A hybrid method for the unsteady compressible Navier-Stokes equations

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1. Motivation and objectives

The finite difference and finite volume methods are two widely used numerical methods employed to solve the Navier-Stokes equations. The high-order finite difference method (HOFDM) can efficiently handle large problems on structured grids, while the finite volume method in combination with unstructured grids (UFVM) can more readily handle complex geometries. There have been many attempts to develop hybrid schemes that combine the strengths of both methods and downplay their weaknesses. Although some of these hybrid schemes have been successfully implemented to improve the accuracy and efficiency of calculations around complex geometries, most of these methods have been known to suffer from late-time instabilities on the interfaces between the structured and unstructured domains.

The present authors have proposed stable hybrid schemes designed to combine the HOFDM and the UFVM. In Nordström & Gong (2006) it was shown how to couple the HOFDM and the UFVM for hyperbolic problems. In Nordström et al. (2007) we applied this hybrid method to the Euler equations, and in Gong & Nordström (2007) a stable interface treatment for viscous problems was studied. It was shown that the hybrid method is an accurate, efficient and a practically useful computational tool that can handle complex geometries as well as wave propagation phenomena. All the techniques employed are based on the so-called summation-by-parts (SBP) operators and impose the boundary and interface conditions weakly (see Kreiss & Schererab (1974)).

In this project we continue the work started in Gong & Nordström (2007) and consider the full Navier-Stokes equations. The main challenge for hybrid methods is to avoid instabilities at the interfaces between subdomains using different methods. We will focus on the stability of the coupling procedure at the interface and treat both the finite difference–finite difference coupling and the finite difference–finite volume coupling. The entire procedure will be exemplified by using finite difference methods.

2. Numerical method

Following the technique of symmetrization developed in Abarbanel & Gottlieb (1981) and Nordström & Svärd (2005), a symmetric form of the time-dependent, compressible,
linearized Navier-Stokes equations in two-dimensions is
\[ u_t + (A_1 u)_x + (A_2 u)_y = \varepsilon \left[ (B_{11} u_x + B_{12} u_y)_x + (B_{21} u_x + B_{22} u_y)_y \right], \quad (2.1) \]
where \( \varepsilon = 1/Re \), the inverse of the Reynolds number, and \( u = \left[ \frac{\varepsilon}{\sqrt{\gamma p}} \hat{\rho}, \hat{u}_1, \hat{u}_2, -\frac{\varepsilon}{\sqrt{\gamma(\gamma-1)}} \hat{p} + \sqrt{\frac{\gamma-1}{\gamma-1} \frac{1}{\rho C_p}} \right]^T. \hat{\rho} \) is the density, \( \hat{u}_1 \) and \( \hat{u}_2 \) are the Cartesian velocity components and \( \hat{p} \) is the pressure. \( A_1, A_2, B_{11}, B_{12}, B_{21} \) and \( B_{22} \) are the coefficient matrices.

Equation (2.1) can be rewritten as
\[ u_t + F_x + G_y = 0, \quad (2.2) \]
with \( F = A_1 u - \varepsilon (B_{11} u_x + B_{12} u_y) = F^I - \varepsilon F^V \) and \( G = A_2 u - \varepsilon (B_{21} u_x + B_{22} u_y) = G^I - \varepsilon G^V. \) Note that \( F^I \) and \( G^I \) contain the inviscid terms, and \( F^V \) and \( G^V \) the viscous terms.

We have the following proposition for Eq. 2.2:

**Proposition 2.1** The continuous problem (2.2) is well posed if the boundary terms are limited by given data since the matrix \( B = [B_{11} B_{12}; B_{21} B_{22}] \) is positive semi-definite.

In order to construct a hybrid method combining both the finite volume method and the finite difference method using a uniform framework of SBP operators, the HOFDM analyzed in Nordström & Carpenter (1999, 2001); Mattsson & Nordström (2004); Strand (1994) and the edge-based node-centered finite volume method (UFVM) considered in Nordström et al. (2003) are used. Without loss of generality, we consider a computational domain consisting of only two subdomains (see Fig. 1), with an interface at \( x = 0. \) Let \( \mathbf{u} \) and \( \mathbf{v} \) be the unknowns in the left and right subdomain, respectively and introduce the superscripts \( ^L \) and \( ^R \) to identify the subdomains. We first discuss the finite difference-finite difference coupling and then use the same technique to tackle the finite volume-finite difference coupling.
2.1. The finite difference-finite difference coupling

The semi-discrete approximation of Eq. 2.1 on two subdomains with an interface can be written as

\[ u_t + D^L_x \mathbf{F}^L + D^L_y \mathbf{G}^L = (M^L)^{-1} \Sigma^L_1 (u_I - v_I) + (M^L)^{-1} \Sigma^L_2 [(\mathbf{F}^V)_I^T - (\mathbf{F}^V)^R_I], \quad (2.3a) \]

\[ v_t + D^R_x \mathbf{F}^R + D^R_y \mathbf{G}^R = (M^R)^{-1} \Sigma^R_1 (v_I - u_I) + (M^R)^{-1} \Sigma^R_2 [(\mathbf{F}^V)^R_I - (\mathbf{F}^V)^L_I], \quad (2.3b) \]

where the subscript \( I \) indicates the interface. Furthermore,

\[
\begin{align*}
D^L_x &= (P^L_x)^{-1} Q^L_x \otimes I_y^L \otimes I_4, & D^R_x &= I_x^L \otimes (P^L_y)^{-1} Q_y^L \otimes I_4, \\
D^L_y &= (P^L_y)^{-1} Q^L_y \otimes I_y^L \otimes I_4, & D^R_y &= I_x^L \otimes (P^R_y)^{-1} Q_y^R \otimes I_4, \\
M^L &= P^L_x \otimes P^L_y \otimes I_4, & M^R &= P^R_x \otimes P^R_y \otimes I_4, \\
\Sigma^L_2 &= (E^L)^T P^L_y \otimes \Sigma^L_2, & \Sigma^R_2 &= (E^R)^T P^R_y \otimes \Sigma^R_2.
\end{align*}
\]

\( D_x \) and \( D_y \) are the finite difference approximations of the first derivatives in \( x \)- and \( y \)-directions, respectively. Note that the operators \( D_x \) and \( D_y \) are SBP operators since the matrices \( P_x \) and \( P_y \) are symmetric and positive definite and \( Q_x^L \) and \( Q_y^L \) are nearly skew-symmetric, that is, \( Q_x + Q_x^T = \text{diag}(-1, 0, \ldots, 0, 1) \) and similarly for \( Q_y \). The size of the identity matrices \( I_x \) and \( I_y \) is equal to the number of grid points. \( \Sigma^L_1 \), \( \Sigma^L_2 \), \( \Sigma^R_1 \), and \( \Sigma^R_2 \) are the interface penalty terms that will be described below. \( E^L \) and \( E^R \) are projection matrices that map \( u \) to \( u_I \) and \( v \) to \( v_I \), respectively.

Applying the energy method to Eqs. 2.3a and 2.3b yields

\[
\frac{d}{dt} \left( \| u \|^2_{M^L} + \| v \|^2_{M^R} \right) + 2\varepsilon \text{Diss} = \mathbf{w}_I^T M \mathbf{w}_I + \text{Pen}^L + \text{Pen}^R, \quad (2.4)
\]

where \( \mathbf{w}_I = [u_I, v_I, (u_x)_I, (v_x)_I, (u_y)_I, (v_y)_I]^T \). \( \text{Pen}^L \) and \( \text{Pen}^R \) are penalty terms that impose the outer boundary conditions weakly; see Nordström & Svärd (2005) for more details. In Eq. 2.4,

\[
\text{Diss} = \begin{bmatrix} u_x^T \\ u_y^T \\ v_x^T \\ v_y^T \end{bmatrix} \begin{bmatrix} P^L_x \otimes P^L_y \otimes B_{11} & P^L_x \otimes P^L_y \otimes B_{12} \\ P^L_x \otimes P^L_y \otimes B_{21} & P^L_x \otimes P^L_y \otimes B_{22} \\ P^R_x \otimes P^R_y \otimes B_{11} & P^R_x \otimes P^R_y \otimes B_{12} \\ P^R_x \otimes P^R_y \otimes B_{21} & P^R_x \otimes P^R_y \otimes B_{22} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} +
\]

and the elements of the symmetric matrix \( M \) are

\[
\begin{align*}
M_{11} &= P^L_y \otimes (-A_1 + \Sigma^L_1 + (\Sigma^L_2)^T), & M_{12} &= P^L_y \otimes -\Sigma^L_1 + P^R_y \otimes -\Sigma^R_1, \\
M_{13} &= P^L_y \otimes (\varepsilon I_4 + \Sigma^L_2 B_{11}), & M_{14} &= P^L_y \otimes -\Sigma^L_2 B_{11}, & M_{15} &= P^L_y \otimes (\varepsilon I_4 + \Sigma^L_2 B_{12}), \\
M_{16} &= P^L_y \otimes -\Sigma^L_2 B_{12}, & M_{22} &= P^R_y \otimes (A_1 + \Sigma^R_1 + (\Sigma^R_2)^T), & M_{23} &= P^R_y \otimes -\Sigma^R_2 B_{11}, \\
M_{24} &= P^L_y \otimes (-\varepsilon I_4 + \Sigma^R_2 B_{11}), & M_{25} &= P^L_y \otimes -\Sigma^R_2 B_{12}, & M_{26} &= P^R_y \otimes (-\varepsilon I_4 + \Sigma^R_2 B_{12}), \\
M_{33} &= M_{34} = M_{35} = M_{36} = 0, & M_{44} &= M_{45} = M_{46} = 0, & M_{55} &= M_{56} = M_{66} = 0.
\end{align*}
\]

Notice that the matrix \( M \) is indefinite and as such Eq. 2.4 does not obey the energy stability criterion. In order to construct a symmetric semi-definite negative matrix on the right-hand side of Eq. 2.4 we must “borrow” interface terms from Diss on the left-
hand side (see Carpenter et al. (1999)). The term \( \text{Diss} \) can be written as
\[
\text{Diss} = \alpha^L p_{MM}^L \left[ \begin{array}{c} (u_x)_I \\ (u_y)_I \end{array} \right]^T \begin{bmatrix} P_{yy}^L \otimes B_{11} & P_{yy}^L \otimes B_{12} \\ P_{yy}^L \otimes B_{21} & P_{yy}^L \otimes B_{22} \end{bmatrix} \left[ \begin{array}{c} (u_x)_I \\ (u_y)_I \end{array} \right] + \beta^R p_{11}^R \left[ \begin{array}{c} (v_x)_I \\ (v_y)_I \end{array} \right]^T \begin{bmatrix} P_{yy}^R \otimes B_{11} & P_{yy}^R \otimes B_{12} \\ P_{yy}^R \otimes B_{21} & P_{yy}^R \otimes B_{22} \end{bmatrix} \left[ \begin{array}{c} (v_x)_I \\ (v_y)_I \end{array} \right],
\]
where \( p_{MM}^L = (P_x^L)_M M, \) and \( p_{11}^R = (P_x^R)_{11}. \) If \( 0 < \alpha^L, \beta^R \leq 1, \) \( \tilde{\text{Diss}} \geq 0 \) since \( \tilde{F}_x^L \) and \( \tilde{F}_x^R \geq 0. \)

As a result, the modified Eq. 2.4 can be written as
\[
\frac{d}{dt} \left( \|u\|_{M^L}^2 + \|v\|_{M^R}^2 \right) + 2\varepsilon \tilde{\text{Diss}} = w_I^T \tilde{M} w_I + \text{Pen}^L + \text{Pen}^R,
\]
where \( \tilde{M} \) is the same as \( M \) except that the zero elements in \( M \) are replaced by
\[
M_{33} = -2\alpha^L p_{MM}^L P_{yy}^L \otimes B_{11}, \quad M_{35} = -2\alpha^L p_{MM}^L P_{yy}^L \otimes B_{12}, \quad M_{53} = M_{35}^T, \\
M_{44} = -2\varepsilon \beta^R p_{11}^R P_{yy}^R \otimes B_{21}, \quad M_{46} = -2\varepsilon \beta^R p_{11}^R P_{yy}^R \otimes B_{22}, \quad M_{64} = M_{46}^T, \\
M_{55} = -2\alpha^L p_{MM}^L P_{yy}^L \otimes B_{22}, \quad M_{66} = -2\varepsilon \beta^R p_{11}^R P_{yy}^R \otimes 2B_{22}.
\]

Before discussing the stability conditions at the interface, we first derive conservation conditions. Let \( \varphi \) be a smooth test function \( (\varphi_I^L = \varphi_I^R = \varphi_I). \) Multiplying Eqs. 2.3a and 2.3b by \( (\varphi_I^L)^T M^L \) and \( (\varphi_I^R)^T M^R, \) respectively, we obtain
\[
(\varphi_I^L)^T M^L u_t + (\varphi_I^R)^T M^R v_t = \varphi_I^T \left[ P_{yy}^L \otimes (A_1 + \Sigma_1^L) - P_{yy}^R \otimes \Sigma_2^R \right] (u_I - v_I) + \varphi_I^T \left[ p_{yy}^L \otimes (\varepsilon I_4 + \Sigma_2^L) - P_{yy}^R \otimes \Sigma_2^R \right] \left( \eta_I^L - \eta_I^R \right) + (\varphi_I^L)^T (P_{yy}^L \otimes I_4) \eta_I^L + (\varphi_I^R)^T (P_{yy}^R \otimes I_4) \eta_I^R - (\varphi_I^L)^T (P_{yy}^L \otimes I_4) \eta_I^L + (\varphi_I^R)^T (P_{yy}^R \otimes I_4) \eta_I^R - (\varphi_I^L)^T (P_{yy}^L \otimes I_4) \eta_I^L + (\varphi_I^R)^T (P_{yy}^R \otimes I_4) \eta_I^R - (\varphi_I^L)^T \left( \eta_I^L + \eta_I^R \right) + (\varphi_I^R)^T \left( \eta_I^L + \eta_I^R \right) - (\varphi_I^L)^T \left( \eta_I^L + \eta_I^R \right) + (\varphi_I^R)^T \left( \eta_I^L + \eta_I^R \right).
\]

If \( P_{yy}^L = P_{yy}^R, \) the conservation conditions
\[
\Sigma_1^L = \Sigma_1^L + A_1, \quad \Sigma_2^R = \Sigma_2^L + \varepsilon I_4,
\]
cancel the interface term in Eq. 2.6. Inserting \( P_{yy}^L = P_{yy}^R = P_y \) and the conservative conditions into Eq. 2.5 results in
\[
\frac{d}{dt} \left( \|u\|_{M^L}^2 + \|v\|_{M^R}^2 \right) + \varepsilon \tilde{\text{Diss}} = \text{Pen}^L + \text{Pen}^R
\]
\[
= - x^T (N \otimes P_y) x.
\]
Here $\Phi$ and $\Psi$ are the permutation matrices used for the transformation of the Kronecker products (see Horn & Johnson (1991)), and

\[
N_{11} = A_1 - \Sigma^L_1 - (\Sigma^L_1 T), \quad N_{13} = -(\varepsilon I_4 + \Sigma^L_2)B_{11}, \quad N_{14} = \Sigma^L_2 B_{11},
\]
\[
N_{15} = -(\varepsilon I_4 + \Sigma^L_2)B_{12}, \quad N_{16} = \Sigma^L_2 B_{12}, \quad N_{33} = 2\varepsilon\alpha^L p^L_{M,M} B_{11},
\]
\[
N_{35} = 2\varepsilon\alpha^L p^L_{M,M} B_{12}, \quad N_{44} = 2\varepsilon\beta^R p^R_{11} B_{11}, \quad N_{46} = 2\varepsilon\beta^R p^R_{11} B_{12},
\]
\[
N_{55} = 2\varepsilon\alpha^L p^L_{M,M} B_{22}, \quad N_{66} = 2\varepsilon\beta^R p^R_{11} B_{22}.
\]

**Remark** The condition $P^L_y = P^R_y$ implies that the same SBP operators should be used in the $y$-direction in both subdomains.

An energy estimation of Eq. 2.8 can be obtained by requiring $N$ to be a positive semi-definite matrix. A transformation matrix $S$ can be constructed such that $ST S = I$ and

\[
S = \begin{bmatrix}
\frac{I_4}{\sqrt{2}} & \frac{I_4}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & I_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_4 & 0 \\
0 & 0 & 0 & 0 & 0 & I_4 \\
\frac{I_4}{\sqrt{2}} & -\frac{I_4}{\sqrt{2}} & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \tilde{N} = SNS^T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{33} & N_{35} & 0 & 0 & \sqrt{2}N_{13} \\
0 & N_{35} & N_{55} & 0 & 0 & \sqrt{2}N_{15} \\
0 & 0 & 0 & N_{44} & N_{46} & \sqrt{2}N_{14} \\
0 & 0 & 0 & 0 & N_{46} & \sqrt{2}N_{16} \\
0 & \sqrt{2}N_{13} & \sqrt{2}N_{15} & \sqrt{2}N_{14} & \sqrt{2}N_{16} & 2N_{11}
\end{bmatrix}.
\]

To simplify the matrix $\tilde{N}$ we introduce

\[
\alpha = \alpha^L p^L_{M,M}, \quad \beta = \beta^R p^R_{11}, \quad \Sigma^L_2 = -\varepsilon \Delta, \quad \Sigma^L_1 = \Sigma^L_1 + \varepsilon \Sigma^L_1 V.
\]

$\tilde{N}$ can be split into inviscid part $\tilde{N}_I$ and viscous part $\tilde{N}_V$,

\[
\tilde{N} = \tilde{N}_I + \tilde{N}_V = \begin{bmatrix}
0_{20,20} & 0_{20,4} \\
0_{4,20} & 2(A_1 - \Sigma^L_1 - (\Sigma^L_1) T)
\end{bmatrix} + \varepsilon \begin{bmatrix}
2\alpha K_{11} & 0_{6,6} & \sqrt{2}K_{13} \\
0_{6,6} & 2\beta K_{11} & \sqrt{2}K_{23} \\
\sqrt{2}K^T_{13} & \sqrt{2}K^T_{23} & 2K_{33}
\end{bmatrix} \tag{2.9}
\]

where

\[
K_{11} = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}, \quad K_{13} = \begin{bmatrix}
(\Delta - I_4)B_{11} \\
(\Delta - I_4)B_{12}
\end{bmatrix}, \quad K_{23} = \begin{bmatrix}
-\Delta B_{11} \\
-\Delta B_{12}
\end{bmatrix}, \quad K_{33} = -[\Sigma^L_1 V + (\Sigma^L_1 V)^T]
\]

The subscripts of 0 in Eq. 2.9 indicate the size of the zero submatrix. A sufficient condition for $\tilde{N}_I$ in Eq. 2.9 to be positive semi-definite is

\[
A_1 - \Sigma^L_1 - (\Sigma^L_1) T \geq 0, \quad \text{or} \quad \Sigma^L_1 + (\Sigma^L_1) T \leq A_1. \tag{2.10}
\]

If $A_1$ is rewritten as

\[
A_1 = X^T \Lambda X = X^T (\Lambda^+ + \Lambda^-) X = X^T \Lambda^+ X + X^T \Lambda^- X = A_1^+ + A_1^-,
\]

where $\Lambda^+ = \text{diag}(\max(\lambda_i, 0))$, $\Lambda^- = \text{diag}(\min(\lambda_i, 0))$ ($\lambda_i$ are the eigenvalues of $A_1$), we obtain

\[
\Sigma^L_1 + (\Sigma^L_1) T \leq A_1^+. \tag{2.11}
\]
Consequently a sufficient condition for positive semi-definiteness of $\hat{N}_V$ is

$$K_{11} \geq 0 \text{ and } - (\Sigma_{14} + (\Sigma_{14})^T) = K_{33} \geq \frac{1}{2\alpha} K_{11}^T K_{11}^{-1} K_{13} + \frac{1}{2\beta} K_{23}^T K_{23}^{-1} K_{23},$$  

(2.12)

because $\hat{N}_V = \varepsilon LDL^T$ with

$$L = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \frac{1}{\sqrt{2\alpha}} K_{13}^T K_{11}^{-1} & \frac{1}{\sqrt{2\beta}} K_{23}^T K_{22}^{-1} & I \end{bmatrix}, \quad D = \begin{bmatrix} 2\alpha K_{11} & 0 & 0 \\ 0 & 2\beta K_{11} & 0 \\ 0 & 0 & 2K_{33} - \frac{1}{\alpha} K_{11}^T K_{11}^{-1} K_{13} - \frac{1}{\beta} K_{23}^T K_{23}^{-1} K_{23} \end{bmatrix}. \quad (2.13)$$

Using the analysis above, the conditions (2.7), (2.10) and (2.12) make the matrix $\hat{N}$ positive semi-definite, which implies that matrix $N$ is positive semi-definite, since for an arbitrary vector $y$,

$$y^T N y = y^T S^T \hat{N} S y = y^T \hat{N} \hat{y} \geq 0.$$

We have proved the following proposition:

**Proposition 2.2** If the conditions

$$\Sigma_{14} \leq \frac{A^T_1}{2}, \quad \Sigma_{14}^T = (\Sigma_{14})^T \quad (2.14a)$$

$$\Sigma_{14} \leq -\frac{\varepsilon}{4\alpha} K_{13}^T K_{11}^{-1} K_{13} - \frac{\varepsilon}{4\beta} K_{23}^T K_{11}^{-1} K_{23}, \quad \Sigma_{14}^T = (\Sigma_{14})^T \quad (2.14b)$$

$$\Sigma_2^L = -\varepsilon \Delta, \quad \text{(free variable)} \quad (2.14c)$$

$$\Sigma_1^R = \Sigma_1^L - A_1, \quad \text{(conservation condition)} \quad (2.14d)$$

$$\Sigma_2^R = \Sigma_2^L + \varepsilon I_4, \quad \text{(conservation condition)} \quad (2.14e)$$

are satisfied, then the scheme (2.3) is stable.

The matrix $K_{11}^{-1}$ can be written in block matrix form as

$$K_{11}^{-1} = \begin{bmatrix} B_{11}^{-1} + B_{11}^{-1}B_{12}\hat{D}^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}\hat{D}^{-1} \\ -\hat{D}^{-1}B_{21}B_{11}^{-1} & \hat{D}^{-1} \end{bmatrix},$$

with $\hat{D} = B_{22} - B_{21}B_{11}^{-1}B_{12}$. When the non-constrained matrix $\Delta$ in Eq. 2.14c is written as $\Delta = \delta I_4$ ($\delta \in \mathcal{R}$), we obtain

$$K_{11}^T K_{11}^{-1} K_{13} = (1 - \delta)^2 B_{11}, \quad \text{and} \quad K_{23}^T K_{23}^{-1} K_{23} = \delta^2 B_{11}.$$

Consequently condition (2.14b) reduces to

$$\Sigma_{14} \leq -\frac{[\varepsilon(1 - \delta)^2 + \alpha \delta^2] \varepsilon}{4\alpha \beta} B_{11}. \quad (2.15)$$

It is easy to verify that the right-hand side of Eq. 2.15 has a minimum value of

$$-\frac{\varepsilon}{4(\alpha + \beta)} B_{11}$$

when $\delta = \beta/(\alpha + \beta)$. Recall that $\alpha = \alpha^L p_{M,M}^L$ and $\beta = \beta^R p_{11}^R$ ($0 < \alpha^L, \beta^R \leq 1$),

$$\alpha \leq p_{M,M}^L = \Delta x^L, \quad \frac{\Delta}{0.17} \quad 2\text{nd-order SBP}, \quad \frac{0.17}{1.13} \quad 4\text{th-order SBP}, \quad \frac{1.13}{1.52} = 6\text{th-order SBP}, \quad \frac{1.52}{1.29} \quad 2\text{nd-order SBP}, \quad \frac{1.29}{1.32} \quad 4\text{th-order SBP}, \quad \frac{1.32}{1.36} = 6\text{th-order SBP}.$$
In order to limit the spectral radius of the problem, the values of $\alpha$ and $\beta$ should be chosen as large as possible, that is, $\alpha = p_{M_M}^L$, $\beta = p_{I_I}^R$.

### 2.2. Finite volume-finite difference coupling

When the finite volume scheme is used on the left subdomain and the finite difference scheme on the right subdomain (see Fig. 1), the semi-discrete approximation of Eq. 2.1 in the two subdomains with an interface can be written as

\[
\begin{align*}
\mathbf{u}_I + D_x^L \mathbf{F}^L + D_y^L \mathbf{G}^L &= (M_L)^{-1} \Sigma^L_1 (\mathbf{u}_I - \mathbf{v}_I) + (M_L)^{-1} \Sigma^L_2 \left[ (\mathbf{F}^V)_I - (\mathbf{F}^V)_I^R \right], \quad (2.16a) \\
\mathbf{v}_I + D_x^R \mathbf{F}^R + D_y^R \mathbf{G}^R &= (M_R)^{-1} \Sigma^R_1 (\mathbf{v}_I - \mathbf{u}_I) + (M_R)^{-1} \Sigma^R_2 \left[ (\mathbf{F}^V)_I^R - (\mathbf{F}^V)_I^L \right], \quad (2.16b)
\end{align*}
\]

Here $D_x^L = (P_L)^{-1} Q_x^L \otimes I_4$ and $D_y^L = (P_L)^{-1} Q_y^L \otimes I_4$ are the finite volume approximations of the first derivative in the $x$- and $y$-directions, respectively. $M_L = P_L \otimes I_4$ is the norm. Note that $D_x^L$ and $D_y^L$ are SBP operators since the matrix $P_L$ is a positive diagonal matrix with the control volumes on the diagonal and $Q_x^L$ and $Q_y^L$ are almost skew-symmetric matrices, that is, $Q_x^L + (Q_x^L)^T = Y$ and $Q_y^L + (Q_y^L)^T = X$ where the non-zero elements in $Y$ and $X$ correspond to the boundary points (see Nordström et al. (2003); Nordström & Gong (2006) for more details). $D_x^R$, $D_y^R$ and $M_R$ are the same as in the definitions used in Section 2.1.

To obtain an energy estimate similar to Eq. 2.8 we also need the condition $P_y^L = P_y^R = P_y$ for the finite volume-finite difference coupling.

**Remark** In Theorem 2.2, $P_y^L = P_y^R$ is implicitly included in the conditions (2.14a)–(2.14c). However, for the finite volume-finite difference coupling, the specific SBP operators that are based on diagonal norms are given in Mattsson & Nordström (2004); Strand (1994). The standard second-, fourth- and sixth-order diagonal norm $P_y^R$ for the finite difference scheme are

\[
\begin{align*}
P_y^R &= \Delta y \cdot \text{diag} \left( \frac{1}{2}, 1, 1, \ldots, 1 \right), \quad (2.17) \\
P_y^R &= \Delta y \cdot \text{diag} \left( 17, 59, 43, 49, \frac{48}{48}, \frac{48}{48}, \frac{48}{48}, 1, 1, \ldots, 1 \right), \quad (2.18) \\
P_y^R &= \Delta y \cdot \text{diag} \left( \frac{13649}{43200}, \frac{12013}{8640}, \frac{2711}{4320}, \frac{5359}{4320}, \frac{7877}{4320}, \frac{43801}{43200}, \frac{1}{8640}, \frac{1}{4320}, \frac{1}{4320}, 1, 1, \ldots, 1 \right), \quad (2.19)
\end{align*}
\]

respectively. Since the old dual grid for the points on the interface consists of the lines between the center of the triangles/rectangles and the midpoints of the edges, the diagonal norm $P_y^L = \Delta y \cdot \text{diag}(1/2, 1, 1, \ldots, 1)$ regardless of the different SBP operators used in the right domain (see Fig. 2.2). Therefore, the relations $P_y^L = P_y^R$ are not automatically satisfied when curvilinear interfaces or high-order SBP operators are used. We need to modify the control volume for the finite volume scheme on the interface to guarantee the condition $P_y^L = P_y^R$ (see Gong & Nordström (2007); Nordström & Gong (2006) for more details). Fig. 2.2 shows the modified control volume when a fourth-order accurate finite-difference SBP operator is used in the right subdomain.

Applying the energy method to Eqs. 2.16a and 2.16b yields

\[
\frac{d}{dt} \left( \| \mathbf{u} \|^2_{M^L} + \| \mathbf{v} \|^2_{M^R} \right) + 2\varepsilon \text{Diss} - \text{Pen}^L - \text{Pen}^R = -\mathbf{x}^T (N \otimes P_y) \mathbf{x}, \quad (2.20)
\]

where the vector $\mathbf{x}$ and matrices $N$ and $P_y$ are the same as those defined in Eq. 2.8, $\text{Pen}^L$ and $\text{Pen}^R$ are positive diagonal matrices.
Figure 2. The UFVM used in the left subdomain and fourth-order accurate HOFDM used in the right subdomain. (left) The old control volume; (right) The modified control volume

Here $p_{11}^R = (P_y^R)_{i_{11}}$ and $p_{LL}^T = \min [(E_P^T P_{y,i})/P_{y,i}^L (i \in \text{Interface})]$, that is, $p_{LL}^T$ is the minimum value of the control volume divided by the norm for all nodes on the interface. If $0 < \alpha^L, \beta^R \leq 1$, $\bar{D}_{\text{iss}} \geq 0$ since $\bar{P}_T$ and $P_y \otimes P_y^R \geq 0$. Observe that $P_T$ has been changed due to the modified control volumes for nodes at the interface.

Remark On an unstructured mesh (see Fig. 2.2), the coefficient $p_{LL}^T$ using the modified
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control volume in the finite volume scheme is

\[ p_{LL} = \min \left[ \frac{E_i^L P^L_{x,i}}{P^L_{x,i}}, \frac{P^L_{y,i}}{P^L_{y,i}} \right] \]

\[ = \min \left[ \left( \frac{L}{L_i} \right)^2 \frac{\Omega'_{f_0}^L}{\Omega'_{f_1}^L}, \frac{\Omega'_{f_2}^L}{\Omega'_{f_2}^L}, \frac{\Omega'_{f_3}^L}{\Omega'_{f_3}^L}, \frac{\Omega'_{f_4}^L}{\Omega'_{f_4}^L}, \frac{\Omega'_{f_5}^L}{\Omega'_{f_5}^L}, \ldots \right], \]

\[ = \min \left[ \frac{48 \Omega'_{f_0}^L}{17 \Delta y}, \frac{48 \Omega'_{f_1}^L}{59 \Delta y}, \frac{48 \Omega'_{f_2}^L}{43 \Delta y}, \frac{48 \Omega'_{f_3}^L}{49 \Delta y}, \frac{\Omega'_{f_4}^L}{\Delta y}, \frac{\Omega'_{f_5}^L}{\Delta y}, \ldots \right]. \quad (i \in \text{Interface}) \]

The interface is modified according to the scheme:

(a) move grid point to the new location given by the finite difference norm \((P^L_y)'_i\)

(b) calculate the new modified control volume \(\Omega'_i\)

(c) compute \((P^L_{LL})'_i = \Omega'_i/(P^L_y)'_i\) at all interface points \(i\)

(d) choose \(p_{LL}^L = \min_i (P^L_{LL})'_i\)

The result is the following Proposition:

**Proposition 2.3** If the conditions

\[ \alpha = \alpha^L p^L_{LL}, \quad p^L_{LL} = \min \left[ \frac{E_i^L P^L_{x,i}}{P^L_{x,i}}, \frac{P^L_{y,i}}{P^L_{y,i}} \right] \quad (i \in \text{Interface}), \quad 0 < \alpha^L \leq 1, \]

\[ \beta = \beta^R p^R_{11}, \quad p^R_{11} = (P^R_y)_{11}, \quad 0 < \beta^R \leq 1, \]

are satisfied, then (2.14a)–(2.14e) in Proposition 2.2 are the stability conditions for the scheme (2.16).

### 3. Results

In this section we validate the theoretical predictions presented in the previous section using the finite difference-finite difference coupling. A stationary viscous shock problem where the middle of the shock is located at the interface is calculated. In the reference frame of the shock the upstream Mach number is 2.0 and the angle of the shock relative to the Cartesian frame is 15°. The Reynolds number \(Re = 50.0\) is based on the upstream Mach number. Moreover the penalty terms in Eq. 2.14 are chosen set to the minimum required values for stability. We integrate the solution to steady-state using the third-order low storage explicit time advancement scheme of Le & Moin (1991).

In the hybrid scheme, the second derivative SBP operator is constructed with \(2p\)-th order accuracy internal and \((p - 1)\)-th order at the boundary by using a diagonal norm. It was proved in Svärd & Nordström (2006) that if the solution is pointwise bounded, the accuracy of the scheme is two orders higher than the accuracy of the second derivative approximation at the boundaries. The convergence rates for the second-, fourth-, sixth- and eighth-order schemes (in the interior points) are 2, 3, 4 and 5, respectively. Since the errors for all variables (density, velocities and energy) are very similar, only the density errors are shown in our calculations. The accuracy of several HOFDM schemes is shown in Table 1. The convergence rates for the second-, fourth-, sixth- and eighth-order schemes (in the interior domain) are 2, 3, 4 and 5, respectively. The results are in agreement with the theory (see Gustafsson (1975); Svärd & Nordström (2006)).

In the following calculations, we used hybrid meshes that consist of two different structured meshes with an interface. The left part of Fig. 3 shows the density isolines using the fourth-order discretization. The corresponding cut at \(y = 0\) can be found in the right part of Fig. 3. The distribution of density close to the interface \(x = 0\) is very smooth, which illustrates that the interface does not introduce large reflections and oscillations.
<table>
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<th>Points</th>
<th>2nd order Err</th>
<th>2nd order q</th>
<th>4th order Err</th>
<th>4th order q</th>
<th>6th order Err</th>
<th>6th order q</th>
<th>8th order Err</th>
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<tr>
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<td>-3.23 2.91</td>
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<tr>
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<td>-4.13 2.98</td>
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<td>-4.48 4.43</td>
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<tr>
<td>$257 \times 257 + 129 \times 257$</td>
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<td>-5.92 4.80</td>
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</table>

Table 1. The convergence rates of density by using the finite difference approximations of Eq. 2.1 on two non-uniform subdomains.

Figure 3. Density isolines with the fourth-order SBP operator. A mesh of 65×65 grid points is used in both subdomains. (left) The whole computational domain; (right) at $y = 0$.

Figure 4. The errors in density at $y = 0$ with SBP operators of different orders. (left) $65 \times 65 + 33 \times 65$ points; (right) $65 \times 65 + 33 \times 65$ points.
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Figure 5. A fourth-order SBP operator is used without viscous penalty terms. (left) The density; (right) The error in density. Note that the situation is shown just before failure of the method. It does not correspond to a converged solution.

The density errors at $y = 0$ with SBP operators of different orders are shown in Fig. 4. On the coarse mesh the higher-order schemes have rather large errors, comparable to the lower order schemes close to the interface $x = 0$. However, when the mesh is refined to $65 \times 129$, the higher-order schemes outperform the lower-order schemes. Table 1 and Fig. 4 illustrate that the interface treatment is stable and accurate for all orders of accuracy.

To further illustrate the necessity of having correct penalty terms, we neglect the viscous penalty term completely. This leads to a complete failure for all schemes (blow up in a couple of time steps) (see Fig. 5).

Finally, we demonstrate the hybrid method on a moving shock problem. In the calculation, the unsteady computation has been carried out on a uniform grid of $65 \times 65$ in each block in combination with the fourth-order accurate SBP operator. All penalty parameters have the same values as for the previous steady case. The shock moves at $\text{Mach}=0.15$ under $45^\circ$. Snapshots of the solution between $t = 0.0$ and $t = 8.0$ are shown in Fig. 6. The shape of the shock through the interface $x = 0$ remains intact, and the corresponding errors are small (see Fig. 7).

4. Conclusions and future work

A stable and conservative hybrid method for solving the full Navier-Stokes equations has been developed. Stable and conservative finite difference-finite difference and finite difference-finite volume coupling at interfaces have been derived. The hybrid method makes it possible to combine the efficiency of the finite difference method and the flexibility of the finite volume schemes using unstructured meshes.

To achieve a stable and conservative interface coupling we have used difference operators of SBP type, a penalty technique for the interface conditions and the energy method.

We have done mesh refinement studies for a steady viscous shock and computations of a moving viscous shock. The numerical experiments support the theoretical conclusions and show that the interface coupling is stable and converge at the correct order.

Future work will include the application of the hybrid scheme to viscous fluid flows on complex geometries by using the finite difference-finite volume coupling at interfaces.

5. Acknowledgments

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Density snapshots of the moving viscous shock problem

Figure 6. The density isolines with the 4th order SBP operator for the unsteady shock problem.

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Density error snapshots of the moving viscous shock problem

Figure 7. The error in density for the 4th order SBP operator for the unsteady shock problem.