Kinetic theory of plasmas: translational energy

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1. Motivation and objectives

Plasmas are ionized gas mixtures (either magnetized or not) that have many practical applications. For instance, lightning is a well-known natural plasma that has been studied for many years (Bazelyan & Raizer 2000). A second application is encountered in hypersonic flows; when a spacecraft enters into a planetary atmosphere at hypervelocity, the gas temperature and pressure strongly rise through a shockwave, consequently, dissociation and ionization of the gas particles occur in the shock layer. Hypersonic flow conditions are reproduced in dedicated windtunnels, such as plasmatrons, arc-jet facilities and shocktubes (Park 1990; Tirsky 1993; Sarma 2000). A third example was discovered approximately two decades ago, when large-scale electrical discharges were discovered in the mesosphere and lower ionosphere above large thunderstorms; these plasmas are now commonly referred to as sprites (Pasko 2007). Fourth, discharges at atmospheric pressure have received renewed attention in recent years due to their ability to enhance the reactivity of a variety of gas flows for applications ranging from surface treatment to flame stabilization and ignition (see Raizer 1991; Starikovskaia 2006, and references cited therein). Fifth, Hall thrusters are being developed to replace chemical systems for many on-orbit propulsion tasks on communications and exploration spacecraft (Boyd 2006). Finally, two important applications of magnetized plasmas are the laboratory thermonuclear fusion (Bobrova *et al.* 2005; Schnak *et al.* 2006) and the magnetic reconnection phenomenom in astrophysics (Yamada 2007).

Depending on the magnitude of the ratio of a reference particle mean free path to the system characteristic length (Knudsen number), two different approaches are generally followed to describe the transport of mass, momentum and energy in a plasma (Bird 1994): either a particle approach at high values of the Knudsen number (solution to the Boltzmann equation using Monte Carlo methods), or a fluid approach at low values (solution to macroscopic conservation equations by means of computational fluid dynamics methods). In this work, we study plasmas that can be described by a fluid approach, thus encompassing most the above-mentioned applications. In this case, kinetic theory can be used to obtain the governing conservation equations and transport fluxes of plasmas. Hence, closure of the problem is realized at the microscopic level by determining from experimental measurements either the potentials of interaction between the gas particles, or the cross-sections.

A complete model of plasmas should allow for the following physical phenomena to be described:

- Thermal non-equilibrium of the translational energy,
- Influence of the electromagnetic field,

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- Occurrence of reactive collisions,
- Excitation of internal degrees of freedom.

So far, no such unified model has been derived by means of kinetic theory. Besides, a derivation of the mathematical structure of the conservation equations also appears to be crucial in the design of the associated numerical methods. Based on our previous work, we investigate (Graille *et al*. 2007) the thermal non-equilibrium of the translational energy (Magin & Degrez 2004a) and the influence of the magnetic field (Giovangigli & Graille 2003). We generalize the Chapman-Enskog method within the context of a dimensional analysis of the Boltzmann equation, emphasizing the role of a multi-scale perturbation parameter on the collisional operator, the streaming operator and the collisional invariants of the Boltzmann equation. We then obtain macroscopic equations eventually leading to a sound entropy structure. Moreover, the system of equations is shown to be conservative and the purely convective system hyperbolic.

2. State-of-the-art and approach followed

Let us now describe in more detail how these issues are currently addressed in the literature. First, a multi-scale analysis is essential to resolve the Boltzmann equation governing the velocity distribution functions. We recall that a fluid can be described in the continuum limit, provided that the Knudsen number is small. Besides, in the case of plasmas, a thermal non-equilibrium may occur between the velocity distribution functions of the electrons and heavy particles (atoms, molecules and ions), given the strong disparity of mass between both types of species. Therefore, the square root of the ratio of the electron mass to a characteristic heavy-particle mass represents an additional small parameter to be accounted for in the derivation of an asymptotic solution to the Boltzmann equation. Literature abounds with expressions of the scaling for the perturbative solution method. For instance, significant results are given in Chmieleski & Ferziger (1967); Daybelge (1970); Devoto (1966); Kolesnikov (1974); Zhdanov (2002). Yet, Petit & Darrozes (1975) have suggested that the only sound scaling is obtained by means of a dimensional analysis of the Boltzmann equation. Subsequently, Degond & Lucquin (1996a,b) have established a formal theory of epochal relaxation based on such a scaling. In their study, the mean velocity of the electrons is shown to vanish in an inertial frame. Moreover, the heavy-particle diffusive fluxes were scarcely dealt with, since their work is restricted to a single type of heavy particles, and thus, no multi-component diffusion was to be found: in such a simplified context, the details of the interaction between the multi-component heavy particles and electrons degenerate and the positivity of the entropy production is straightforward. We will establish a theory based on a multi-scale analysis for a multi-component plasma (which includes the single heavy-particle case) where the mean electron velocity is the mean heavy-particle velocity in the absence of external forces. As an alternative, Magin & Degrez (2004a) have also proposed a model for a multi-component plasma based on a hydrodynamic referential. They have applied a multi-scale analysis to the derivation of the multi-component transport fluxes and coefficients. However, the proposed treatment of the collision operators is heuristic. Moreover, since the hydrodynamic velocity is used to define the referential instead of the mean heavy-particle velocity, the Chapman-Enskog method requires a transfer of lower order terms in the integral equation for the electron perturbation function to ensure mass conservation. Finally, we also emphasize that the development of models for plasmas in thermal equilibrium shall always be obtained as a particular case of the general theory.

Second, the magnetic field induces anisotropic transport fluxes when the electron collision frequency is lower than the electron cyclotron frequency of gyration around the magnetic lines. Braginskii (1958) has studied the case of fully ionized plasmas composed of one single ion species. Recently, Bobrova *et al.* (2005) have generalized the previous work to multi-component plasmas. However, the scaling used in both contributions does not comply with a dimensional analysis of the Boltzmann equation. Lucquin (1998, 2000) has investigated magnetized plasmas in the latter framework. Nevertheless, the same limitation is found for the diffusive fluxes as in Degond & Lucquin (1996a,b). Finally, Giovangigli & Graille (2003) have studied the Enskog expansion of magnetized plasmas and obtained macroscopic equations together with expressions of the transport fluxes and coefficients. Unfortunately, the difference of mass between electrons and heavy particles is not accounted for in their work.

Third, plasmas are strongly reactive gas mixtures. The kinetic mechanism comprises numerous reactions (see Capitelli *et al.* 2000): dissociation of molecules by electron and heavy-particle impact, three body recombination, ionization by electron and heavy-particle impact, associative ionization, dissociative recombination, radical reactions, charge exchange. Giovangigli & Massot (1998) have derived a formal theory of chemically reacting flows for the case of neutral gases. Subsequently, Giovangigli & Graille (2003) have studied the case of ionized gases. We recall that their scaling does not take into account the mass disparity between electrons and heavy particles. Besides, in chemical equilibrium situations, a long-standing theoretical debate in the literature deals with non-uniqueness of the two-temperature Saha equation. Recently, Giordano & Capitelli (2001) have emphasized the importance of the physical constraints imposed on the system by using a thermodynamic approach. A derivation based on kinetic theory should further improve the understanding of the problem. Choquet & Lucquin (2005) have already studied the case of ionization reactions by electron impact.

Fourth, molecules rotate and vibrate, and moreover, the electronic energy levels of atoms and molecules are excited. Generally, the rotational energy mode is considered to be fully excited above a few Kelvins. In a plasma environment, the vibrational and electronic energy modes are also significantly excited. The detailed treatment of the internal degrees of freedom is beyond the scope of the present study, where we will address only the translational energy in the context of thermal non-equilibrium. The reader is thus referred to the specialized literature (Brun 2006; McCourt *et al.* 1990; Nagnibeda & Kustova 2003).

Fifth, the development of numerical methods to solve conservation equations relies on the identification of their intrinsic mathematical structure. For instance, the system of conservation equations of mass, momentum and energy is found to be nonconservative for thermal non-equilibrium ionized gases. This formulation is therefore not suitable for numerical approximations of discontinuous solutions. Coquel & Marmignon (1998) have derived a well-posed conservative formulation based on a phenomenological approach. Nevertheless, their derivation is not consistent with a scaling issued from a dimensional analysis. We will show that kinetic theory, based on first principles, naturally allows for an adequate mathematical structure to be obtained, as opposed to the phenomenological approach.

In Graille *et al.* (2007), we propose to derive the multi-component plasma conservation equations of mass, momentum and energy, as well as the expressions of the associated multi-component transport fluxes and coefficients. The multi-component Navier-Stokes regime is reached for the heavy particles, which follow a hyperbolic scaling, and is coupled with first-order drift-diffusion equations for the electrons, which follow a parabolic scaling. Here, we deal with equations at order ε^1, hence one order beyond the expansion commonly investigated in the literature. The derivation relies on kinetic theory and is based on the ansatz that the particles constitutive of the plasma are inert and only possess translational degrees of freedom. The electromagnetic field influence is accounted for. We express the Boltzmann equation in a noninertial reference frame. We show that the mean heavy-particle velocity is a suitable choice for the referential velocity. This step is essential to establish a formalism where the electrons follow the bulk movement of the plasma. Then, we define the reference quantities of the system in order to derive the scaling of the Boltzmann equation from a dimensional analysis. The multi-scale aspect occurs in both the streaming operator and collision operator of the Boltzmann equation. Consequently, the scaling of the partial collision operators between unlike particles requires a special treatment. In addition, we determine the space of collisional invariants associated with the electrons and the heavy particles, respectively. We then use a Chapman-Enskog method to derive macroscopic conservation equations. The system is examined at successive orders of approximation, each corresponding to a physical time scale. For that purpose, scalar products and linearized collision operators are introduced. We also establish the formal existence and uniqueness of a solution to the Boltzmann equation. The multi-component transport coefficients are then calculated in terms of bracket operators whose mathematical structure allows for the sign of the transport coefficients to be determined; in particular the Kolesnikov effect, or the expressions of the crossed contributions to the mass and energy transport fluxes coupling the electrons and heavy particles. The explicit expressions of the transport coefficients can be obtained by means of a Galerkin spectral method disregarded in this contribution. Finally, the first and second laws of thermodynamics are proved to be satisfied by deriving a total energy equation and an entropy equation. Then, we establish a conservative formulation, from a fluid standpoint, of the system of macroscopic equations. We also identify the mathematical structure of the purely convective system. Hence, we demonstrate that kinetic theory shall be used as a guideline in the derivation of the macroscopic conservation equations as well as in the design of the associated numerical methods.

Beyond the obvious interest of such a study from the perspective of the applications and design of numerical schemes, the present contribution also yields a formal kinetic theory of mixtures of separate masses, where the light species obey a parabolic scaling and the heavy species obey a hyperbolic scaling. The original treatment of the two different scalings for fluid flows was first provided by Bardos *et al*. (1991). In their study, the purely hyperbolic scaling yields the compressible gas dynamics equations, whereas the purely parabolic scaling leads to the low Mach number limit. These scalings are quite classical and both of them can be used for various asymptotics such as the Vlasov-Navier-Stokes equations in different regimes investigated by Goudon *et al*. (2005a,b). A rigourous derivation of a set of macroscopic equations in the situation where the hyperbolic and parabolic scalings are entangled in the same problem is an original result obtained in the present work.

3. Boltzmann equation

3.1. *Assumptions*

(a) Our plasma is described by the kinetic theory of gases based on classical mechanics, provided that: 1) The mean distance between the gas particles $1/(n^0)^{1/3}$ is larger than the thermal de Broglie wavelength, where n^0 is a reference number density, 2) The square of the ratio of the electron thermal speed V_{e}^0 to the speed of light is small.

(b) The reactive collisions and particle internal energy are not accounted for.

(c) The particle interactions are modeled as binary encounters by means of a Boltzmann collision operator, provided that: 1) The gas is sufficiently dilute, i.e., the mean distance between the gas particles $1/(n^0)^{1/3}$ is larger than the particle interaction distance $(\sigma^0)^{1/2}$, where σ^0 is a reference differential cross-section common to all species, 2) The plasma parameter, quantity proportional to the number of electrons in a sphere of radius equal to the Debye length, is supposed to be large. Hence, multiple charged particle interactions are treated as equivalent binary collisions by means of a Coulomb potential screened at the Debye length.

(d) A plasma is composed of electrons and a multi-component mixture of heavy particles (atoms, molecules and ions). The ratio of the electron mass m_{e}^0 to a characteristic heavy-particle mass m_h^0 is such that the nondimensional number $\varepsilon = \sqrt{m_{\mathrm{e}}^0/m_h^0}$ is small.

(e) The pseudo-Mach number, defined as a reference hydrodynamic velocity divided by the heavy-particle thermal speed $M_h = v^0/V_h^0$, is supposed to be of the order of 1.

(f) The macroscopic time scale t^0 is assumed to be comparable with the heavy-particle kinetic time scale t_h^0 divided by ε. The macroscopic length scale is based on a reference convective length $L^0 = v^0 t^0$.

(g) The reference electrical and thermal energies of the system are of the same order of magnitude.

The mean free path l^0 and macroscopic length scale L^0 allow for the Knudsen number to be defined $Kn = l^0/L^0$. It can be shown that this quantity is small, provided that assumptions *d*-*f* are satisfied. A continuum description of the system is therefore deemed to be possible.

3.2. *Dimensional analysis*

The temporal evolution of the velocity distribution function f_i of the plasma particle i is governed in the phase space $(\boldsymbol{x}, \boldsymbol{c}_i)$ by the Boltzmann equation, where quantity $\boldsymbol{c}_i$ stands for the particle velocity in an inertial frame. The choice of an adequate referential proves to be essential to conduct a rigorous multi-scale analysis. Given the strong disparity of mass between the electrons and heavy particles, a frame linked with the heavy particles appears to be a natural choice for plasmas. Based on the following definition of the peculiar velocities

$$\boldsymbol{C}_i = \boldsymbol{c}_i - \boldsymbol{v}_h, \tag{3.1}$$

where $\boldsymbol{v}_h$ is the mean heavy-particle velocity, the Boltzmann equation can be expressed in nondimensional form, respectively for the electrons and heavy particles, as

$$\begin{aligned}\partial_t f_{\mathrm{e}} + \tfrac{1}{\varepsilon M_h}(\boldsymbol{C}_{\mathrm{e}} + \varepsilon M_h \boldsymbol{v}_h)\cdot\boldsymbol{\partial_x} f_{\mathrm{e}} + \varepsilon^{-(1+b)} q_e \big[(\boldsymbol{C}_{\mathrm{e}} + \varepsilon M_h \boldsymbol{v}_h)\wedge\boldsymbol{B}\big]\cdot\boldsymbol{\partial}_{\boldsymbol{C}_{\mathrm{e}}} f_{\mathrm{e}} \\ + \left(\tfrac{1}{\varepsilon M_h} q_e \boldsymbol{E} - \varepsilon M_h \tfrac{\mathrm{D}\boldsymbol{v}_h}{\mathrm{D}t}\right)\cdot\boldsymbol{\partial}_{\boldsymbol{C}_{\mathrm{e}}} f_{\mathrm{e}} - \boldsymbol{\partial}_{\boldsymbol{C}_{\mathrm{e}}} f_{\mathrm{e}} \otimes \boldsymbol{C}_{\mathrm{e}} : \boldsymbol{\partial_x} \boldsymbol{v}_h = \tfrac{1}{\varepsilon^2}\mathcal{J}_{\mathrm{e}},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\partial_t f_i + \tfrac{1}{M_h}(\boldsymbol{C}_i + M_h \boldsymbol{v}_h)\cdot\boldsymbol{\partial_x} f_i + \varepsilon^{1-b} \tfrac{q_i}{m_i} \big[(\boldsymbol{C}_i + M_h \boldsymbol{v}_h)\wedge\boldsymbol{B}\big]\cdot\boldsymbol{\partial}_{\boldsymbol{C}_i} f_i \\ + \left(\tfrac{1}{M_h}\tfrac{q_i}{m_i}\boldsymbol{E} - M_h \tfrac{\mathrm{D}\boldsymbol{v}_h}{\mathrm{D}t}\right)\cdot\boldsymbol{\partial}_{\boldsymbol{C}_i} f_i - \boldsymbol{\partial}_{\boldsymbol{C}_i} f_i \otimes \boldsymbol{C}_i : \boldsymbol{\partial_x}\boldsymbol{v}_h = \tfrac{1}{\varepsilon}\mathcal{J}_i, \quad i \in \mathrm{H},\end{aligned} \tag{3.3}$$

where symbol H denotes the set of indices of the heavy particles and $\mathcal{J}_{\mathrm{e}}$, $\mathcal{J}_i$, $i \in \mathrm{H}$, the electron and heavy-particle collision operators (see Graille *et al.* 2007). Symbol $\boldsymbol{E}$ stands for the electric field; $\boldsymbol{B}$, the magnetic field; q_i, the particle charge; and m_i, its mass. The integer parameter $b \leq 1$ describes the intensity of the magnetic field. Note that Eq. 3.2 for the light species is typical of a parabolic scaling which corresponds to the low Mach number limit for the electron gas, whereas Eq. 3.3 for the heavy species is typical of a hyperbolic scaling that corresponds to compressible gas dynamics for the heavy-species gas mixture (Bardos *et al*. 1991). The present scaling is thus intermediate between the usual cases; we will have to identify the mathematical structure of the resulting system of macroscopic equations.

3.3. *Collisional invariants*

Definition 3.1 *The space of scalar electron collisional invariants $\mathcal{I}_{\mathrm{e}}$ is spanned by the following elements*

$$\begin{cases} \hat{\psi}_{\mathrm{e}}^1 = 1, \\ \hat{\psi}_{\mathrm{e}}^2 = \frac{1}{2}\boldsymbol{C}_{\mathrm{e}}\cdot\boldsymbol{C}_{\mathrm{e}}, \end{cases} \tag{3.4}$$

and the space of scalar heavy-particle collisional invariants $\mathcal{I}_h$ by

$$\begin{cases} \hat{\psi}_h^j = \big(m_i\delta_{ij}\big)_{i\in\mathrm{H}}, & j \in \mathrm{H}, \\ \hat{\psi}_h^{n^{\mathrm{H}}+\nu} = \big(m_i C_{i\nu}\big)_{i\in\mathrm{H}}, & \nu \in \{1,2,3\}, \\ \hat{\psi}_h^{n^{\mathrm{H}}+4} = \big(\frac{1}{2}m_i\boldsymbol{C}_i\cdot\boldsymbol{C}_i\big)_{i\in\mathrm{H}}, \end{cases} \tag{3.5}$$

where symbol n^{H} denotes the cardinality of the set H.

For the families ξ_{e}, ζ_{e}, concerning the electrons, and $\xi_h = (\xi_i)_{i\in\mathrm{H}}$, $\zeta_h = (\zeta_i)_{i\in\mathrm{H}}$, concerning the heavy particles, we introduce two scalar products

$$\langle\!\langle \xi_{\mathrm{e}}, \zeta_{\mathrm{e}} \rangle\!\rangle_{\mathrm{e}} = \int \xi_{\mathrm{e}} \odot \bar{\zeta}_{\mathrm{e}} \, \mathrm{d}\boldsymbol{C}_{\mathrm{e}}, \qquad \langle\!\langle \xi_h, \zeta_h \rangle\!\rangle_h = \sum_{j\in\mathrm{H}} \int \xi_j \odot \bar{\zeta}_j \, \mathrm{d}\boldsymbol{C}_j, \tag{3.6}$$

where symbol $\odot$ stands for the maximum contracted product in space and symbol $^-$, the transpose conjugate operation. Then, macroscopic properties are expressed as partial scalar products of the distribution functions and collisional invariants

$$\begin{cases} \langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^1 \rangle\!\rangle_{\mathrm{e}} = \rho_{\mathrm{e}}, \\ \langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^2 \rangle\!\rangle_{\mathrm{e}} = \rho_{\mathrm{e}} e_{\mathrm{e}}, \end{cases} \tag{3.7}$$

and

$$\begin{cases} \langle\!\langle f_h, \hat{\psi}_h^i \rangle\!\rangle_h = \rho_i, & i \in \mathrm{H}, \\ \langle\!\langle f_h, \hat{\psi}_h^{n^{\mathrm{H}}+\nu} \rangle\!\rangle_h = 0, & \nu \in \{1,2,3\}, \\ \langle\!\langle f_h, \hat{\psi}_h^{n^{\mathrm{H}}+4} \rangle\!\rangle_h = \rho_h e_h. \end{cases} \tag{3.8}$$

Symbol ρ_i, stands for the mass density of particle i; $\rho_h = \sum_{j\in\mathrm{H}} \rho_j$, the heavy-particle mass density; e_{e}, the electron thermal energy per unit mass; and e_h, the heavy-particle thermal energy per unit mass. Moreover, translational temperatures are introduced as averaged thermal energies

$$T_{\mathrm{e}} = \frac{2}{3n_{\mathrm{e}}} \langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^2 \rangle\!\rangle_{\mathrm{e}}, \tag{3.9}$$

$$T_h = \frac{2}{3n_h} \langle\!\langle f_h, \hat{\psi}_h^{n^{\mathrm{H}}+4} \rangle\!\rangle_h, \tag{3.10}$$

where the heavy-particle number density reads $n_h = \sum_{j\in\mathrm{H}} n_j$.

Table 1. Chapman-Enskog steps.

Order	Time	Heavy particles	Electrons
ε^{-2}	t_{e}^0		Expression of f_{e}^0 Thermalization (T_{e})
ε^{-1}	t_h^0	Expression of f_i^0, $i \in \mathrm{H}$ Thermalization (T_h)	Equation for ϕ_{e} Zero-order momentum relation
ε^0	t^0	Equation for ϕ_i, $i \in \mathrm{H}$ Euler equations	Equation for ϕ_{e}^2 Zero-order drift-diffusion equations First-order momentum relation
ε	$\frac{t^0}{\varepsilon}$	Navier-Stokes equations	First-order drift-diffusion equations

4. Generalized Chapman-Enskog method

We use an Enskog expansion to derive an approximate solution to the Boltzmann Eqs. 3.2-3.3 by expanding the species distribution functions as

$$f_{\mathrm{e}} = f_{\mathrm{e}}^0(1 + \varepsilon\phi_{\mathrm{e}} + \varepsilon^2\phi_{\mathrm{e}}^2 + \varepsilon^3\phi_{\mathrm{e}}^3) + \mathcal{O}(\varepsilon^4), \tag{4.1}$$

$$f_i = f_i^0(1 + \varepsilon\phi_i + \varepsilon^2\phi_i^2) + \mathcal{O}(\varepsilon^3), \qquad i \in \mathrm{H}, \tag{4.2}$$

and by imposing that the zero-order contributions f_{e}^0 and $f_h^0 = (f_i^0)_{i\in\mathrm{H}}$ yield the local macroscopic properties

$$\langle\!\langle f_{\mathrm{e}}^0, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}} = \langle\!\langle f_{\mathrm{e}}, \hat{\psi}_{\mathrm{e}}^l \rangle\!\rangle_{\mathrm{e}}, \qquad l \in \{1, 2\}, \tag{4.3}$$

$$\langle\!\langle f_h^0, \hat{\psi}_h^l \rangle\!\rangle_h = \langle\!\langle f_h, \hat{\psi}_h^l \rangle\!\rangle_h, \qquad l \in \{1, \dots, n^{\mathrm{H}}{+}4\}. \tag{4.4}$$

Hence, based on the dimensional analysis, the electron Boltzmann equation 3.2 appears to be

$$\begin{aligned}\varepsilon^{-2}\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^0) + \varepsilon^{-1}\mathscr{D}_{\mathrm{e}}^{-1}(f_{\mathrm{e}}^0, \phi_{\mathrm{e}}) + \mathscr{D}_{\mathrm{e}}^0(f_{\mathrm{e}}^0, \phi_{\mathrm{e}}, \phi_{\mathrm{e}}^2) + \varepsilon\mathscr{D}_{\mathrm{e}}^1(f_{\mathrm{e}}^0, \phi_{\mathrm{e}}, \phi_{\mathrm{e}}^2, \phi_{\mathrm{e}}^3)\\ = \varepsilon^{-2}\mathscr{J}_{\mathrm{e}}^{-2} + \varepsilon^{-1}\mathscr{J}_{\mathrm{e}}^{-1} + \mathscr{J}_{\mathrm{e}}^0 + \varepsilon\mathscr{J}_{\mathrm{e}}^1 + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{4.5}$$

Likewise, the heavy-particle Boltzmann equation 3.3 is found to be

$$\mathscr{D}_i^0(f_i^0) + \varepsilon\mathscr{D}_i^1(f_i^0, \phi_i) = \varepsilon^{-1}\mathscr{J}_i^{-1} + \mathscr{J}_i^0 + \varepsilon\mathscr{J}_i^1 + \mathcal{O}(\varepsilon^2), \qquad i \in \mathrm{H}. \tag{4.6}$$

In the Chapman-Enskog method, the plasma is observed at successive orders of the ε parameter equivalent to as many time scales: the electron kinetic time scale t_{e}^0, the heavy-particle kinetic time scale t_h^0, the macroscopic time scale t^0, and the macroscopic time scale divided by ε. The micro- and macroscopic equations derived at each order are reviewed in Table 1. The resolubility of the electron and heavy-particle perturbation functions is classically based on the identification of the kernel of the linearized collision operators and space of scalar collisional invariants of both types of species. The quasi-equilibrium solutions are Maxwell-Boltzmann velocity distribution functions at the electron temperature or the heavy-particle temperature depending on the type of species, Hence allowing for thermal non-equilibrium to occur.

Proposition 4.1 *The zero-order electron distribution function f_{e}^0, solution to Eq. 4.5 at order ε^{-2}, i.e., $\mathscr{D}_{\mathrm{e}}^{-2}(f_{\mathrm{e}}^0) = \mathcal{J}_{\mathrm{e}}^{-2}$, that satisfies the scalar constraints 4.3 is a Maxwell-Boltzmann distribution function at the electron temperature*

$$f_{\mathrm{e}}^0 = n_{\mathrm{e}} \left(\frac{1}{2\pi T_{\mathrm{e}}}\right)^{3/2} \exp\left(-\frac{1}{2T_{\mathrm{e}}}\boldsymbol{C}_{\mathrm{e}}\cdot\boldsymbol{C}_{\mathrm{e}}\right). \tag{4.7}$$

Proposition 4.2 *Considering f_{e}^0 given by Eq. 4.7, the zero-order family of heavy-particle distribution functions f_h^0 solution to Eq. 4.6 at order ε^{-1}, i.e., $\mathcal{J}_i^{-1} = 0$, $i \in \mathrm{H}$, that satisfies the scalar constraints 4.4 is a family of Maxwell-Boltzmann distribution functions at the heavy-particle temperature*

$$f_i^0 = n_i \left(\frac{m_i}{2\pi T_h}\right)^{3/2} \exp\left(-\frac{m_i}{2T_h}\boldsymbol{C}_i\cdot\boldsymbol{C}_i\right), \quad i \in \mathrm{H}. \tag{4.8}$$

At order ε^1, the set of macroscopic conservation equations of mass, momentum and energy comprises multi-component Navier-Stokes equations for the heavy particles, which follow a hyperbolic scaling, and first-order drift-diffusion equations for the electrons, which follow a parabolic scaling. The first-order conservation equations of heavy-particle mass, momentum and energy read

$$\partial_t \rho_i + \boldsymbol{\partial_x}\cdot(\rho_i \boldsymbol{v}_h + \tfrac{\varepsilon}{M_h}\rho_i \boldsymbol{V}_i) = 0, \quad i \in \mathrm{H}, \tag{4.9}$$

$$\begin{aligned}\partial_t(\rho_h \boldsymbol{v}_h) + \boldsymbol{\partial_x}\cdot(\rho_h \boldsymbol{v}_h\otimes\boldsymbol{v}_h + \tfrac{1}{M_h^2}p\mathbb{1}) = -\tfrac{\varepsilon}{M_h^2}\boldsymbol{\partial_x}\cdot\boldsymbol{\Pi}_h + \tfrac{1}{M_h^2}nq\boldsymbol{E}&\\ + [\delta_{b0}\mathbf{I}_0 + \delta_{b1}\mathbf{I}]\wedge\boldsymbol{B},&\end{aligned} \tag{4.10}$$

$$\begin{aligned}\partial_t(\rho_h e_h) + \boldsymbol{\partial_x}\cdot(\rho_h e_h \boldsymbol{v}_h) = -(p_h\mathbb{1} + \varepsilon\boldsymbol{\Pi}_h):\boldsymbol{\partial_x}\boldsymbol{v}_h - \tfrac{\varepsilon}{M_h}\boldsymbol{\partial_x}\cdot\boldsymbol{q}_h + \tfrac{\varepsilon}{M_h}\boldsymbol{J}_h\cdot\boldsymbol{E}'&\\ + \Delta E_h^0 + \varepsilon\Delta E_h^1.&\end{aligned} \tag{4.11}$$

The first-order conservation equations of electron mass and energy read

$$\partial_t \rho_{\mathrm{e}} + \boldsymbol{\partial_x}\cdot\Big[\rho_{\mathrm{e}}\big(\boldsymbol{v}_h + \tfrac{1}{M_h}(\boldsymbol{V}_{\mathrm{e}} + \varepsilon\boldsymbol{V}_{\mathrm{e}}^2)\big)\Big] = 0, \tag{4.12}$$

$$\begin{aligned}\partial_t(\rho_{\mathrm{e}} e_{\mathrm{e}}) + \boldsymbol{\partial_x}\cdot(\rho_{\mathrm{e}} e_{\mathrm{e}} \boldsymbol{v}_h) = -p_{\mathrm{e}}\boldsymbol{\partial_x}\cdot\boldsymbol{v}_h - \tfrac{1}{M_h}\boldsymbol{\partial_x}\cdot\left(\boldsymbol{q}_{\mathrm{e}} + \varepsilon\boldsymbol{q}_{\mathrm{e}}^2\right)&\\ + \tfrac{1}{M_h}\left(\boldsymbol{J}_{\mathrm{e}} + \varepsilon\boldsymbol{J}_{\mathrm{e}}^2\right)\cdot\boldsymbol{E}' + \delta_{b0}\varepsilon M_h \boldsymbol{J}_{\mathrm{e}}\cdot\boldsymbol{v}_h\wedge\boldsymbol{B} + \Delta E_{\mathrm{e}}^0 + \varepsilon\Delta E_{\mathrm{e}}^1.&\end{aligned} \tag{4.13}$$

Quantity $p = p_{\mathrm{e}}+p_h$ is the mixture pressure, where p_{e} is the electron partial pressure and p_h, the heavy-particle partial pressure. The transport fluxes are defined for the electrons as first- and second-order diffusion velocity, $\boldsymbol{V}_{\mathrm{e}}$, $\boldsymbol{V}_{\mathrm{e}}^2$, heat flux, $\boldsymbol{q}_{\mathrm{e}}$, $\boldsymbol{q}_{\mathrm{e}}^2$, and conduction current, $\boldsymbol{J}_{\mathrm{e}}$, $\boldsymbol{J}_{\mathrm{e}}^2$, as well as for the heavy particles as first-order species diffusion velocities, $\boldsymbol{V}_i$, $i \in \mathrm{H}$, heat flux, $\boldsymbol{q}_h$, viscous tensor $\boldsymbol{\Pi}_h$, and conduction current, $\boldsymbol{J}_h$. The total conduction current reads $\mathbf{I}_0$ in the case where $b = 0$ (strong ionization) and $\mathbf{I}$ in the case $b = 1$ (weak ionization). The zero- and second-order terms of energy exchanged in collisions between electrons and heavy particles read ΔE_h^0, ΔE_h^1 from the heavy-particle standpoint, and ΔE_{e}^0, ΔE_{e}^1 from the electron standpoint. The transport coefficients have been written in terms of bracket operators in Graille *et al.* (2007); both electron and heavy-particle transport coefficients exhibit anisotropy, provided that the magnetic field is strong. We have also proposed a complete description of the Kolesnikov effect, i.e., the crossed contributions to the mass and energy transport fluxes coupling the electrons and heavy particles. This effect, appearing in multi-component plasmas, is essential to obtain a positive entropy production. In the case of single heavy species plasmas considered by Degond & Lucquin (1996a,b), the Kolesnikov effect is not present. Therefore, the details of the interaction between the multi-component heavy particles and electrons degenerate and the positivity of the entropy production is straightforward.

5. Conservation equations and mathematical structure

The properties of electron and heavy-particle transport matrices can be established by using the mathematical structure of the bracket operators. In particular, the properties of symmetry and positivity imply that the second law of thermodynamics is satisfied as shown by deriving an equation of conservation of the entropy s having a positive entropy production rate $\Upsilon \geq 0$. Moreover, the first law of thermodynamics is also verified by deriving an equation for the total energy $\mathcal{E} = \rho_{\mathrm{e}} e_{\mathrm{e}} + \rho_h e_h + M_h^2 \rho_h \frac{1}{2}|\boldsymbol{v}_h|^2$. The system of mass, momentum, total energy and entropy equations is conservative from a fluid standpoint in the variables

$$U = [\rho_{\mathrm{e}},\ (\rho_i)_{i\in\mathrm{H}},\ \rho_h \boldsymbol{v}_h,\ \mathcal{E},\ \rho s]^T,$$

that reads

$$\partial_t U + \boldsymbol{\partial_x}\cdot\boldsymbol{F} + \boldsymbol{\partial_x}\cdot\boldsymbol{\mathcal{F}} = \Omega, \tag{5.1}$$

with the convective fluxes

$$\boldsymbol{F} = [\rho_{\mathrm{e}} \boldsymbol{v}_h,\ (\rho_i)_{i\in\mathrm{H}} \boldsymbol{v}_h,\ \rho_h \boldsymbol{v}_h \otimes \boldsymbol{v}_h + \frac{1}{M_h^2} p \mathbb{1},\ \mathcal{H} \boldsymbol{v}_h,\ \rho s \boldsymbol{v}_h]^T,$$

where the total enthalpy reads $\mathcal{H} = \rho\mathcal{E} + p$, the diffusive fluxes

$$\boldsymbol{\mathcal{F}} = [\frac{\rho_{\mathrm{e}}}{M_h}(\boldsymbol{V}_{\mathrm{e}} + \varepsilon \boldsymbol{V}_{\mathrm{e}}^2), \frac{\varepsilon}{M_h}(\rho_i \boldsymbol{V}_i)_{i\in\mathrm{H}},\ \frac{\varepsilon}{M_h^2}\boldsymbol{\Pi}_h,\ \frac{\varepsilon}{M_h^2}\boldsymbol{\Pi}_h\cdot\boldsymbol{v}_h + \frac{1}{M_h}\boldsymbol{\mathcal{Q}},\ \boldsymbol{\mathcal{J}}]^T,$$

where the total heat flux reads $\boldsymbol{\mathcal{Q}}$ and the entropy flux $\boldsymbol{\mathcal{J}}$, and finally the source terms

$$\Omega = [0, 0, \frac{nq}{M_h^2}\boldsymbol{E} + (\delta_{b0}\mathbf{I}_0 + \delta_{b1}\mathbf{I})\wedge\boldsymbol{B},\ \mathbf{I}\cdot\boldsymbol{E},\ \Upsilon]^T,$$

where symbol n stands for the mixture number density and q, its charge. We then extract a purely convective system from Eq. 5.1

$$\partial_t U + \boldsymbol{\partial_x}\cdot\boldsymbol{F} = \Omega', \tag{5.2}$$

where the convective source terms are given by

$$\Omega' = [0,\ 0,\ \frac{nq}{M_h^2}\boldsymbol{E} + (\delta_{b0} + \delta_{b1})\mathbf{I}'\wedge\boldsymbol{B},\ \mathbf{I}'\cdot\boldsymbol{E},\ \Upsilon']^T,$$

the current $\mathbf{I}' = nq\boldsymbol{v}_h$, and the entropy production rate

$$\Upsilon' = \frac{(T_{\mathrm{e}} - T_h)^2}{T_{\mathrm{e}} T_h} \sum_{j\in\mathrm{H}} \frac{n_j}{m_j}\nu_{j\mathrm{e}}.$$

Symbol $\nu_{i\mathrm{e}}$, $i \in \mathrm{H}$, is the collision frequency of the electron heavy-particle interaction. The purely convective system given in Eq. 5.2 is rewritten in a quasi-linear form

$$\partial_t W + \boldsymbol{A}\cdot\boldsymbol{\partial_x} W = \Omega'_W, \tag{5.3}$$

by means of the variables

$$W = [\rho_{\mathrm{e}},\ (\rho_i)_{i\in\mathrm{H}},\ \boldsymbol{v}_h,\ p_{\mathrm{e}},\ p_h]^T,$$

the source terms

$$\Omega'_W = [0,\ 0,\ \frac{nq}{M_h^2\rho_h}\boldsymbol{E} + \frac{1}{\rho_h}(\delta_{b0}+\delta_{b1})\mathbf{I}'\wedge\boldsymbol{B},\ \tfrac{2}{3}\Delta E_{\mathrm{e}}^0,\ \tfrac{2}{3}\Delta E_h^0]^T,$$

and the Jacobian matrices

$$A_\nu = \begin{pmatrix} v_{h\nu} & 0 & \rho_{\mathrm{e}}\boldsymbol{e}_\nu^T & 0 & 0 \\ 0 & v_{h\nu}(\delta_{ij})_{i,j\in\mathrm{H}} & (\rho_i)_{i\in\mathrm{H}}\boldsymbol{e}_\nu^T & 0 & 0 \\ 0 & 0 & v_{h\nu}\mathbb{1} & \frac{1}{M_h^2\rho_h}\boldsymbol{e}_\nu & \frac{1}{M_h^2\rho_h}\boldsymbol{e}_\nu \\ 0 & 0 & \gamma p_{\mathrm{e}}\boldsymbol{e}_\nu^T & v_{h\nu} & 0 \\ 0 & 0 & \gamma p_h\boldsymbol{e}_\nu^T & 0 & v_{h\nu} \end{pmatrix}, \quad \nu\in\{1,2,3\}, \tag{5.4}$$

where the specific heat ratio reads $\gamma = 5/3$ and symbol $\boldsymbol{e}_\nu$ stands for the unit vector in the ν direction.

For any direction defined by the unit vector $\boldsymbol{n}$, the matrix $\boldsymbol{n}\cdot\boldsymbol{A}$ is shown to be diagonalizable with real eigenvalues and a complete set of eigenvectors. There are two nonlinear acoustic fields with the eigenvalues $\boldsymbol{v}_h\cdot\boldsymbol{n}\pm c$, where the sound speed is given by $c^2 = p/(\rho_h M_h^2)$, and linearly degenerate fields with the eigenvalue $\boldsymbol{v}_h\cdot\boldsymbol{n}$ of multiplicity $n^{\mathrm{H}}+4$. Thus, the macroscopic system of conservation equations derived from kinetic theory in the proposed mixed hyperbolic-parabolic scaling has a hyperbolic structure, as far as the convective part of the system is concerned. Such a property is far from being obvious since the obtained sound speed involves the electron pressure (considering that the rigorous derivation of the momentum equation of the heavy particles involves many analytical steps).

6. Further development

The explicit expressions of the diffusion coefficients, thermal diffusion coefficients, viscosity, and partial thermal conductivities can be obtained by means of a variational procedure based on a Galerkin spectral method (Chapman & Cowling 1939) used to solve the integral equations. The expressions of the thermal conductivity, thermal diffusion ratios and Stefan-Maxwell equations for the diffusion velocities can be derived by means of a Goldstein expansion of the perturbation functions, as proposed by Kolesnikov & Tirskiy (1984). Finally, the mathematical structure of the transport matrices obtained by the variational procedure can readily be used to build efficient transport algorithms, as already shown by Ern & Giovangigli (1994) for neutral gases, or Magin & Degrez (2004b) for unmagnetized plasmas.

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