A stable, efficient, and adaptive hybrid method for unsteady aerodynamics

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1. Motivation and objectives

The generation and transportation of vortices from wingtips, rotors, and wind mills, and the generation and propagation of sound from aircraft, cars, and submarines require methods that can handle locally highly non-linear phenomena in complex geometries as well as efficient and accurate signal transportation in domains with smooth flow and geometries.

These demands require a hybrid between a finite volume method on an unstructured grid (for the non-linear phenomena and complex geometries) and a high-order finite difference method on the structured part (for the wave propagation).

There are essentially two different types of hybrid methods. The most common one employs different governing equations in different parts of the computational domain. A typical example is noise generated in an isolated part of the flow, considered as the sound source. The nonlinear phenomenon in the complex geometry is often computed by the Euler or Navier-Stokes equations. The sound propagation to the far field is considered governed by the linear wave equation with source terms from the Euler or Navier-Stokes calculation, see Lyrintzis (1994); Wells & Renaut (1997).

All coupling procedures that involve different governing equations suffer from one major problem. A stable and accurate numerical procedure does not suffice for convergence to the true solution, even if accurate data is at hand. Convergence to the true solution requires *a priori* knowledge of exactly where and how the solution shifts from being governed by one equation set to being goverened by the other. This *a priori* knowledge cannot be obtained as part of the coupling procedure.

In this project we intend to develop another type of hybrid method that avoids the artificial decoupling mentioned above and uses the same governing equations (in this case the Euler or Navier-Stokes equations) in the whole computational domain, not just close to the source. The word hybrid points in this case to the use of different numerical methods in different parts of the computational domain. Examples of this type of hybrid method can be found in Burbeau & Sagaut (2005); Rylander & Bondeson (2000). In this type of coupling procedure (provided that accurate data is known), a stable and accurate numerical procedure does suffice for convergence to the true solution.

Strict stability, which prevents error growth on realistic mesh sizes, is very important for calculations over long times. We have derived and studied strictly stable unstruc-

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tured finite volume methods (Nordström *et al.* 2003; Svärd & Nordström 2004; Svärd *et al.* 2006) and higher-order finite difference methods (Carpenter *et al.* 1999; Nordström & Carpenter 1999, 2001; Mattsson & Nordström 2004; M. & Nordström 2006) for hyperbolic, parabolic, and incompletely parabolic problems. These methods employ so-called summation-by-parts (SBP) operators and impose the boundary conditions weakly (Nordström *et al.* 2003; Carpenter *et al.* 1994).

In (Nordström & Gong 2006) it was proven that a specific interface procedure is stable for hyperbolic systems of equations. This project will rely heavily on these results; we will apply the theoretical results to the Euler equations. In a forthcoming paper we will include the treatment of the viscous terms in the Navier-Stokes equations.

A general 3-D code (CDP) that uses the node-centered finite volume method mentioned above has been developed by the Center for Turbulence Research (CTR) at Stanford University. A 3-D multi-block code (SUmb) that uses the finite difference technique discussed above is available at the Department of Aeronautics & Astronautics at Stanford University. These codes compute approximations to the Euler or Navier-Stokes equations and are the initial building blocks for the new hybrid method. A third coupling code (CHIMPS-lite, a simplified version of CHIMPS) will administer the coupling procedure and make it possible for the two solvers to communicate in an efficient and scalable way (Alonso *et al.* 2006).

2. Analysis

The material in this section is based on Nordström & Gong (2006). To introduce our technique we will consider the hyperbolic system

$$u_t + Au_x + Bu_y = 0, \qquad -1 \leq x \leq 1, 0 \leq y \leq 1 \tag{2.1}$$

with suitable initial and boundary conditions. A and B are constant symmetric matrices with k rows and columns. We will also consider a simplified computational domain that is divided into two subdomains. A so-called edge-based unstructured finite volume method will be used to discretize (2.1) on subdomain $[-1, 0] \times [0, 1]$ with an unstructured mesh, while a high-order finite difference method will be used on subdomain $[0, 1] \times [0, 1]$ with a structured mesh (see Fig. 1).

The fact that the unknowns in the finite volume and the finite difference methods are located in the nodes and can be collocated at the interface is a key ingredient in the coupling procedure presented below.

2.1. *The edge-based finite volume method*

In Nordström *et al.* (2003); Nordström & Gong (2006) it was shown that the semi-discrete finite volume form of (2.1) on subdomain $[-1, 0] \times [0, 1]$ can be written

$$\mathbf{u}_t + \{[(P^L)^{-1}Q_x^L] \otimes A\}\mathbf{u} + \{[(P^L)^{-1}Q_y^L] \otimes B\}\mathbf{u} = \{[(P^L)^{-1}(E_I^L)^T P_y^L] \otimes \Sigma^L\}(\mathbf{u}_I - \mathbf{v}_I) + \mathrm{SAT}^L, \tag{2.2}$$

where SAT^L is the penalty term that imposes the outer boundary conditions weakly. The SAT technique is a penalty procedure that can be used to specify outer boundary conditions as well as treating block interfaces. $\mathbf{u}_I$ and $\mathbf{v}_I$ are vectors that represent $\mathbf{u}$ and $\mathbf{v}$ ($\mathbf{v}$ is the discrete finite difference solution that will be presented below) on the interface respectively. E_I^L is a projection matrix that maps $\mathbf{u}$ to $\mathbf{u}_I$ such that $\mathbf{u}_I = (E_I^L \otimes I_k)\mathbf{u}$.

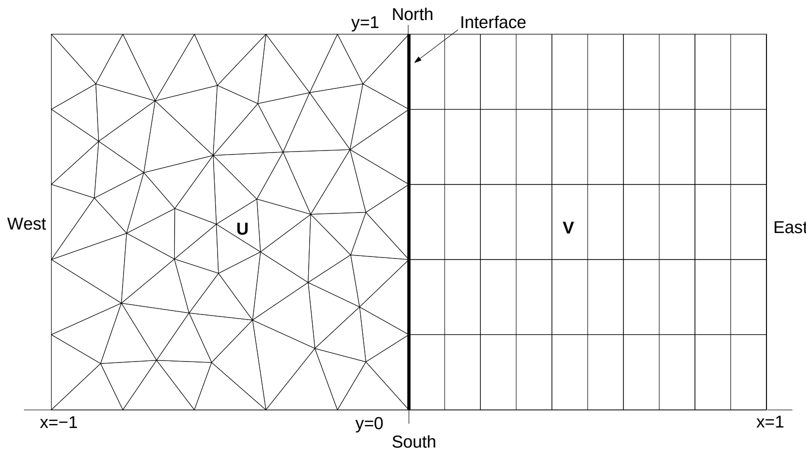
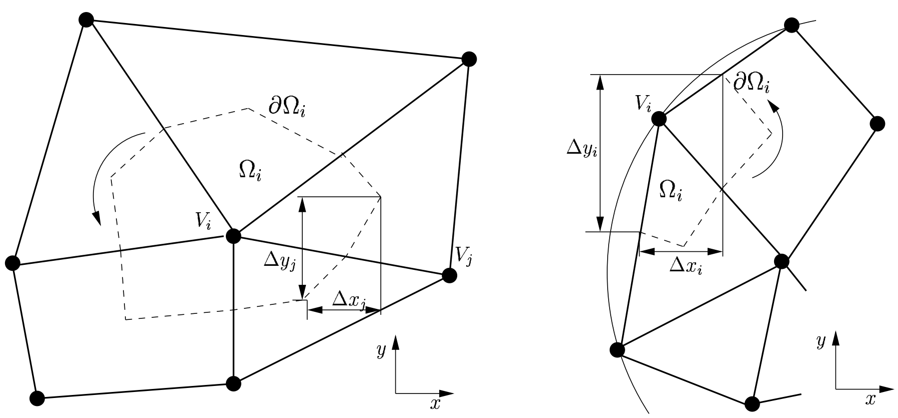


FIGURE 1. The hybrid mesh on the computational domain.



(a) in the interior

(b) on the boundary

FIGURE 2. The grid (solid lines) and the dual grid (dashed lines).

The non-zero components of E_I^L have the value 1 and appear at the interface. $P_y^L \otimes \Sigma^L$ is a penalty matrix that will be determined below by stability requirements.

P^L is a positive diagonal $m \times m$ matrix with the control volumes Ω_i on the diagonal and Q_x^L and Q_y^L are almost skew symmetric $m \times m$ matrices. The matrices Q_x^L and Q_y^L have the components

$$(Q_x^L)_{ij} = \frac{\Delta y_j}{2} = -(Q_x^L)_{ji}, \quad (Q_x^L)_{ii \notin \partial\Omega} = 0, \quad (Q_x^L)_{ii \in \partial\Omega} = \frac{\Delta y_i}{2}, \tag{2.3}$$

$$(Q_y^L)_{ij} = -\frac{\Delta x_j}{2} = -(Q_y^L)_{ji}, \quad (Q_y^L)_{ii \notin \partial\Omega} = 0, \quad (Q_y^L)_{ii \in \partial\Omega} = -\frac{\Delta x_i}{2}. \tag{2.4}$$

The definition of Δx_j and Δy_j is presented in Fig. reffig:grid. Moreover, (2.3) and (2.4) imply that Q_x^L and Q_y^R satisfy

$$Q_x^L + (Q_x^L)^T = Y, \qquad Q_y^L + (Q_y^L)^T = X, \tag{2.5}$$

where the non-zero elements in Y and X are Δy_i, $-\Delta x_i$ and correspond to the boundary points. For more details on the SBP properties of the finite volume scheme, see Nordström *et al.* (2003).

2.2. *The high-order finite difference method*

Consider the subdomain $[0,1]\times[0,1]$ with a structured mesh of $n\times l$ points. The finite difference approximation of u at the grid point (x_i, y_j) is a $k\times 1$ vector denoted $\mathbf{v}_{ij}$. We organize the solution in the global vector $\mathbf{v}=[\mathbf{v}_{11},\ldots,\mathbf{v}_{1l},\mathbf{v}_{21},\ldots,\mathbf{v}_{2l},\ldots,\mathbf{v}_{n1},\ldots,\mathbf{v}_{nl}]^T$. $\mathbf{v}_x$ and $\mathbf{v}_y$ are approximations of u_x and u_y and are approximated using the high-order accurate SBP operators for the first derivative constructed in Mattsson & Nordström (2004); Kreiss & Scherer (1974); Strand (1994). The difference operators in the x and y direction on the right subdomain are denoted $(P_x^R)^{-1}Q_x^R$ and $(P_y^R)^{-1}Q_y^R$, respectively.

The semi-discrete approximation of (2.1) on subdomain $[0,1]\times[0,1]$ can be written,

$$\begin{aligned}\mathbf{v}_t + \{[(P_x^R)^{-1}Q_x^R]\otimes I_y^R\otimes A\}\mathbf{v} + \{I_x^R\otimes[(P_y^R)^{-1}Q_y^R]\otimes B\}\mathbf{v} = \mathrm{SAT}^R \\ + \{[(P_x^R\otimes P_y^R)^{-1}(E_I^R)^T]P_y^R\otimes\Sigma^R\}(\mathbf{v}_I-\mathbf{u}_I),\end{aligned} \tag{2.6}$$

where the sizes of the identity matrices I_x^R and I_y^R are $n\times n$ and $l\times l$ respectively. SAT^R is the SAT penalty term for the outer boundary conditions. E_I^R is a projection matrix that maps $\mathbf{v}$ to $\mathbf{v}_I$, that is, $\mathbf{v}_I=(E_I^R\otimes I_k)\mathbf{v}$. Σ^R is a penalty matrix that will be determined below by stability requirements.

Note that $\mathbf{u}_I$ and $\mathbf{v}_I$ in (2.2) and (2.6) are collocated at the interface. This is absolutely essential for the accuracy of the hybrid scheme. It will be shown that it is also necessary for stability.

Note that the operators $(P_x^R)^{-1}Q_x^R$ and $(P_y^R)^{-1}Q_y^R$ are SBP operators since matrices P_x^R and P_y^R are symmetric and positive definite and the matrices Q_x and Q_y are nearly skew-symmetric, that is,

$$Q_x^R+\left(Q_x^R\right)^T = D_x^R = diag(-1,0,...0,1), Q_y^R+\left(Q_y^R\right)^T = D_y^R = diag(-1,0,...0,1), \tag{2.7}$$

where D_x^R and D_y^R are $n\times n$ and $l\times l$ matrices, respectively.

2.3. *Stable interface treatment*

We define the norms $N^L=P^L\otimes I_k$ and $N^R=(P_x^R\otimes P_y^R)\otimes I_k$, where $N^L=(N^L)^T>0$ and $N^R=(N^R)^T>0$. We also define an inner product and a norm for discrete real vector-functions $\mathbf{a},\mathbf{b}\in\mathbf{R}^n$ by

$$(\mathbf{a},\mathbf{b})_H=\mathbf{a}^TH\mathbf{b}, \qquad \|\mathbf{a}\|_H^2=(\mathbf{a},\mathbf{a}), \qquad H=H^T>0. \tag{2.8}$$

We apply the energy method by multiplying (2.2) and (2.6) with $\mathbf{u}^TN^L$ and $\mathbf{v}^TN^R$ respectively. We also use (2.5), (2.7), (2.8), (2.5) and assume that the terms including $\mathbf{u}_B$, $\mathbf{v}_E$, $\mathbf{v}_S$, $\mathbf{v}_N$ at the outer boundaries are precisely cancelled by the SAT terms (Carpenter *et al.* 1999; Nordström & Carpenter 1999). This yields the energy estimate

$$\frac{d}{dt}\left(\|u\|_{N^L}^2+\|u\|_{N^R}^2\right)=[\mathbf{u}_I,\mathbf{v}_I]^TM_I[\mathbf{u}_I,\mathbf{v}_I], \tag{2.9}$$

where

$$M_I=\begin{bmatrix} -P_y^L\otimes A+P_y^L\otimes\Sigma^L+P_y^L\otimes(\Sigma^L)^T & -P_y^L\otimes\Sigma^L-P_y^R\otimes\Sigma^R \\ -P_y^L\otimes\Sigma^L-P_y^R\otimes\Sigma^R & P_y^R\otimes A+P_y^R\otimes\Sigma^R+P_y^R\otimes(\Sigma^R)^T \end{bmatrix}.$$

We need M_I to be negative semi-definite for stability. Consider a simplified case where

$$P_y^L = P_y^R = P_y, \qquad \Sigma^L = (\Sigma^L)^T, \qquad \Sigma^R = (\Sigma^R)^T. \tag{2.10}$$

This yields,

$$M_I = P_y \otimes \begin{bmatrix} -A + 2\Sigma^L & -\Sigma^L - \Sigma^R \\ -\Sigma^L - \Sigma^R & A + 2\Sigma^R \end{bmatrix} = P_y \otimes M.$$

To obtain stability M has to be negative semi-definite. We can diagonalize A by $X^T A X = \Lambda$, where X is an orthogonal matrix consisting of the eigenvectors of A. Moreover, consider penalty parameters Σ^L and Σ^R of the form $X^T \Sigma^L X = \Lambda^L$ and $X^T \Sigma^R X = \Lambda^R$. We denote by λ_i the ith diagonal component of Λ and similarly λ_i^L and λ_i^R for Λ^L and Λ^R. Then we obtain a negative semi-definite M if

$$\lambda_i^R = \lambda_i^L - \lambda_i, \quad \lambda_i^L \le \frac{\lambda_i}{2}, \quad i = 1, \dots, k. \tag{2.11}$$

The first condition in (2.11) is recognized as the condition for a conservative interface treatment. The second condition in (2.11) leads to stability if conservation is guaranteed. For more details, see Nordström & Carpenter (1999).

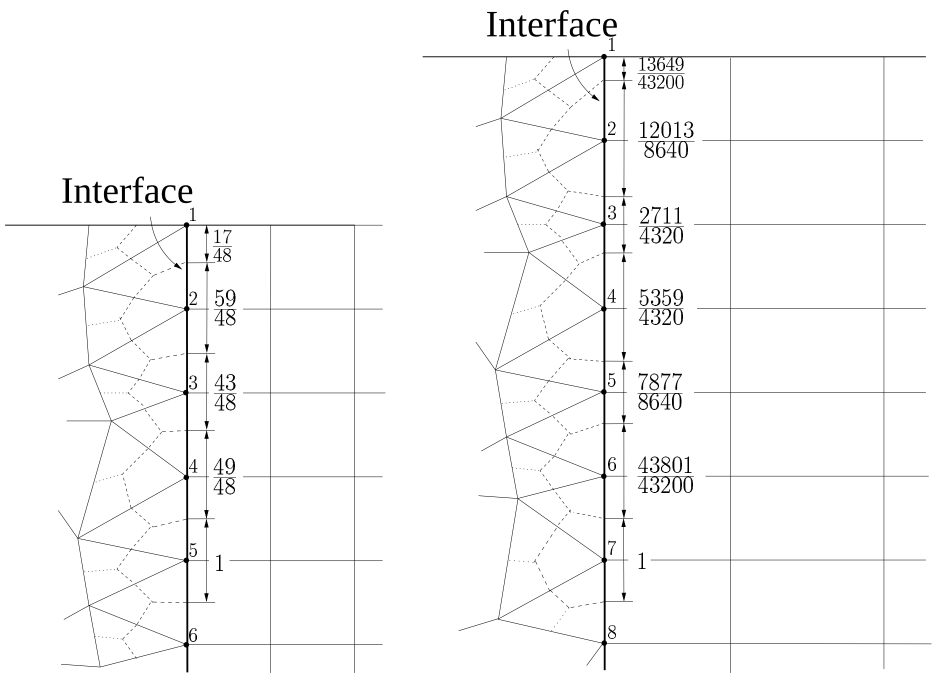
We have proved the following proposition:

Proposition 2.1. *If the conditions (2.10)-(2.11) hold, (2.9) leads to a bounded energy and (2.2), (2.6) have a stable and conservative interface treatment.*

The specific SBP operators that are based on diagonal norms are given in Mattsson & Nordström (2004); Strand (1994). When we use the second-order diagonal norm $P_y^R = \mathrm{diag}[1/2, 1, \dots, 1, 1/2]/h$ on the right subdomain, we do not need to change the control volumes on the left domain, since $P_y^L = P_y^R$. But the standard fourth- and sixth-order diagonal norms are

$$\frac{1}{h}\begin{bmatrix} \frac{17}{48} & & & & & \\ & \frac{59}{48} & & & & \\ & & \frac{43}{48} & & & \\ & & & \frac{49}{48} & & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}, \frac{1}{h}\begin{bmatrix} \frac{13649}{43200} & & & & & & & \\ & \frac{12013}{8640} & & & & & & \\ & & \frac{2711}{4320} & & & & & \\ & & & \frac{5359}{4320} & & & & \\ & & & & \frac{7877}{8640} & & & \\ & & & & & \frac{43801}{43200} & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix}, \tag{2.12}$$

respectively. In both cases we need to modify the control volume for the finite volume method at the points on the interface to guarantee $P_y^L = P_y^R$. The old dual grid for the points at the interface consists of the lines between the center of the triangles and the midpoints of the edges. In order to match P_y^L and P_y^R, the new lines will connect the center of the triangles and the points at the interface that correspond to the P_y^R (see Fig. 3).



(a) Fourth-order SBP (b) Sixth-order SBP

FIGURE 3. The modified control volumes for the points on the interface.

2.4. *The coupling code, CHIMPS-lite*

A general 3-D code (CDP) that uses the node-centered finite volume method previously mentioned has been developed by the Center for Turbulence Research (CTR) at Stanford University. Also available at the Department of Mechanical Engineering at Stanford University is a 3-D multi-block code (SUmb) based on high order finite difference methods.

These two codes compute approximations to the Euler or Navier-Stokes equations and are the initial building blocks for the new hybrid method. The codes are node-based and use SBP operators and penalty techniques for imposing the boundary and interface conditions weakly. This numerical technique enables coupling of the two codes by sending the value of the dependent variables in the nodes located on the interface to the other code while simultaneously recieving the colocated data at the interface from the other code. Each code provides boundary data to the other code.

A third coupling code (CHIMPS-lite) administers the coupling procedure and makes it possible for the two solvers to communicate in an efficient and scalable way. CHIMPS-llite is a simplified version of CHIMPS (Alonso *et al.* 2006) designed specifically for interfaces with collocated nodes where no interpolation is required. In an initial setup phase, both codes register their interface nodes with CHIMPS-lite, and the parallel communication pattern is built. Using this communication pattern, CHIMPS-lite then facilitates the exchange of interface data at each stage in the Runge-Kutta scheme. The development of coupling software like CHIMPS and CHIMPS-lite is an essential new ingredient that will

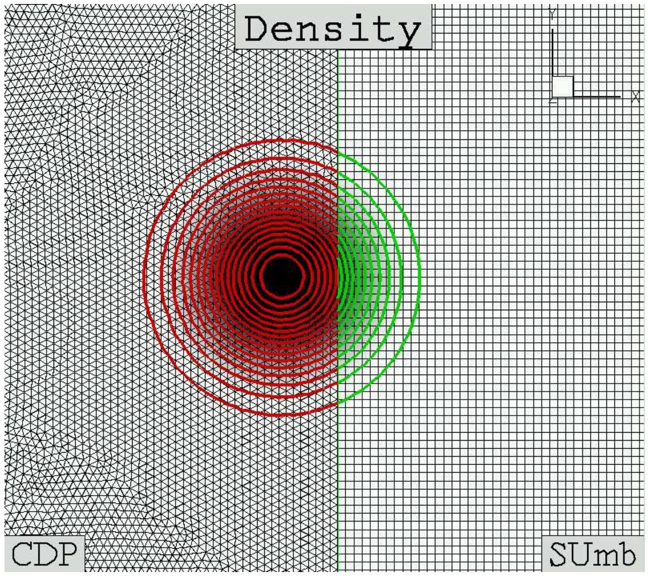


FIGURE 4. Transport of a vortex across an interface for the Euler equations.

take the coupling idea from theoretical concept to practical tool for fluid flow investigations.

3. Results

We consider this project as work in progress; only a few preliminary results currently exist. Fig. 4 presents a calculation using the unstructured finite volume code CDP coupled to the high order finite difference code SUmb. The calculation is fourth order accurate and shows the transport of a vortex across an interface for the Euler equations. Other similar results have been produced. The results indicate that the procedure is stable and useful.

4. Future work

Future work involves verifying the computational procedure against exact solutions to ensure that it converges at the correct rate. We also intend to apply the method to a high-lift device problem with complex geometry and high accuracy requirements. These results will be presented at the 2007 SIAM Conference on Computational Science and Engineering, in Costa Mesa, California.

Viscous terms will then be included and verified in the same manner. A stable and accurate operational hybrid method for the Navier-Stokes equations will allow for the analysis of very demanding fluid flow problems involving complex geometries and wave propagation effects that are not possible to address today.

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