Localized artificial viscosity and diffusivity scheme for capturing discontinuities on curvilinear and anisotropic meshes

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1. Motivation and objectives

Due to advances in computational power and numerical algorithms, the application of large-eddy simulation (LES) to transitional and turbulent compressible flows is the focus of significant recent research. The engineering motivation for compressible LES is to provide a more realistic turbulent flowfield than RANS simulations and to elucidate the unsteady phenomena (mixing, combustion, sound-generation, unsteady load, etc.).

Because of their spectral-like resolution, high-order compact difference schemes (Lele 1992) are an attractive choice for LES of transitional and turbulent flows to reduce dispersion, anisotropy and dissipation errors associated with the spatial discretization. However, these central difference schemes cannot be applied directly to flows that contain discontinuities. When flows contain steep gradients, such as shock waves and contact surfaces, non-physical spurious oscillations are generated that make the simulation unstable. The development of numerical algorithms that capture discontinuities and also resolve the scales of turbulence in compressible turbulent flows remains a significant challenge.

Several techniques to extend the compact schemes to discontinuous flows have been proposed. Lee et al. (1997) proposed a hybrid approach to capture discontinuities. In regions of strong shock waves, the compact differencing of convective fluxes is replaced locally by the essentially non oscillatory (ENO) scheme. Visbal & Gaitonde (2005) developed an adaptive filter method in which the compact scheme is coupled with a locally reduced-order filter to capture discontinuities. Both approaches require a detector to identify the smooth and non-smooth regions in the flow. The requirement of a discontinuity detector is a bottleneck for these methods when applied to complex applications.

An attractive alternative to these methods has been proposed by Cook & Cabot (2004, 2005) and in follow-on work by Fiorina & Lele (2007) by dynamically adding spectral-like high-wavenumber biased artificial viscosity and diffusivity where needed, to capture discontinuities using high-order compact difference schemes. The method does not require a discontinuity detector or weighting scheme and was shown to work well on 1- and 2-D shock-related problems. However, most of the test cases in those researches used uniformly spaced Cartesian coordinate systems. Therefore, the extension of the artificial viscosity and diffusivity method to curvilinear and anisotropic meshes is still an open issue. This extension is necessary for the method to be used in practical applications.

In the present study, an extension of the localized high-wavenumber biased artificial viscosity and diffusivity to curvilinear and anisotropic meshes is proposed. The original formulation is also simplified to reduce the computational costs while achieving better representation of high-order derivatives. The performance of the method will be assessed on several 1- and 2-D shock-related problems. An accompanying paper (Kawai & Lele...
S. Kawai and S. K. Lele 2007) discusses the application of this method to the problem of sonic jet injection in a supersonic crossflow.

2. Mathematical models

2.1. Governing equations

The Navier-Stokes equations including artificial viscosity (Cook & Cabot 2004, 2005) and diffusivity (Fiorina & Lele 2007) terms for an ideal non-reactive gas are:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - \nabla (\chi_p \nabla \rho) = 0, \tag{2.1}
\]

\[
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + p \delta - \tau) = 0, \tag{2.2}
\]

\[
\frac{\partial e}{\partial t} + \nabla \cdot [e \mathbf{u} + (p \delta - \tau) \cdot \mathbf{u} - \lambda \nabla T] = 0, \tag{2.3}
\]

\[
e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u}, \quad p = \rho RT, \tag{2.4}
\]

where \(\rho\) is the density, \(\mathbf{u}\) is the velocity vector, \(p\) is the pressure, \(e\) is the total energy, \(T\) is the temperature, \(\gamma (=1.4)\) is the ratio of specific heats, \(R\) is the gas constant, \(\lambda\) is the thermal conductivity, \(\delta\) is the unit tensor and \(\chi_p\) is the artificial diffusivity. The viscous stress tensor \(\tau\) is

\[
\tau = (\mu_s + \mu_l)(2S) + (\mu_b - \frac{2}{3} (\mu_s + \mu_l))((\nabla \cdot \mathbf{u}) \delta), \tag{2.5}
\]

where \(\mu_l\) is the fluid viscosity, \(\mu_s\) and \(\mu_b\) are the artificial shear and bulk viscosities, and \(S\) is the symmetric strain rate tensor, \(S = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\).

2.2. Artificial viscosity and diffusivity

When a high-order compact scheme is applied to solve flows that involve steep gradients such as those due to shock waves and contact surfaces, non-physical spurious oscillations are generated that make the simulation unstable. A key issue here is how to properly remove the non-physical spurious oscillations without damping the resolved scales of turbulence.

Cook & Cabot (2004, 2005) introduced a high-wavenumber biased artificial shear and bulk viscosities \(\mu_s\) and \(\mu_b\) to suppress the spurious oscillations. These artificial shear and bulk viscosities are defined by:

\[
\mu_s = C^s\mu \alpha_r, \quad \mu_b = C^b\mu \alpha_r, \quad \alpha_r = \rho \Delta^{r+2} |\nabla \cdot \mathbf{S}|, \tag{2.6}
\]

where \(C^s\mu\) and \(C^b\mu\) are user-specified constants, \(\Delta\) is the local grid spacing, and \(S\) is the magnitude of the strain rate tensor. If \(r\) is sufficiently high, the high-wavenumber biased \((k^r)\) artificial viscosity only damps wavenumbers close to the unresolved wavenumbers. The overbar denotes an approximate truncated-Gaussian filter along each grid line (Cook & Cabot 2004).

Fiorina & Lele (2007) extended the method to suppress the numerical oscillations across the steep gradients of temperature and species by adding a similar artificial diffusivity. A grid-dependent artificial diffusivity is defined by:

\[
\chi_p = C_p \beta_r, \quad \beta_r = \frac{\alpha_r}{c_p} \Delta^{r+2} |\nabla \cdot \mathbf{S}|, \tag{2.7}
\]
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where \( C \) is a user-specified constant, \( a_0 \) is a reference speed of sound, \( c_p \) is the specific heat at constant pressure, and \( |\nabla s| \) is the norm of the fluid entropy gradient.

Typically, in their method, \( \Delta \) is defined as the geometrically-averaged local grid spacing (cube-root of cell volume), and a value of \( r=4 \) is chosen and \( \nabla^4 f \) is decomposed to a series of Laplacians, \( \nabla^4 f = \nabla^2 (\nabla^2 f) \). The recommended values for the user-specified constants with \( r=4 \) are \( C_\mu^0=0.002 \) and \( C_b^0=1 \) (Cook & Cabot 2005) and \( C_\rho=0.01 \) (Fiorina & Lele 2007).

2.3. Extension to curvilinear and anisotropic meshes

The original properties of the 1-D formulation of artificial viscosity and diffusivity that capture discontinuities with minimal effects on vorticity should be preserved when the method is extended to multi-dimensional curvilinear and anisotropic meshes.

In order to extend the original method to a multi-dimensional generalized coordinate system, the \( \Delta^r \) scaling appearing in Eqs. 2.6 and 2.7 has to be generalized. The use of geometrically-averaged grid spacing as proposed in the original paper (Cook & Cabot 2005) is one choice that worked well on an isotropic grid. However, the geometrically-averaged grid spacing introduces an undesirable mesh dependence when an anisotropic mesh is used. As a simple example, consider the problem of capturing a steady shock wave in 1-D flow on a grid spacing defined by \( \Delta x \), for which the 1-D formulation of artificial shear viscosity will be

\[
\mu_{s,1D} = C_\mu^0 \rho \Delta x^{r+2} \frac{\partial^2 S}{\partial x^2},
\]  

When the same 1-D shock is considered on a 2-D domain, the artificial shear viscosity using the geometrically-averaged grid spacing will be

\[
\mu_{s,2D} = C_\mu^0 \rho (\Delta x \Delta y)^{r+2} |\nabla^r S| = \mu_{s,1D} \times \left( \frac{\Delta y}{\Delta x} \right)^{r+2}.
\]

If an isotropic grid is used (\( \Delta x = \Delta y \)), \( \mu_{s,1D} = \mu_{s,2D} \) and the artificial dissipation terms behave consistently with the original 1-D formulation. However, when an anisotropic grid (\( \Delta x \neq \Delta y \)) is used, \( \mu_{s,1D} \neq \mu_{s,2D} \) and more or less than the necessary dissipation will be introduced. This might cause significant numerical damping of resolved scales of turbulence due to the excessive dissipation or spurious non-physical oscillations across the steep gradients.

To construct a consistent multi-dimensional artificial viscosity and diffusivity, each derivative in \( \nabla^r S \) needs to be grid-dependent for all possible discontinuity directions. That is, each derivative in \( \nabla^r S \) should be scaled by the grid spacing in the derivative direction to avoid the undesirable effects from the grid spacing in other directions. The Gaussian filter removes the cusps introduced by the high-order derivative operators used to compute the artificial viscosity and diffusivity. Therefore, the grid spacing dependence should be located in the operand of the Gaussian filter. As discussed, in the original methods, the \( \nabla^r f \) appearing in Eqs. 2.6 and 2.7 is computed by a series of Laplacians and evaluated using a high-order compact difference scheme. The evaluation of the series of Laplacians induces significant computational cost. Furthermore, considering a multi-dimensional generalized coordinate extension of a series of Laplacians, the full implementation of the Laplacians induces further computational costs to evaluate additional cross-derivative terms. To reduce the computational costs and achieve better representation of the high derivatives for more practical use, an alternative simplification
is the direct evaluation of \( \sum_{j=1}^{3} \frac{\partial^r f}{\partial \xi_i^r} \) instead of a series of Laplacians. Hence we evaluate the artificial viscosity and diffusivity on a multi-dimensional generalized coordinate system defined by:

\[
\alpha_r = \rho \sum_{l=1}^{3} \sum_{m=1}^{3} \Delta_l^{r+2} \left( \frac{\partial \xi_l}{\partial x_m} \right) r \frac{\partial^r S}{\partial \xi_i^r}, \quad (2.10)
\]

\[
\beta_r = \frac{a_0}{c_p} \sum_{l=1}^{3} \sum_{m=1}^{3} \Delta_l^{r+2} \left( \frac{\partial \xi_l}{\partial x_m} \right) r \frac{\partial^r |\nabla s|}{\partial \xi_i^r}, \quad (2.11)
\]

where \( \xi, \eta \) and \( \zeta \) and \( x_m \) refers to \( x, y \) and \( z \) when \( l \) and \( m \) are 1, 2 and 3, respectively. \( \Delta_l \) is the grid spacing in the physical space along with the grid line in the \( \xi_l \) direction and is defined by \( \Delta_l^2 = \sum_{n=1}^{3} \left( \frac{x_{n+1} - x_{n-1}}{2} \right)^2 \), where \( x_{n,i} \) refers to \( x_i, y_i \) and \( z_i \) when \( n = 1, 2 \) and 3 and \( i \) is a node index in the \( \xi_l \) direction. Note that due to computational cost, all the cross-derivative terms are neglected in Eqs. 2.10 and 2.11. This could potentially have a significant detrimental effect. However, the performance of the overall scheme examined here with test cases for 2-D shocks on Cartesian and curvilinear grids show that this assumption does not have a major detrimental effect. In the limit of \( \Delta \xi_l \to 0 \), Eqs. 2.1–2.3 converge to the original Navier-Stokes equations.

In the present study \( r=4 \) is adopted in Eqs. 2.10 and 2.11. The fourth derivatives, \( \frac{\partial^4 S}{\partial \xi_i^4} \) and \( \frac{\partial^4 |\nabla s|}{\partial \xi_i^4} \), are evaluated by (Lele 1992):

\[
\alpha f_{i-1}^{m} + f_i^{m} + \alpha f_{i+1}^{m} = b f_{i+3} - 9 f_{i+1} + 16 f_i - 9 f_{i-1} + f_{i-3}
\]

\[
\frac{6 h^4}{b f_{i+2} - 4 f_{i+1} + 6 f_i - 4 f_{i-1} + f_{i-2}}, \quad (2.12)
\]

Sixth- and fourth-order tridiagonal schemes and a fourth-order explicit scheme at interior points are obtained by setting the parameters as: sixth-order tridiagonal, \( a=\frac{7}{13}, a=\frac{14}{13} \) and \( b=\frac{1}{13} \) (C6); fourth-order tridiagonal, \( a=\frac{1}{4}, a=\frac{3}{4} \) and \( b=0 \) (C4); fourth-order explicit, \( a=0, a=2 \) and \( b=1 \) (E4). These schemes are used in the present assessment.

At a near-boundary point \( i \), one-sided explicit formulas are utilized:

\[
f_i^{m} = \frac{1}{h^4} \sum_{n=1}^{8} a_{n,i} f_n, \quad i \in 1, 2 \quad (2.13)
\]

\[
f_i^{m} = \frac{1}{h^4} \sum_{n=0}^{7} a_{imax-n,i} f_{imax-n}, \quad i \in (imax - 1, imax). \quad (2.14)
\]

Second-order boundary schemes are used in the present study. Coefficients for the second-order boundary schemes at each left, near-boundary point, 1 and 2 are \( (a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}, a_{5,1}, a_{6,1}, \text{truncation error}) = (3, -14, 26, -24, 11, -2, \frac{15}{16} h^2 f^{(6)}) \) and \( (a_{1,2}, a_{2,2}, a_{3,2}, a_{4,2}, a_{5,2}, a_{6,2}, \text{truncation error}) = (2, -9, 16, -14, 6, -1, \frac{15}{16} h^2 f^{(6)}) \). With regard to the C6 and E4, a second-order central scheme is used for point 3. The user-specified constants are set to \( C^\mu=0.002, C^b=1 \) and \( C^p=0.01 \).

2.4. Numerical scheme

The equations are solved in generalized curvilinear coordinates, where spatial derivatives for convective terms, viscous terms, metrics and Jacobian are evaluated by the sixth-order
compact difference scheme (Lele 1992). A fourth-order Runge-Kutta method is used for temporal integration. The eighth-order low-pass spatial filtering scheme (Gaitonde & Visbal 2000) is used on the conservative variables once in each direction after each final Runge-Kutta step in order to ensure numerical stability.

3. Numerical results

The numerical results with the original double Laplacian formulation of artificial viscosity and diffusivity that are evaluated by using sixth-order compact difference scheme are denoted AVD-C6 where the $\Delta^6$ scaling is evaluated by the geometrically-averaged grid spacing. The results from using the proposed simplified artificial viscosity and diffusivity for multi-dimensional generalized coordinate system (Eqs. 2.10 and 2.11) are denoted SAVD-C6, SAVD-C4 and SAVD-E4 where the last two characters such as C6, C4 and E4 denote the schemes used to evaluate the fourth derivatives in the equations: C6, sixth-order tridiagonal scheme; C4, fourth-order tridiagonal scheme; E4, fourth-order explicit scheme.

3.1. One-dimensional shock tube problems

3.1.1. Sod shock tube problem on an isotropic mesh

The first 1-D shock tube test case for the simplified artificial viscosity and diffusivity is the shock tube problem introduced by Sod (1978). This problem contains shock and contact discontinuities. Initial left- and right-side conditions are: $\rho = 1.0$, $u = 0.0$ and $p = 1.0$ for $x \leq 0$, and $\rho = 0.125$, $u = 0.0$ and $p = 0.1$ for $x > 0$. Simulations are performed on a uniformly spaced grid with 201 grid points where $\Delta x = 0.005$.

Figure 1 shows the comparison between the exact solution, the original artificial viscosity and diffusivity method AVD-C6, and the simplified methods SAVD-C6, SAVD-C4 and SAVD-E4 for the density, the artificial viscosity and diffusivity at the time of $\tau = 0.2$. The artificial viscosity and diffusivity are normalized by the maximum obtained by AVD-C6. The shock and contact discontinuities are captured well without significant spurious oscillations and show good agreement with the exact solution. The AVD-C6, SAVD-C6, SAVD-C4 and SAVD-E4 show almost identical results as expected. The artificial viscosities and diffusivities obtained by the AVD-C6, SAVD-C6 and SAVD-C4 are also nearly identical. The SAVD-E4 shows lower artificial viscosity and diffusivity than the other schemes but the spatial extent is similar. Overall, there is no significant difference in the performance of the schemes.
3.1.2. Sod shock tube problem on an anisotropic mesh

The second test case is conducted on a 2-D mesh with an anisotropic mesh, aspect ratio of 3 \((\Delta y = 3\Delta x)\). Initial left- and right-side conditions are the same as the first test case. This is used to confirm the consistency between the original 1-D formulation and the simplified model on a multi-dimensional formulation. Simulations are carried out on a uniformly spaced Cartesian grid with 201 and 13 grid points in the \(x\)- and \(y\)-direction where \(\Delta x = 0.005\) and \(\Delta y = 0.015\). Periodic boundary conditions are applied on the boundaries in the \(y\)-direction.

Figure 2 shows the comparison between the exact solution and the numerical simulations at the time \(\tau = 0.2\). The artificial viscosity and diffusivity is normalized by using the same maximum values obtained by AVD-C6 in the first test case (3.1.1). The simplified multi-dimensional formulations consistently maintain the original properties of the 1-D formulation. The results of SAVD-C6, SAVD-C4 and SAVD-E4 are identical to the results obtained in the first test case on the 1-D mesh. On the other hand, the AVD-C6, which uses geometrically-averaged grid spacing for the \(\Delta^6\) scaling, introduces excessive artificial dissipation at the discontinuity. The excessive viscosity and diffusivity causes more diffuse expansion waves and a non-physical development of the contact surface.

3.1.3. Shock tube problem with strong temperature discontinuity

The third test case is the shock tube problem with strong temperature discontinuity at the contact surface introduced by Fiorina & Lele (2007). This test case is used to evaluate the capability of the simplified formulation of artificial diffusivity. Initial left- and right-side conditions are: \(\rho = 1.0, u = 0.0\) and \(p = 1.1\) for \(x \leq 0\), and \(\rho = 0.1, u = 0.0\) and \(p = 1.0\) for \(x > 0\). Simulations are performed on a uniformly spaced grid with 201 grid points where \(\Delta x = 0.005\).

Comparison between the exact solution and the numerical simulations for the temperature and the normalized artificial diffusivity at \(\tau = 0.12\) are shown in Fig. 3. The AV-C6 is the simulation without the original artificial diffusivity model. Recall that the AV-C6 introduces spurious oscillations due to the lack of artificial diffusivity. The AVD-C6, SAVD-C6, SAVD-C4 and SAVD-E4 show nearly identical results and the temperature discontinuity is well-captured. The artificial viscosity and diffusivity obtained by the AVD-C6, SAVD-C6 and SAVD-C4 are also almost identical. Consistently with the first test case (3.1.1), the SAVD-E4 shows lower artificial viscosity and diffusivity with similar spatial extent as the others, but this does not have a detrimental effect on the discontinuity capturing.
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Figure 3. Numerical simulations of the 1-D shock tube problem with a density factor of 10. Temperature (close-up view in the contact discontinuity region) and artificial diffusivity are presented at \( \tau = 0.12 \). Thin solid line, exact; \( \ldots \), AVD-C6; \( \ldots \), SAVD-C6; \( \ldots \), SAVD-C4; \( \ldots \), SAVD-E4; \( \ldots \), AV-C6.

Figure 4. Numerical simulations of the 1-D Shu-Osher shock turbulence interaction. Density is presented at \( \tau = 1.8 \). \( \bullet \), reference solution (Adams & Stolz 2002) obtained on 1600 grid points with fifth-order ENO-Roe scheme; \( \ldots \), AVD-C6; \( \ldots \), SAVD-C6; \( \ldots \), SAVD-C4; \( \ldots \), SAVD-E4.

3.1.4. One-dimensional Shu-Osher problem

The 1-D shock-entropy wave interaction introduced by Shu & Osher (1989) is investigated to assess the capability of the simplified formulation of localized artificial viscosity and diffusivity on shock-turbulence interaction. Because the entropy waves are sensitive to the numerical dissipation, the schemes using an upwinding to capture discontinuities usually introduce excessive numerical dissipation and the entropy waves are damped. Initial left- and right-side conditions are: \( \rho = 3.857143 \), \( u = 2.629369 \) and \( p = 10.33333 \) for \( x < -4 \), and \( \rho = 1+0.2\sin(5x) \), \( u = 0.0 \) and \( p = 1.0 \) for \( x \geq -4 \). Simulations are performed on a uniformly spaced grid with 201 grid points where the computational domain is \(-5 \leq x \leq 5\) with \( \Delta x = 0.05 \).

Figure 4 shows the comparison between the reference solution and the numerical simulations for the density at the time of \( \tau = 1.8 \). The reference solution is obtained on 1600 grid points with fifth-order ENO-Roe scheme (Adams & Stolz 2002). Numerical simulations of the original and all the simplified formulations capture the shock and entropy waves well. The AVD-C6, SAVD-C6, SAVD-C4 and SAVD-E4 show almost identical density distributions and reasonable agreement with the reference solution.
Two-dimensional double Mach reflection

The first 2-D shock test case on a uniformly spaced Cartesian mesh is the double Mach reflection problem initially used to compare several numerical schemes by Woodward & Collela (1984). Since the shock waves are not aligned to the grid, this problem can be used to assess the impact of the cross-derivative terms on the results using a Cartesian mesh. The original artificial viscosity and diffusivity formulations in Eqs. 2.6 and 2.7 include the cross-derivative term of $\frac{\partial^2 f}{\partial x \partial y}$, whereas the simplified multi-dimensional formulations in Eqs. 2.10 and 2.11 do not have the cross term.

Initially a Mach 10 shock wave is at a 60° angle with a reflecting wall and intersects at the bottom boundary $x=1/6$ and $y=0$. The air ahead of the shock is stationary with a density of 1.4 and a pressure of 1. The same boundary conditions as Woodward & Collela (1984) are employed. The conditions at the top boundary are set to describe the exact motion of the Mach 10 shock. Therefore, the Mach 10 shock keeps the 60° angle and moves to the right of the domain, which creates a double Mach reflection of the shock at the wall. The conditions from $x=0$ to $x=1/6$ at the bottom boundary are fixed as the conditions of the initial post-shock flow and reflecting wall conditions are used from $x=1/6$. The values at the left boundary are fixed to the initial post-shock values, and zero-gradient conditions are employed at the right boundary. Simulations are carried out on a uniformly spaced Cartesian grid with $241 \times 121$ grid points in the $x$- and $y$-directions where the computational domain extends from $x=0$ to $x=4$ and $y=0$ to $y=2$ with $\Delta x = \Delta y = 1/60$.

Density contours in the region $x \in [0,3]$ and $y \in [0,1]$ at the time $\tau = 0.2$ are plotted in Fig. 5. All the results are nearly identical. The results of the SAVDs that ignore the cross-derivative terms do not show a major detrimental effect on the discontinuity-capturing. Two dimensional shock interactions and contact discontinuities including the near-wall jet are well captured at the proper locations in the simulations, which are sensitive to the numerics and difficult to capture. High-resolution characteristics of the present sixth-order compact difference scheme can achieve better representation of the wall jet compared with the high-order upwind-weighted schemes such as a fifth-order WENO scheme (Jiang & Shu 1996), while shock and contact discontinuities are captured by adding the artificial viscosity and diffusivity to suppress high-wavenumber wiggles.
3.3. Two-dimensional oblique shock reflection

The second 2-D shock test case is the oblique shock reflection on an inviscid wall. The shock angle is $33^\circ$ with the Mach 3 freestream. Isotropic and anisotropic Cartesian meshes are used to test the simplified model on a multi-dimensional formulation. This problem can be used to assess the capability of the simplified method on isotropic and anisotropic Cartesian meshes.

The computational domain extends from $x=-1.5$ to $x=1.5$ and $y=0$ to $y=1$ where the isotropic mesh consists of $301 \times 101$ grid points in the $x$- and $y$-directions ($\Delta x = \Delta y = 0.01$) and the mesh aspect ratio of the anisotropic mesh is 5 ($\Delta x = 0.05$, $\Delta y = 0.01$). The shock jump conditions across one grid point and slip-wall conditions are imposed on the upper and lower boundaries, respectively. Inflow conditions are fixed to the freestream and outflow conditions are extrapolated.

3.3.1. An isotropic mesh

Pressure contours in the region $x \in [0,1.5]$ and $y \in [0,0.5]$ obtained by the AVD-C6 and SAVD-C6 on an isotropic mesh are plotted in Fig. 6. The results of the SAVD-C4 and SAVD-E4 are nearly identical to the SAVD-C6 (not shown here). Pressure profiles along $y=0.18$ line for the several schemes are shown in Fig. 7. All the simulations allow for converged solutions without significant wiggles even though the shock wave is not aligned with the mesh. Almost identical results are obtained by the AVD and SAVD methods. Although the shock wave is slightly smeared compared with Roe’s third-order upwind scheme, the proper post-shock conditions are well-recovered.
3.3.2. An anisotropic mesh

Pressure contours and pressure profiles along $y = 0.18$ line obtained by the several schemes on the anisotropic mesh ($\Delta x = 5\Delta y$) are plotted in Figs. 8 and 9. The pressure contours of SAVD-C4 and SAVD-E4 are not presented here because all the SAVD methods show almost identical results. The shock wave obtained by the AVD-C6 is considerably smeared and is shifted upstream because of the undesirable mesh dependency. It also appears that the shock is slightly unsteady and cannot achieve a converged solution. On the other hand, the SAVD methods capture the shock wave well at the location similar to that of the Roe scheme and show converged solutions. Consistent with the isotropic test case (3.3.1), the shock is slightly smeared compared with the Roe scheme but the the proper post-shock conditions are well recovered.

3.4. Two-dimensional supersonic blunt body flow

The last test case is the 2-D blunt body in a Mach 3 inviscid flow. The simulations are carried out on a curvilinear and non-unity aspect ratio mesh where the grid spacing perpendicular to the front bow shock is smaller than that in the other direction. This test case allows us to investigate the capability of the multi-dimensional curvilinear and anisotropic mesh formulations of the simplified artificial viscosity and diffusivity. The effect of the cross-derivative terms can also be assessed for a generalized coordinate system.

An impulsive start of freestream Mach number 3 is imposed on the simulations. Therefore the Mach 3 front bow shock gradually develops from the blunt body toward the left. Reflecting wall conditions are imposed on the blunt body, and the inflow boundary conditions are fixed to freestream conditions. Fourth-order extrapolation is employed at the
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Figure 10. Numerical simulations of the 2-D Mach 3 blunt body flow. Pressure, 20 contours from 0.94 to 12, (a) AVD-C6, (b) SAVD-C6, (c) SAVD-C4, (d) SAVD-E4. Artificial viscosity, (e) AVD-C6, (f) SAVD-C6, (g) computational grid.

Figure 11. Pressure profiles at centerline $y=0$ of the 2-D Mach 3 blunt body flow. —— with $\bullet$, AVD-C6; —— with filled $\Delta$, SAVD-C6; ——, SAVD-C4; ———, SAVD-E4; ———, AV-C6.

outflow boundaries. A curvilinear and anisotropic mesh of $81 \times 61$ is analytically (Jiang & Shu 1996) generated by using:

\begin{align}
    x &= -(R_x - (R_x - 1)\eta)\cos(\theta(2\xi - 1)), \\
    y &= (R_y - (R_y - 1)\eta)\sin(\theta(2\xi - 1)),
\end{align}

where the parameters are set to $R_x = 3$, $R_y = 6$, and $\theta = 5\pi/12$.

Figure 10 displays pressure contours (a)-(d), artificial viscosity distributions (e)-(f) and the computational grid (g). Pressure profiles along the centerline $y=0$ are plotted in Fig. 11. Only the AVD-C6 could not achieve a converged solution. Therefore the result of AVD-C6 shows the snapshot after a long period at the computational time $\tau = 25$. The AVD-C6 shows relatively high artificial viscosity around the centerline of the bow shock region compared with the SAVD. This is due to the undesirable effects from the geometrically-averaged grid spacing. For AVD-C6, the artificial viscosity smears the bow shock over 10 grid points and induces non-physical oscillation in the front bow shock. On the other hand, the multi-dimensional curvilinear and anisotropic mesh formulations of the simplified artificial viscosity and diffusivity works well on a curvilinear and non-unity...
aspect ratio mesh. The SAVD-C6, SAVD-C4 and SAVD-E4 show the converged solutions without spurious oscillations and provide almost identical results. The SAVD-C6, SAVD-C4 and SAVD-E4 capture the shock over 4–5 grid points. The shock location at \( x/R = -1.7 \) and post-shock conditions are in good agreement with the results obtained by the simulations with 201 × 121 mesh (only upper-half domain) using the hybrid compact-Roe and Roe schemes (Visbal & Gaitonde 2005).

### 4. Conclusions

Simple and efficient localized high-wavenumber biased artificial viscosity and diffusivity schemes in a multi-dimensional generalized coordinate framework have been developed for capturing discontinuities based on the original 1-D formulation proposed by Cook & Cabot (2004, 2005) and Fiorina & Lele (2007). Computational efficiency benefit came from approximating the double Laplacians with the direct implementation of fourth derivatives.

The method has been successfully applied to 1- and 2-D shock-related problems on isotropic and anisotropic Cartesian meshes and an anisotropic curvilinear mesh. The extension of the method allows us to apply the method on multi-dimensional curvilinear and anisotropic meshes while maintaining the original properties of the 1-D formulation of artificial viscosity and diffusivity. The simplification of the method reduces computational cost and does not show any major detrimental effect on the discontinuity-capturing under the conditions examined. Almost identical results are obtained for the problems by using the different schemes that evaluate the fourth derivatives in the formulations of the simplified artificial viscosity and diffusivity. Therefore the SAVD-E4 method can be an attractive choice for reducing computational costs when the method is applied to real, practical applications on a multi-dimensional generalized coordinate system. The SAVD-E4 reduces the cost for calculating the artificial viscosity and diffusivity by a factor of 2 compared with the AVD-C6. This leads to a reduction of approximately 14% in the total computational cost. The authors have applied the SAVD-E4 method to the practical problem of an under-expanded sonic jet injection into a supersonic crossflow and the results show the capability of the method for the LES (Kawai & Lele 2007).

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### REFERENCES


