Asymptotic analysis of the constant pressure turbulent boundary layer

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1. Motivation and objectives

The Navier-Stokes equations are expanded in asymptotic power series in a small parameter $\epsilon = u_e/U_e$ which is determined as a function of Reynolds number by an asymptotic matching procedure. The present matched asymptotic expansion analysis differs from the more traditional approach by employing the unsteady Navier-Stokes equations instead of the unclosed Reynolds averaged equations. It is therefore not necessary to expand the Reynolds stress separately in the small parameter. The analysis is therefore simpler and requires fewer assumptions. The main result of this analysis is an instantaneous log-law in the overlap region, of form $u^+ = \kappa^{-1} \ln(y^+) + B$ where the additive “constant” $B$ is independent of $y$ but depends on the outer scaled $x, z, t$ variables.

In this paper the constant pressure flat plate turbulent boundary layer is studied by the method of matched asymptotic expansions. This was first done as a formal procedure by Yajnik (1970) and Mellor (1972). They expanded the two-dimensional Reynolds-averaged Navier-Stokes equation, which has always defined this problem, in asymptotic power series in a small parameter, employing expansions in inner and outer regions which were then matched in an overlap region where both expansions are assumed valid. Since the equations are unclosed, the Reynolds stress terms must be expanded in addition to the velocity components. The present paper follows the same basic procedure, but makes use of the complete unsteady Navier-Stokes equations, and is therefore properly posed. This approach requires fewer assumptions. In particular no assumptions are required for the Reynolds stresses.

The present analysis is restricted to incompressible turbulent flow along a flat plate with constant free stream velocity $U_e$. A constant length scale $l$ is introduced which is the boundary layer thickness at some position $x_0$, $l = \delta(x_0)$. In the outer expansion all the spatial coordinates are scaled with $l$, not just the wall normal coordinate. This implies a turbulent boundary layer in which large scale vortical structures are convected with the free stream. It is assumed that the length scales of these structures are of the same order as the thickness of the boundary layer. These can be envisioned as long waves on the turbulent shear layer which originate from instabilities near the wall and interact back on the near wall disturbances in a complicated manner.

The inner layer is scaled with wall variables in the traditional way.

2. Outer expansion

The Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

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are scaled to outer variables by the change of variables
\[ U = u / U_e, \quad P = p / U_e^2, \quad X = x / l, \quad Y = y / l, \quad Z = z / l, \quad T = U_e t / l. \]  
(3)

This makes
\[ \frac{\partial U}{\partial T} + U \cdot \nabla U = -\nabla P + R_e^{-1} \nabla^2 U, \quad R_e = U_e l / \nu \]  
(4)
\[ \nabla \cdot U = 0 \]  
(5)

Now make a change of variables to
\[ U = \hat{i} + \epsilon U_1(\xi, Y, Z, T_1), \quad \xi = X - T, \quad T_1 = \epsilon T, \quad P = \epsilon^2 P_1, \]  
(6)
which puts the problem in a coordinate frame moving with the free stream; \( T_1 \) is the natural time scale in this frame. The unit vector \( \hat{i} \) is in the x direction. The velocity \( U_1 \) satisfies the exact Navier-Stokes equation,
\[ \frac{\partial U_1}{\partial T_1} + U_1 \cdot \nabla U_1 = -\nabla P_1 + (\epsilon R_e)^{-1} \nabla^2 U_1, \]  
(7)

so no approximation has been made. At this stage \( \epsilon \) is an arbitrary constant parameter. Now if we assume that \( \epsilon \) is a function of \( R_e \) which tends to zero as \( R_e \to \infty \) (tentatively identified as \( u_r / U_e \)) and expand in an asymptotic power series, we find a time dependent “defect law”,
\[ U = \hat{i} + \epsilon U_1(X - T, Y, Z) + \cdots \]  
(8)

In this approximation \( U_1(X - T, Y, Z) \), without the explicit dependence on \( T_1 \), satisfies the steady nonviscous version of (7). The vorticity is frozen into the external stream. At fixed \( x \) the velocity would be seen as a fluctuating function of time as vortical structures are convected past the observer. Viscosity is neglected since Mellor (1972) showed that \( R_e^{-1} \) is smaller than any power of \( \epsilon \) as \( \epsilon \to 0 \) (transcendentally small), which will be verified below. This means that viscosity would be negligible to any order in the asymptotic expansion of (7) in powers of \( \epsilon \). With the viscous second derivative term neglected this is a singular perturbation problem which calls for a rescaling of the equations in order to describe a viscous boundary layer near the wall which can satisfy the no slip boundary condition.

3. Inner expansion

Rescale the Navier-Stokes equations using a velocity scale \( v_i \) and a length scale \( l_i \) in such a way as to retain the viscous terms in the infinite Reynolds number limit. Inner variables are defined by
\[ u_i = u / v_i, \quad p_i = p / v_i^2, \quad y_i = y / l_i, \quad x_i = x / l_i, \quad z_i = z / l_i, \quad t_i = v_i t / l_i \]  
(9)

When \( l_i v_i / \nu = 1 \) the rescaled equations become
\[ \frac{\partial u_i}{\partial t_i} + u_i \cdot \nabla u_i = -\nabla_i p_i + \nabla_i^2 u_i. \]  
(10)

with the viscous terms of the same order as inertial terms, without identifying \( v_i \); any constant velocity would do this. To make a definite physical identification the friction velocity will be used, \( v_i = u_r \), where \( u_r = \sqrt{\langle \tau_w \rangle / \rho} \), with \( \langle \tau_w \rangle \) the average shear stress at the wall. This is of course the scaling velocity which is always used for the wall layer, but it may not be the only possibility.

We reason as follows: \( \langle \tau_w \rangle / \rho = \nu < du / dy >_0 = \nu (v_i / l_i) < du_i / dy_i >_0 = v_i^2 < \nu \).
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\( \frac{du_i}{dy_i} > 0 \). A scaling condition can be taken as \( < \frac{du_i}{dy_i} > 0 = 1 \), so \( v_i = u_\tau \). Thus we have \( v_i = u_\tau \) and \( i = \nu / u_\tau \).

Solutions of (10) are sought in an asymptotic power series in \( \epsilon \). The lowest order term in the expansion, \( u_{i,1} \), satisfies the same equation and could depend parametrically on the outer variables which are slowly varying in inner variables. For instance the convecting vorticity in the outer flow can induce long irrotational waves near the wall which would enter the problem through boundary conditions. Townsend (1961) described this outer influence as an “inactive” component of the inner flow. Presumably the inner layer responds to this by an “active” ejection of vorticity and low speed fluid.

4. Matching

We have a two term outer expansion and a one term inner expansion,

\[
\begin{align*}
\text{two term outer : } U &= \hat{1} + \epsilon U_1(X - T, Y, Z) \\
\text{one term inner : } U &= \epsilon u_{i,1}(x_i, y_i, z_i, t_i)
\end{align*}
\]

where the inner and outer variables are related by \( X = x_i/(\epsilon Re) \), \( Y = y_i/(\epsilon Re) \), \( Z = z_i/(\epsilon Re) \). These will be matched in the \( y \) variables holding \( x, z, t \) fixed, assuming that they both express the same function in an overlap region. We will use the VanDyke (1964) matching principle, which states that the one term inner expansion of the two term outer expansion has to equal the two term outer expansion of the one term inner expansion. This is a little tricky because the matching process, as conceived here, also has to determine the \( Re \) dependence of \( \epsilon(Re) \).

The two term outer expansion, expressed in the inner y variable is

\[
U = \hat{1} + \epsilon U_1(X - T, y_i/(\epsilon Re), Z) .
\]

This is to be expanded in inner variables, retaining one term in an inner expansion. Expressing this back in outer variables

\[
U = \hat{1} + \epsilon U_1(X - T, Y, Z) \text{ asymptotically as } Y \to 0.
\]

The one term inner expansion expressed in the outer y variable is

\[
U = \epsilon u_{i,1}(x_i, \epsilon Re Y, z_i, t_i) .
\]

This is to be expanded in outer variables retaining two terms in an outer expansion. This generates the large \( y_i \) asymptote of \( u_{i,1} \) which has to depend the same way for large \( y_i \) as the two term outer expansion does for small \( Y \). (This is a little confusing because one can’t see two terms explicitly. We will come back to this.) The matching principle means that (14) and (15) have to be equal for all \( Y \) when \( Re \) is large. In general this will not be true without restrictions on the functions. For the x-component this requires

\[
1 + \epsilon U_1(X - T, Y, Z) = \epsilon u_{i,1}(x_i, \epsilon Re Y, z_i, t_i) .
\]

This matches exactly for all \( Y \) if

\[
u_{i,1} = \kappa^{-1} \ln(\epsilon Re Y) + B_i(x_i, z_i, t_i)
\]

and

\[
U_1 = \kappa^{-1} \ln(Y) + B_o(X - T, Z)
\]

since \( \ln Y \) will cancel from both sides for any \( Re \) and no other functional form can
accomplish this. Substituting (17) and (18) into (16) with \( B = B_i - B_o \) results in the identity
\[
1 = \epsilon \left( \kappa^{-1} \ln(\epsilon R_e) + B \right).
\]
(19)

This gives the required relationship between \( \epsilon \) and \( R_e \), with the added requirement that \( \kappa^{-1} \) and \( B \) must be constants (independent of coordinates and time), since \( \epsilon \) and \( R_e \) are constants. This is the standard form for the friction law. It may be solved for \( R_e \) as a function of \( \epsilon \) as
\[
R_e = (be)^{-1} \exp(\kappa/\epsilon) \quad \text{with} \quad b = \exp(\kappa B).
\]
(20)

This shows that \( R_e^{-1} \) is transcendentally small compared to powers as was shown by Mellor.

Having determined \( \epsilon(R_e) \) it is instructive to verify formally that the asymptotic matching procedure is satisfied. For the inner expansion in outer variables use (17) in (15) (for the x component) with \( \epsilon R_e = b^{-1} \exp(\kappa/\epsilon) \):
\[
U = \epsilon \left( \kappa^{-1} \ln(b^{-1}Y \exp(\kappa/\epsilon)) + B_i \right).
\]
(21)

This is supposed to be expanded to two terms in \( \epsilon \) holding \( Y \) fixed. Because \( \ln(\exp(\kappa/\epsilon)) \equiv \kappa/\epsilon \) this gives
\[
U = 1 + \epsilon \left( \kappa^{-1} \ln(b^{-1}Y) + B_i \right) \equiv 1 + \epsilon \left( \kappa^{-1} \ln(Y) + B_o \right)
\]
(22)

which is the two term outer expansion, as was to be shown. In the same way the outer expansion in inner variables is
\[
U = 1 + \epsilon \left( \kappa^{-1} \ln(b\epsilon \exp(-\kappa/\epsilon)) + B_o \right)
\]
(23)

This is to be expanded to one term in \( \epsilon \) holding \( y_i \) fixed. Here it gives exactly one term,
\[
U = \epsilon \left( \kappa^{-1} \ln(b\epsilon \exp(Y)) + B_o \right) \equiv \epsilon \left( \kappa^{-1} \ln(y_i) + B_i \right),
\]
(24)

because the “1” is cancelled by the singular term in the logarithm.

For the \( y \)-component of velocity the matching condition is
\[
0 = \epsilon \left( v_{i,1} (x_i, \epsilon R_e, z_i, t_i) - V_1 (X - T, Y, Z) \right).
\]
(25)

This is only possible for all \( Y \) as \( R_e \) gets large if both functions are independent of \( y \),
\[
V_1 = v_{i,1} = A_o (X - T, Z).
\]
(26)

Similarly, for the \( z \)-component
\[
W_1 = w_{i,1} = C_o (X - T, Z).
\]
(27)

5. Conclusion

Therefore in the overlap region, we have
\[
U = 1 + \epsilon \left( \kappa^{-1} \ln(Y) + B_o (X - T, Z) \right)
\]
(28)
\[
V = \epsilon A_o (X - T, Z)
\]
(29)
\[
W = \epsilon C_o (X - T, Z)
\]
(30)
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where the functions \( A_o, B_o, C_o \) are unknown. These can also be expressed in inner variables:

\[
\begin{align*}
    u^+ &= \kappa^{-1} \ln(y^+) + B_o(X - T, Z) + B \\
    v^+ &= A_o(X - T, Z) \\
    w^+ &= C_o(X - T, Z)
\end{align*}
\]

(31)

(32)

(33)

At fixed \( X, Z \) the functions \( A_o, B_o, C_o \) are fluctuating random functions of time. Note that the logarithmic part is independent of time. Also note that in (31) \( B_o(X - T, Z) + B \equiv B_i \) depends on outer variables because of the matching condition. The dependence on inner variables has dropped out as \( y_i \to \infty \). This asymptotic boundary condition is where a parametric dependence on outer variables can enter solutions of the inner equations.

The averages of these expressions are:

\[
\begin{align*}
    \langle U \rangle &= 1 + \epsilon \left( \kappa^{-1} \ln(Y) + \langle B_o \rangle \right) \\
    \langle V \rangle &= \epsilon \langle A_o \rangle \\
    \langle W \rangle &= \epsilon \langle C_o \rangle
\end{align*}
\]

(34)

(35)

(36)

where the angle brackets denote ensemble or time averages.

The fluctuating parts are obtained by subtracting:

\[
\begin{align*}
    U' &= U - \langle U \rangle = \epsilon (B_o - \langle B_o \rangle) \equiv \epsilon B' \\
    V' &= V - \langle V \rangle = \epsilon (A_o - \langle A_o \rangle) \equiv \epsilon A' \\
    W' &= W - \langle W \rangle = \epsilon (C_o - \langle C_o \rangle) \equiv \epsilon C'
\end{align*}
\]

(37)

(38)

(39)

There are some empirical results which describe fluctuations of this kind. Lindgren (2003) have shown, using the KTH database for high Reynolds number zero pressure gradient turbulent boundary layers, that the PDF of fluctuations of \( u \) normalized by its rms value, i.e. \( B'^2 / \langle (B')^2 \rangle^{1/2} \), is self similar and approximately Gaussian in the overlap region and independent of Reynolds number and \( y \). One might reasonably assume that the \( A' \) and \( C' \) fluctuations are of a similar nature.

It follows from (37), (38), and (39) that Reynolds stresses, which depend on averages of \( A', B', C' \), are all constants in the overlap region:

\[
\begin{align*}
    \langle U' V' \rangle &= \epsilon^2 \langle A'B' \rangle \\
    \langle V' W' \rangle &= \epsilon^2 \langle A'A' \rangle \\
    \langle W' W' \rangle &= \epsilon^2 \langle C'C' \rangle \\
    \langle U' U' \rangle &= \epsilon^2 \langle B'B' \rangle
\end{align*}
\]

(40)

(41)

(42)

(43)

The method of matched asymptotic expansions has been used here to study the constant pressure fully developed turbulent boundary layer. The new idea in the present analysis is the use of the unsteady Navier-Stokes equation. The conventional treatment of this problem has always made use of the Reynolds-averaged momentum equation, which requires a separate expansion of the Reynolds stresses. This is not required in the present approach. The result of the analysis is an instantaneous logarithmic law in the overlap region described by, (28), (29), and (30), in which the velocity fluctuations, (37), (38), and (39), are independent of the distance from the wall and are functions which
are “frozen” into the flow so that the dependence on \( t \) implies a dependence on \( x \). This is Taylor’s frozen hypothesis, which is often assumed for processing experimental data. The analysis presented here, while making use of formal matched asymptotic methods, strongly resembles the Millikan (1939) classical overlap treatment in which empirical defect and wall laws were matched. Here (11) is the defect law and (12) the wall law; solutions of the Navier-Stokes equations have been substituted for the empirical laws and matching was done by a formally different method.

The number of assumptions required for the analysis is quite small. It was assumed that the boundary layer thickness is the proper outer scaling for all the coordinates, implying that the large eddies of the flow scale with the boundary layer thickness. It is also assumed that the velocity tends to uniform flow in the outer scaling as the Reynolds number tends to infinity.

6. Discussion

The existence of an overlap region implies a coupling or interaction between an outer problem and an inner problem. The inner problem is to solve the Navier-Stokes equations in inner variables with asymptotic boundary conditions as \( y_i \) tends to infinity. This looks much like a conventional boundary layer problem. The outer problem is to solve the inviscid Euler equations with boundary condition at \( Y = 0 \). The nature of the coupling between the two problems is described below.

There are three unknown functions of outer variables \( A_o, B_o, C_o \) which result from this analysis. However, it appears that only two of them are independent. We reason as follows. Suppose that the inner scaled Navier-Stokes equations (10) are solved with boundary conditions on the horizontal velocity components at infinity, therefore involving the \( B_o \) and \( C_o \) functions. These are the usual boundary conditions for boundary layers. Using this solution integrate the continuity equation from zero to infinity:

\[
v_i(y_i \to \infty) = -\int_0^\infty \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial w_i}{\partial z_i} \right) dy_i.
\]

But \( v_i(y_i \to \infty) = A_o \) and the right hand side of (44) depends only on the functions \( B_o \) and \( C_o \). Therefore by this “displacement” argument \( A_o \) is a function of \( B_o \) and \( C_o \). The implication of this is that the inner layer provides a boundary condition for the outer scaled equations at \( Y = 0 \) as in second order laminar boundary layer theory. Actually, one more condition is needed since there are two unknown functions. The continuity equation can be used in the overlap region. Since \( A_o \) is independent of \( Y \), we have the condition \( \partial B_o / \partial X + \partial C_o / \partial Z = 0 \).

Fluctuations in the wall stress are related to fluctuations in the additive function \( B_0 \). Use \( \tau_w / \rho = u_i^2 \frac{du_i}{dy_i} \) or the equivalent \( \tau_w / < \tau_w > = (du_i / dy_i)_0 \). Now near the wall \( u_i = y_i (du_i / dy_i)_0 \) and from (31) \( u_i = \kappa^{-1} \ln(y_i) + B_o + B \). Equating these two expressions at \( y_i = 1 \) gives the approximate result \( \tau_w / < \tau_w > = (du_i / dy_i)_0 = B_o + B \). This can be written as

\[
\frac{\tau_w - < \tau_w >}{< \tau_w >} = B_o - < B_o >.
\]

This suggests that fluctuations in the shear stress at the wall are related to the outer problem boundary condition on \( V \) at \( Y = 0 \).
REFERENCES


