Passive scalar power law

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1. Motivation and objectives

Lundgren (2003) derived a formula for the instantaneous velocity difference in the inertial range of scales by analysis of the Navier-Stokes equation. That result is

$$\mathbf{v} = U\mathbf{V}(\hat{\mathbf{r}})(r/L)^q, \tag{1}$$

where $\mathbf{V}(\hat{\mathbf{r}})$ and q are dimensionless random variables which require statistical specification. The parameter $\hat{\mathbf{r}}$ in $\mathbf{V}$ is a unit vector along $\mathbf{r}$ indicating the orientation of the two points, but both $\mathbf{V}$ and q are independent of the separation magnitude r. The longitudinal (along the direction $\hat{\mathbf{r}}$) component of (1) is given by

$$v = UV(r/L)^q \quad . \tag{2}$$

The purpose of this article is to derive the corresponding result for the difference in passive scalar between these two points. This result is

$$\Delta\theta(\mathbf{r}, \tau) = \Theta C_o(\hat{\mathbf{r}}, D/\nu)\left(\frac{r}{L}\right)^{q_s}, \tag{3}$$

where q_s is undetermined and could therefore be a random variable.

The analysis which leads to (3) is given in sections 2 and 3 along with the analysis of the velocity difference problem since the scalar difference problem cannot really be treated separately.

2. Partial Lagrangian coordinate system

2.1. *Velocity difference*

The starting point for an analysis of structure functions is an equation for the difference in velocity between two points. Such an equation may be derived by letting one of the points, $\mathbf{x}$, be a Lagrangian fluid particle, moving with the fluid. The second point, which is not Lagrangian, is tied to the motion of the first with fixed separation $\mathbf{r}$. It was shown in Lundgren (2003) that $\mathbf{v}$ satisfies the Navier-Stokes equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial \mathbf{r}} + \nu\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{v} \quad , \qquad \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{v} = 0 \quad , \tag{4}$$

in which only derivatives with respect to $\mathbf{r}$ occur; none with respect to the Lagrangian variable.

A formal continuum mechanics derivation is given here. Let one of the points be a Lagrangian fluid particle represented by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where $\mathbf{X}$ is a Lagrangian variable. The second point at $\mathbf{x} + \mathbf{r}$ is not Lagrangian. The velocity difference between these two points is defined by

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$$\mathbf{v}(\mathbf{r},t;\mathbf{X}) = \mathbf{u}(\mathbf{x}(\mathbf{X},t)+\mathbf{r},t) - \mathbf{u}(\mathbf{x}(\mathbf{X},t),t) \quad . \tag{5}$$

The velocity field $\mathbf{u}(\mathbf{x},t)$ is a realization of a turbulent incompressible flow. The vector $\mathbf{r}$ is an independent spatial variable. The usual Lagrangian kinematic notation and definitions are employed. The transfomation $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ gives the position at time t of the fluid particle which was at $\mathbf{X}$ at an initial time. The velocity field at point $\mathbf{x}$ is defined in terms of this transformation by

$$\mathbf{u}(\mathbf{x},t) = \left.\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} \equiv \frac{d\mathbf{x}}{dt} \ , \tag{6}$$

i.e., the velocity at a point $\mathbf{x}$ at time t is defined as the velocity of a Lagrangian particle which passes through $\mathbf{x}$ at this time. The inverse transformation $\mathbf{X} = \mathbf{X}(\mathbf{x},t)$ gives the initial position of the particle which is at $\mathbf{x}$ at time t.

The key calculation is

$$\left.\frac{\partial(\mathbf{v}(\mathbf{r},t;\mathbf{X})}{\partial t}\right|_{\mathbf{X}} = \frac{\partial \mathbf{u}(\mathbf{x}+\mathbf{r},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\frac{\partial \mathbf{u}(\mathbf{x}+\mathbf{r},t)}{\partial \mathbf{r}} - \frac{d\mathbf{u}(\mathbf{x},t)}{dt}. \tag{7}$$

Substituting the Navier-Stokes equation for $\partial \mathbf{u}(\mathbf{x}+\mathbf{r},t)/\partial t$, gives

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = -\frac{1}{\rho}\frac{\partial p(\mathbf{x}+\mathbf{r},t)}{\partial \mathbf{r}} + \nu\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{u}(\mathbf{x}+\mathbf{r},t) - \frac{d\mathbf{u}}{dt}, \tag{8}$$

where $\partial \mathbf{v}/\partial t$ is the derivative with $\mathbf{X}$ fixed. Since $\mathbf{u}(\mathbf{x},t)$ is independent of $\mathbf{r}$ and $d\mathbf{u}/dt$ is just a function of t when $\mathbf{X}$ is fixed, this can be written as (4), with $P = p(\mathbf{x}+\mathbf{r},t) + \rho\mathbf{r}\cdot d\mathbf{u}/dt$. The inertial force $d\mathbf{u}/dt$ on the right-hand-side of (8) has been absorbed into the pressure term.

2.2. *Scalar difference*

An equation for the difference in value of a passive scalar between two points, one of the points being the Lagrangian fluid particle, may be derived in the same way as for the velocity difference. A passive scalar $\theta(\mathbf{x},t)$ satisfies the equation

$$\frac{\partial \theta(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\theta(\mathbf{x},t) = D\nabla^2\theta(\mathbf{x},t) \quad , \tag{9}$$

while $\theta(\mathbf{x}+\mathbf{r},t)$ satisfies

$$\frac{\partial \theta(\mathbf{x}+\mathbf{r},t)}{\partial t} + \mathbf{u}(\mathbf{x}+\mathbf{r},t)\cdot\nabla\theta(\mathbf{x}+\mathbf{r},t) = D\nabla^2\theta(\mathbf{x}+\mathbf{r},t) \quad . \tag{10}$$

Now introduce the Lagrangian coordinate $\mathbf{X}$ with $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ being its position at time t, then

$$\left.\frac{\partial \theta(\mathbf{x}(\mathbf{X},t)+\mathbf{r},t)}{\partial t}\right|_{\mathbf{X},\mathbf{r}} = \frac{\partial \theta(\mathbf{x}+\mathbf{r},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\frac{\partial \theta(\mathbf{x}+\mathbf{r},t)}{\partial \mathbf{r}} \quad , \tag{11}$$

and substitute $\partial\theta(\mathbf{x}+\mathbf{r},t)/\partial t$ from (10), obtaining

$$\left.\frac{\partial \theta(\mathbf{x}(\mathbf{X},t)+\mathbf{r},t)}{\partial t}\right|_{\mathbf{X},\mathbf{r}} = -\mathbf{v}(\mathbf{r},t)\cdot\frac{\partial \theta(\mathbf{x}+\mathbf{r},t)}{\partial \mathbf{r}} + D\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\theta(\mathbf{x}+\mathbf{r},t) \quad . \tag{12}$$

Using the difference in θ between two points separated by distance $\mathbf{r}$, $\Delta\theta = \theta(\mathbf{x}(\mathbf{X},t) + \mathbf{r}, t) - \theta(\mathbf{x}(\mathbf{X},t),t)$, we obtain

$$\left.\frac{\partial \Delta\theta}{\partial t}\right|_{\mathbf{X},\mathbf{r}} + \left.\frac{\partial \theta(\mathbf{x},t)}{\partial t}\right|_{\mathbf{X}} + \mathbf{v}(\mathbf{r},t)\cdot\frac{\partial \Delta\theta}{\partial \mathbf{r}} = D\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\Delta\theta \quad . \tag{13}$$

This follows because $\theta(\mathbf{x},t)$ doesn't depend on $\mathbf{r}$. This may be written in the final form

$$\frac{\partial \Delta\theta}{\partial t} + \mathbf{v}(\mathbf{r},t)\cdot\frac{\partial \Delta\theta}{\partial \mathbf{r}} = D\frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\Delta\theta - \frac{d\theta(\mathbf{x},t)}{dt}, \tag{14}$$

where $\partial\Delta\theta/\partial t$ is understood to be the derivative with $\mathbf{X}$ and $\mathbf{r}$ fixed, a notation which can be used because the equation has no derivatives with respect to $\mathbf{X}$. The material derivative $d\theta/dt$ on the right-hand-side of (14) is just a function of time since $\mathbf{X}$ is fixed.

3. Matched asymptotic expansions

In this section the method of matched asymptotic expansions is used to derive the inertial range power law (1) for velocity differences. A similar power law for the scalar difference is derived in a subsection 3.2.

3.1. *Velocity power law*

First change variables in the Navier-Stokes equation (4). Use outer variables, with the rms velocity U as the characteristic velocity, and $L = U^3/\bar{\epsilon}$ as the characteristic length. Define dimensionless outer variables $\mathbf{v}_o = \mathbf{v}/U,\ \mathbf{r}_o = \mathbf{r}/L,\ \tau = tU/L$. Scaling with these variables gives

$$\frac{\partial \mathbf{v}_o}{\partial \tau} + \mathbf{v}_o\cdot\nabla_o\mathbf{v}_o = -\nabla_o P_o + R_L^{-1}\nabla_o^2\mathbf{v}_o \quad , \quad \nabla_o\cdot\mathbf{v}_o = 0, \tag{15}$$

where $R_L = UL/\nu$ and $\nabla_o = \partial/\partial\mathbf{r}_o$. An outer expansion as $R_L \to \infty$ may be taken in the form

$$\mathbf{v} = U\mathbf{v}_o, \quad \mathbf{v}_o = \mathbf{v}_{o,1} + R_L^{-1}\mathbf{v}_{o,2} + \cdots \ . \tag{16}$$

The first term $\mathbf{v}_{o,1}$ satisfies (15) without the viscous term.

Now rescale the variables so that the viscous term is retained as $R_L \to \infty$, by defining inner variables

$$\mathbf{r}_i = R_L^{-\alpha}\mathbf{r}_o, \quad \mathbf{v}_i = R_L^{-\beta}\mathbf{v}_o, \quad P_i = R_L^{-2\beta}P_o \quad . \tag{17}$$

Then (15) may be rewritten as

$$R_L^{-\beta+\alpha}\frac{\partial \mathbf{v}_i}{\partial \tau} + \mathbf{v}_i\cdot\nabla_i\mathbf{v}_i = -\nabla_i P_i + R_L^{-1-\beta-\alpha}\nabla_i^2\mathbf{v}_i \quad . \tag{18}$$

The conditions that the viscous terms be retained as $R_L \to \infty$ gives the condition $\beta + \alpha = -1$. Clearly another condition would be required to complete this, but it is left undetermined. Equation (18) takes the form (using $\beta = -1 - \alpha$)

$$R_L^{2\alpha+1}\frac{\partial \mathbf{v}_i}{\partial \tau} + \mathbf{v}_i\cdot\nabla_i\mathbf{v}_i = -\nabla_i P_i + \nabla_i^2\mathbf{v}_i, \tag{19}$$

which suggests an inner expansion

$$\mathbf{v} = UR_L^{-1-\alpha}\mathbf{v}_i, \quad \mathbf{v}_i = \mathbf{v}_{i,1} + R_L^{1+2\alpha}\mathbf{v}_{i,2} + \cdots \quad . \tag{20}$$

These two asymptotic expansions describe the same function in different variables.

If there is an overlap region where both are valid the method of matched asymptotic expansions may be used. Van Dyke's matching principle states that one should take the inner expansion of the outer expansion and set that equal to the outer expansion of the inner expansion. That prescription was carried out in detail in Lundgren (2003), yielding the functional relation

$$\mathbf{v}_{o,1}(\mathbf{r}_o) = R_L^{-1-\alpha}\mathbf{v}_{i,1}(R_L^{-\alpha}\mathbf{r}_o) \quad . \tag{21}$$

This can be satisfied in the limit $R_L \to \infty$ only if

$$\mathbf{v}_{o,1} = \mathbf{V}r_o^q \quad \text{and} \quad \mathbf{v}_{i,1} = \mathbf{V}r_i^q \quad .$$

Here $q = -(1+\alpha)/\alpha$ is undetermined (as is α). That is, substituting these into (21), with $r_i = R_L^{-1/(1+q)}r_o$ satisfies it exactly and any power other than q would give zero or infinity as $R_L \to \infty$. In these equations $\mathbf{V} = \mathbf{V}(\mathbf{x}, t, \hat{\mathbf{r}})$ is a dimensionless random variable; $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. The significant result is

$$\mathbf{v} = U\mathbf{V}\Big(\frac{r}{L}\Big)^q \tag{22}$$

in which both $\mathbf{V}$ and q are random variables, i.e. they depend on the realization of the turbulent flow. Longitudinal structure function ensemble averages at a fixed point $\mathbf{x}$ are then given by

$$B_p = \big\langle(\mathbf{v}\cdot\hat{\mathbf{r}})^p\big\rangle = U^p\big\langle(\mathbf{V}\cdot\hat{\mathbf{r}})^p\big(\frac{r}{L}\big)^{qp}\big\rangle \quad . \tag{23}$$

Since $\alpha = -1/(1+q)$ and $\beta = -q/(1+q)$ the inner and outer variables are related (from (17)) by $\mathbf{r}_i = R_L^{1/(1+q)}\mathbf{r}_o$ and $\mathbf{v}_i = R_L^{q/(1+q)}\mathbf{v}_o$, from which one sees that the inner length scale is $l_q = LR_L^{-1/(1+q)}$ and the inner velocity scale is $v_q = UR_L^{-q/(1+q)}$. (These satisfy the essential physical condition $l_q v_q/\nu \equiv 1$.) If q is *not* a r.v. but a simple constant, then it was shown in Lundgren (2003) that the additional physical hypothesis that the mean dissipation is finite as $R_L \to \infty$ implies that $q = 1/3$. In this case the inner scales are the Kolmogorov variables: $l_{1/3} = \eta = (\nu^3/\bar{\epsilon})^{1/4}$ and $v_{1/3} = v_K = (\nu\bar{\epsilon})^{1/4}$.

3.2. *Scalar power law*

An analysis of the scalar equation (14) proceeds in a similar manner. Outer scaling of length, velocity and time are the same as for the Navier-Stokes equation: $\mathbf{v}_o = \mathbf{v}/U,\ \mathbf{r}_o = \mathbf{r}/L,\ \tau = tU/L$. The scalar variables are made dimensionless with $\Theta =< \theta(\mathbf{x})^2 >^{1/2}$; $\Delta\theta_o = \Delta\theta/\Theta,\ \theta_o = \theta/\Theta$. Changing variables in (14) gives the outer equation for the scalar difference

$$\frac{\partial\Delta\theta_o}{\partial\tau} + \mathbf{v}_o(\mathbf{r}_o, t)\cdot\frac{\partial\Delta\theta_o}{\partial\mathbf{r}_o} = \frac{D}{\nu}R_L^{-1}\frac{\partial}{\partial\mathbf{r}_o}\cdot\frac{\partial}{\partial\mathbf{r}_o}\Delta\theta_o - \frac{d\theta_o}{d\tau} \quad . \tag{25}$$

Expanding this in an asymptotic series of form

$$\Delta\theta = \Theta\Delta\theta_o, \quad \Delta\theta_o = \Delta\theta_{o,1} + R_L^{-1}\Delta\theta_{o,2} + \cdots \tag{26}$$

results in the lowest order equation

$$\frac{\partial\Delta\theta_{o,1}}{\partial\tau} + \mathbf{v}_o(\mathbf{r}_o, t)\cdot\frac{\partial\Delta\theta_{o,1}}{\partial\mathbf{r}_o} = -\frac{d\theta_o}{d\tau}. \tag{27}$$

To get an inner equation rescale in such a way as to retain the diffusion term. This is the same as for the velocity: $\mathbf{r}_i = \mathbf{r}/l_q = R_L^{1/1+q}\mathbf{r}_o$, $\mathbf{v}_i = \mathbf{v}/v_q = R_L^{-q/1+q}\mathbf{v}_o$. The new one is $\Delta\theta_i = R_L^{-\beta_s}\Delta\theta_o$, where β_s is as yet undetermined. Making this change of variables gives

$$R_L^{-(1-q)/(1+q)}\frac{\partial\Delta\theta_i}{\partial\tau} + \mathbf{v}_i(\mathbf{r}_i,\tau)\cdot\frac{\partial\Delta\theta_i}{\partial\mathbf{r}_i} = \frac{D}{\nu}\frac{\partial}{\partial\mathbf{r}_i}\cdot\frac{\partial}{\partial\mathbf{r}_i}\Delta\theta_i - R_L^{\gamma}\frac{d\theta_i}{d\tau}, \tag{28}$$

where $\gamma = \beta_s - (1-q)/(1+q)$ is assumed to be negative. Expanding in asymptotic series

$$\Delta\theta = \Theta R_L^{-\beta_s}\Delta\theta_i \quad \Delta\theta_i = \Delta\theta_{i,1} + R_L^{\gamma}\Delta\theta_{i,2} + \cdots. \tag{29}$$

In the limit as $R_L \to \infty$ we have

$$\mathbf{v}_i(\mathbf{r}_i,\tau)\cdot\frac{\partial\Delta\theta_{i,1}}{\partial\mathbf{r}_i} = \frac{D}{\nu}\frac{\partial}{\partial\mathbf{r}_i}\cdot\frac{\partial}{\partial\mathbf{r}_i}\Delta\theta_{i,1}. \tag{30}$$

The matching arguement gives

$$\Delta\theta_{o,1}(\mathbf{r}_o,\tau) = R_L^{-\beta_s}\Delta\theta_{i,1}(R_L^{1/1+q}\mathbf{r}_o,\tau;D/\nu) \tag{31}$$

to be satisfied as $R_L \to \infty$. This requires

$$\Delta\theta_{o,1} = C_o r_o^{q_s}, \quad \Delta\theta_{i,1} = C_i r_i^{q_s} \quad \text{with } r_i = R_L^{1/1+q} r_o \tag{32}$$

in order for the powers to match. Substituting these back into (32) gives the added result $C_o = C_i R_L^{-\beta_s + q_s/(1+q)}$, from which we see $C_o = C_i$ and $\beta_s = q_s/(1+q)$, because C_o and C_i must be independent of R_L.

The major result is that in the inertial range (the overlap region)

$$\Delta\theta(\mathbf{r},\tau) = \Theta C_o(\hat{\mathbf{r}}, D/\nu)\left(\frac{r}{L}\right)^{q_s}, \tag{33}$$

where q_s is undetermined and could therefore be a random variable: C_o depends on the direction $\hat{\mathbf{r}}$ and perhaps on D/ν and it could also be a random variable which could depend on the random variable q_s.

4. Conclusions

This instantaneous power law can be used to study scalar intermittency by specifying appropriate statistical properties for the random variable q_s. That will be done in a future article. The power law will be used here only to derive the one dimensional scalar spectrum for isotropic homogeneous turbulence. We will use Yaglom's equation, the scalar equivalent of the Karman-Howarth equation,

$$< \mathbf{v}\cdot\hat{\mathbf{r}}\ (\Delta\theta)^2 > -2D\frac{\partial < (\Delta\theta)^2 >}{\partial r} = -\frac{4}{3}\epsilon_\theta r, \tag{34}$$

where ϵ_θ is the scalar dissipation. In the inertial convection range both viscosity and diffusivity are neglegible, so the diffusion term may be neglected and $\mathbf{v}\cdot\hat{\mathbf{r}} \sim U(r/L)^{1/3}$ (using the K41 exponent 1/3). Since $\Delta\theta \sim \Theta(r/L)^{q_s}$ it is clear from (34) that $q_s = 1/3$ and therefore

$$\left\langle(\Delta\theta)^2\right\rangle \sim \Theta^2(r/L)^{2/3} \quad . \tag{35}$$

The one dimensional scalar spectrum is given by

$$E_\theta = 2\int_0^\infty R_\theta \cos(kr)dr \tag{36}$$

where the autocorrelation R_θ is related to the second order structure function by

$$R_\theta = \frac{1}{2}\left(\Theta^2 - < (\Delta\theta)^2 >\right) \quad . \tag{37}$$

Taking $< (\Delta\theta)^2 >= \Theta^2(r/L)^{2/3}$ when $r/L < 1$ and $< (\Delta\theta)^2 >= \Theta^2$ when $r/L > 1$, i.e. extending the inertial range result out to $r/L = 1$, E_θ is given by

$$E_\theta = \int_0^L \Theta^2\left(1-(r/L)^{2/3}\right)\cos(kr)dr \quad . \tag{38}$$

Integrating this by parts gives

$$E_\theta = \frac{2\Theta^2}{3kL^{2/3}}\int_0^L r^{-1/3}\sin(kr)dr \quad . \tag{39}$$

Since the integral will exist, we can take the limit as $L \to \infty$ and obtain the result

$$E_\theta = \frac{2\Theta^2}{3L^{2/3}}\Gamma(2/3)\cos(\pi/6)\; k^{-5/3} \quad . \tag{40}$$

(Here $\int_0^\infty r^{-\nu}\sin(kr)dr = k^{\nu-1}\Gamma(1-\nu)\cos(\pi\nu/2)$, with $\nu = 1/3$ was used.) Thus we have

$$E_\theta = .78\frac{\Theta^2}{L^{2/3}}k^{-5/3}. \tag{41}$$

This is the same as Batchelor's result (1959), since $\Theta^2/L^{2/3}$ is equal to $\epsilon^{-1/3}\epsilon_\theta$. Batchelor obtained this by dimensional analysis, here we have derived it by analysis of the Navier-Stokes equation.

The spectrum in the viscous convective range is more problematic. this is where r is in the dissipative range, $r = O(\eta)$, but $D << \nu$ so it is still a convective range. In this case $\mathbf{v}\cdot\hat{\mathbf{r}} \sim (\epsilon/\nu)^{1/2}\, r$ in (34), with the diffusion term still neglected. A little inspection shows $\Delta\theta$ independent of r. This is where the problem lies in this approach since we need a power law. This is resolved by taking

$$\Delta\theta = \Theta_2(r/L_2)^{q_2} \tag{42}$$

and then taking the limit as $q_2 \to 0$. We can take $L_2 = b\eta$, with some fairly large constant b, and Θ_2 so it would approximately patch to $\Delta\theta = \Theta(r/L)^{1/3}$ when $r = L_2$, that is $\Theta_2 = \Theta(L_2/L)^{1/3}$. Then we can set this up with $< (\Delta\theta)^2 >= \Theta_2^2(r/L_2)^{2q_2}$ when $r < L_2$ and $< (\Delta\theta)^2 >= \Theta_2^2$ when $r > L_2$, and proceed as with the inertial convective range. We find

$$E_\theta = \frac{2q_2\Theta_2^2}{kL_2^{2q_2}}\int_0^{L_2} r^{2q_2-1}sin(kr)dr \quad . \tag{43}$$

Taking $L_2 \to \infty$ in the integral it gives $\pi k^{2q_2}/2$ and therefore we get

$$E_\theta = \frac{\pi q_2\Theta^2(L_2/L)^{2/3}}{L_2^{2q_2}}k^{-1-2q_2}. \tag{44}$$

As $q_2 \to 0$, with the appropriate choice of b, and with $(\eta/L)^{2/3} = R_L^{-1/2}$, this can be written

$$E_\theta = A\Theta^2 R_L^{-1/2} k^{-1}. \tag{45}$$

This is the same as Batchelor's $\epsilon_\theta \nu^{1/2} \epsilon^{-1/2} k^{-1}$.

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