariani (1995), Duraisamy (2005))
3. Baseline SST $k - \omega$ model
4. Baseline $v^2 - f$ model
5. Modified $v^2 - f$ model with no curvature correction (CC) (Peterson Reif et al. (1999))
6. Modified $v^2 - f$ model with CC.

All computations were performed using the Stanford Multi-Block structured mesh solver SUmB, which is a scalable compressible RANS code.

Figure 7 shows the eddy viscosity contours (normalized by molecular viscosity) for different $v^2 - f$ models at a streamwise station 0.246 chords behind the wing trailing edge. The baseline model predicts maximum eddy viscosity in the center of the core, which is unphysical in light of the stabilizing effects of solid body rotation. The modified $v^2 - f$ model without CC results in very low eddy viscosity levels in the solid body rotation region of the core, but as predicted in the analysis, the level of eddy viscosity is found to be very high at the edge of the core. The modified $v^2 - f$ with CC is found to reduce drastically the eddy viscosity levels in the rotational region AND in the curved shear layer. Note that in the planar part of the shear layer, the eddy viscosity level is almost similar to that of the other models. Though not shown here, the eddy viscosity contours of the SA and SST models were qualitatively similar to Fig. 7a, even for the corrected SA, though the levels were generally higher.

Figure 8 shows the vertical component of velocity, which is the primary quantity of interest in typical applications, along a horizontal line passing through the core of vortex at different streamwise locations. Clearly, inclusion of the CC improves the predictions. As expected, the modification without CC proves to be detrimental. The effect of the curvature correction is more drastic in the axial velocity comparisons, as seen from Fig. 9. Accurate computation of the axial velocity is critical to the correct prediction of the swirl velocity because of the radial transport of angular momentum that is associated with the presence of strong gradients of axial velocity. Figure 10 demonstrates the performance of the other turbulence models. It is evident that the baseline models perform worse than does the baseline $v^2 - f$ model.

Figure 11 compares the computed turbulent kinetic energy with experimental data for the modified $v^2 - f$ model at three different streamwise locations. Reasonable quantitative and qualitative agreement is achieved with experiments.

† The production term is based on a function $|\omega| - 2 \min(|S| - |\omega|, 0)$ instead of $\omega$ in the baseline model.
Figure 7. Eddy viscosity contours (normalized by molecular viscosity) at x/c=0.246.

7. Summary

Streamline curvature correction is incorporated within the existing framework of the $v^2 - f$ closure. When applied to fixed wing tip vortex formation, the curvature correction considerably improves the prediction of the mean velocity field. In addition, reasonable agreement was achieved for the turbulent kinetic energy. This work should be considered preliminary for a variety of reasons:

- The present form of curvature correction has not been tested in different flow problems.
- Effect on quantities such as surface pressure and skin friction coefficient is yet to be determined.
- Application of curvature correction was seen to degrade convergence to steady state; hence, the utility in unsteady flows is questionable.

Future work will also be directed towards evaluating the performance of the model in
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Figure 8. Vertical velocity (normalized by free-stream velocity) along a line passing through the vortex core at different streamwise stations.

much simpler situations, for instance, in diffusion of an isolated turbulent vortex (Qin (1998)) and curved flow in a pipe (Rumsey & Gatski (2001)). It remains to be seen whether additional information such as $v^2 / k$ and $\nabla f$ can be used to obtain a better representation of streamline curvature/anisotropy.

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8. Appendix: Curvature correction in EARSMs

A general model for the evolution of the anisotropy tensor is given by Gatski & Jongen (2000):

$$\frac{D b}{Dt} = - \frac{b}{a_4} - a_3 \left( bS + Sb - \frac{2}{3} (bS)I \right) + a_2 (bW - Wb) + \frac{a_2}{\tau} \left( b^2 - \frac{1}{3} (b^2)I \right), \quad (8.1)$$

where $a_i$ are calibrated coefficients, $\tau = k/\epsilon$, and $\{\}$ represents the trace of a matrix. EARSMs are generally derived by assuming weak equilibrium of (8.1) (Gatski & Jongen et al., 2000).
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Figure 11. Turbulent kinetic energy (normalized by free-stream velocity squared) along 3 streamwise stations.

(2000)):

$$\frac{Db}{Dt} = 0. \quad (8.2)$$

This leads to an algebraic system of equations for the anisotropy tensor, an explicit solution of which results in an EARS model. Therefore, EARSs determine the equilibrium state of the anisotropy tensor and use it as an approximation to the time evolved value of anisotropy (Girimaji (1997)). Though reasonable in homogeneous shear flows, this assumption is dependent on the choice of coordinate system. In a steady flow with circular streamlines, the above assumption will be clearly violated if the coordinate system is not aligned with the streamlines (for instance, if a Cartesian system is used). Hence, a good EARS model can be obtained by imposing (8.2) only in a curvilinear coordinate system that “follows” the flow locally in some sense (Wallin & Johansson (2002), Gatski & Jongen (2000)).

Consider the solution of the equations in a Cartesian coordinate system. Let $T$ be the transformation matrix from the original inertial system to a local curvilinear system. Then, taking the material derivative of $TbT^T$ and transforming back to the Cartesian frame (Hellsten et al. (2002)), the anisotropy transport is given by:

$$\frac{Db}{Dt} = T^T \left[ \frac{D}{Dt} (TbT^T) \right] T - (b\bar{\Omega} - \bar{\Omega}b), \quad (8.3)$$

where

$$\bar{\Omega} = T^T \frac{DT^T}{Dt} \quad (8.4)$$

If the $T$ is such that the anisotropy can be neglected in the transformed coordinate system, then (8.3) becomes

$$\frac{Db}{Dt} = -(b\bar{\Omega} - \bar{\Omega}b) \quad (8.5)$$
By inspection, the extra terms can be incorporated into (8.1) just by replacing $W$ with

$$W' = W + \frac{1}{a_2} \Omega.$$  \hfill (8.6)

In the particular case of the $v^2 - f$, $1/a_2 = C_w = 2.25$. Therefore, it becomes clear that the effect of curvature and frame rotation enters the equation in a similar way. Hence, incorporation of the effects of streamline curvature reduces to the problem of finding the appropriate transformation $T$ and evaluation of (8.4).

REFERENCES


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