Dimensional reduction approach to shock-capturing

By P. Stinis†

1. Motivation and objectives

Compact (Padé) finite difference schemes are attractive because of their spectral like properties. Unfortunately, they cannot be used for shock-capturing purposes. Inspired by recent work on dimensional reduction (Chorin & Stinis (2005)), we propose a shock-capturing algorithm which combines a compact finite difference scheme with a Riemann solver. In a compact finite difference, the flux derivative at each grid point is a function of the fluxes at all the points. The algorithm divides the available grid into resolved and unresolved points. The flux on the resolved points is evaluated by the current value of the solution there. The necessary flux values on the unresolved points are provided by a Riemann solver. In essence, the Riemann solver acts as a projection of the flux at an unresolved point on the fluxes of the neighboring resolved points.

2. The shock-capturing algorithm

Suppose that we want to solve with a compact finite difference scheme a system of time-dependent partial differential equations whose solution develops shocks. For the sake of clarity we limit our attention to the case of one spatial dimension and the equations are in conservative form. Suppose that the spatial domain has length L, and let the grid on which we want to compute the solution consist of N points. This results in a grid spacing $h = \frac{L}{N}$. If we use the compact scheme on all the points, then the predicted solution will develop oscillations around the location of the shock. A compact scheme is essentially equivalent to an expression of the flux derivative at one point as a linear combination of the flux values at all the points on the grid. For example, for a flux function $f(x)$, the standard fourth-order Pade scheme (Lele (1992)), for the flux derivative $f'_i = f'(x_i)$ at a point $x_i,\ i = 1, \dots, N$ is given by

$$\frac{1}{4} f'_{i-1} + f'_i + \frac{1}{4} f'_{i+1} = \frac{3}{2} \frac{f_{i+1} - f_{i-1}}{2h}. \tag{2.1}$$

Divide the grid points into two subsets, the resolved points and the unresolved points. We will evolve the solution on the resolved points *only*. However, this will require the specification of the flux at *all* the points, both resolved and unresolved. For the sake of concreteness, choose every other point on the grid as resolved. This would result in $\frac{N}{2}$ resolved points and $\frac{N}{2}$ unresolved points. If x_i is a resolved point, we see from Eq. (2.1) that the r.h.s. of the equation for the flux derivative involves the flux values at unresolved points. Similarly, the RHS of the equation of the flux derivative for an unresolved point involves the flux values at resolved points. In order to solve the tridiagonal system for the

† Current address: Department of Mathematics, University of Minnesota, Minneapolis, MN 55455

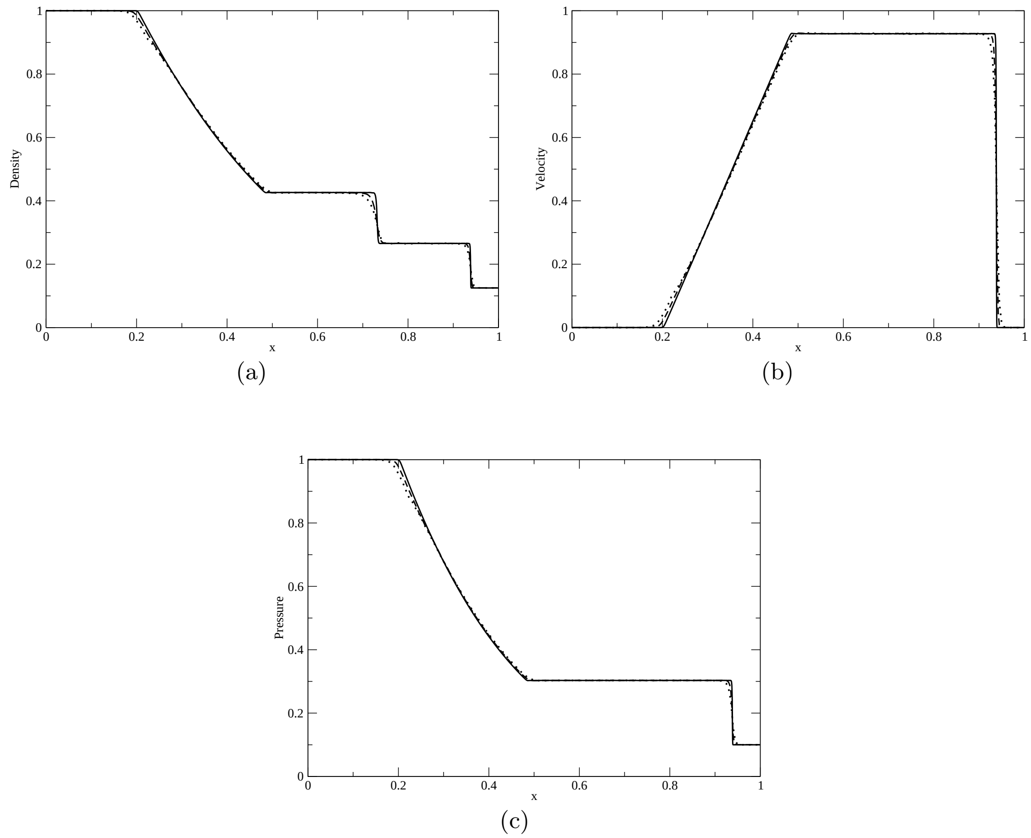


FIGURE 1. Sod problem. (a) Density at time t=.25, (b) Velocity at time t=.25, (c) Pressure at time t=.25. $N = 400$ ($\cdots$), $N = 800$ (- - -) , $N = 5000$ (—).

flux derivatives one has to specify the flux at *all* the points, resolved and unresolved. For the resolved points, the flux is evaluated directly using the current value of the solution at the resolved point. For the unresolved points, we provide the flux value by a Riemann solver. A Riemann solver expresses the value of the flux at a point as a function of the values at points to the left and right of the point. In our case, we build the Riemann problem at an unresolved point using the flux values at the neighboring resolved points. Thus, the Riemann solver in essence provides us with a projection of the flux value at the unresolved points on the flux values at the resolved points.

3. Numerical results

We present results for three standard 1-D test cases, namely the Sod problem (Sod (1978)), the Shu-Osher (turbulence-shock wave interaction) problem (Shu & Osher (1989)) and the Colella-Woodward (intense blast) problem (Woodward & Colella (1984)). We use the standard fourth-order compact scheme coupled to Roe's approximate Riemann solver along with the standard entropy fix (Roe (1981); Toro (1999)). The approximate Riemann solver requires the flux values to the left and right of the point where the flux will

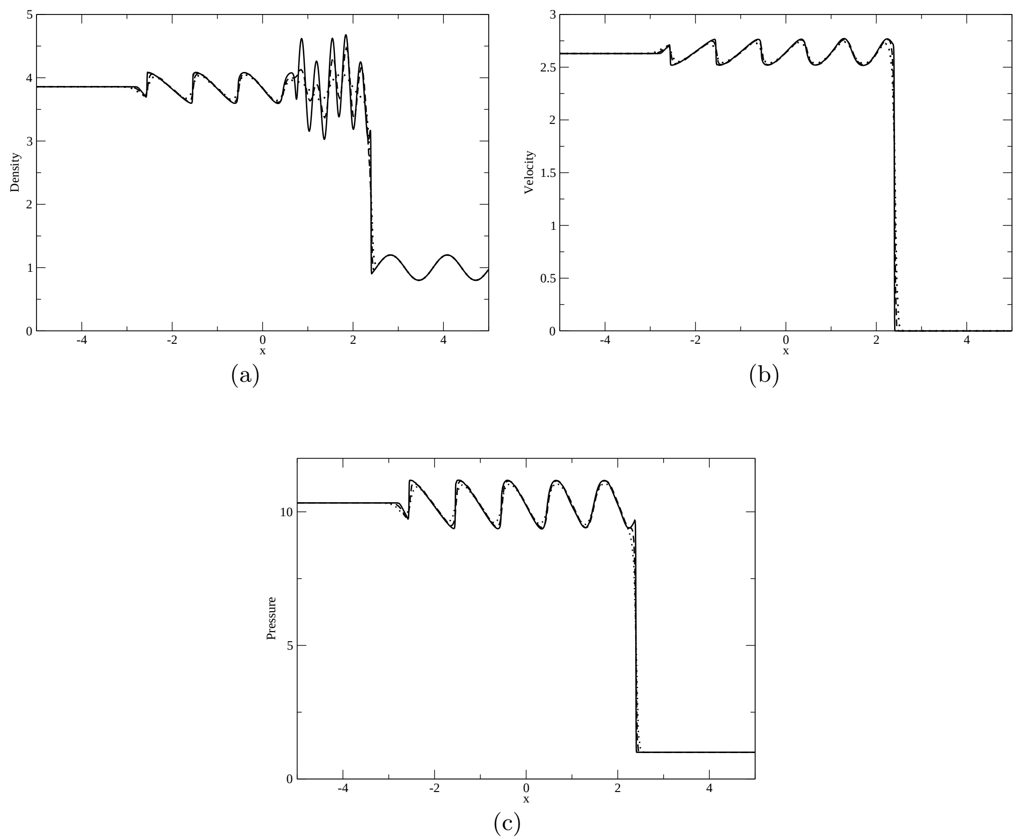
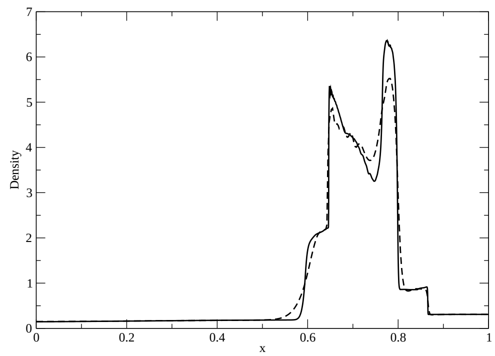


FIGURE 2. Shu-Osher problem. (a) Density at time t=1.8, (b) Velocity at time t=1.8, (c) Pressure at time t=1.8. $N = 400$ $(\cdots)$, $N = 800$ (- - -) , $N = 5000$ (—).

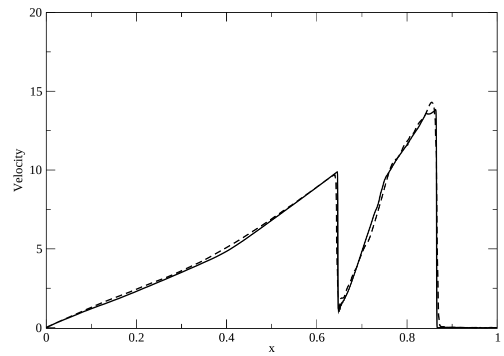
be evaluated. We provide these values using ENO reconstruction based on the values of the solution at the resolved points (Shu & Osher (1989)). In the numerical experiments we use ENO reconstruction of order 3 with the reconstruction performed on the characteristic variables (Toro (1999)). From the Figs.1-3 it is obvious that the incorporation of the Riemann solver eliminates any spurious oscillations around the shocks.

4. Future work

We have presented a hybrid compact finite difference-Riemann solver scheme which was constructed in order to alleviate the spurious oscillation problems of compact finite-difference schemes around shocks. After completing the current project we discovered three publications that present coupled compact scheme-Riemann solver algorithms (Adams & Shariff (1996); Deng & Maekawa (1997); Wang & Huang (2002)) which however are different from the current approach. In particular, they either involve the application of the Riemann solver only around the shocks or the use of staggered compact schemes. Our approach stems from the standard practice of dimensional reduction where one follows the solution on a coarser grid than one can afford and uses the rest of the available



(a)



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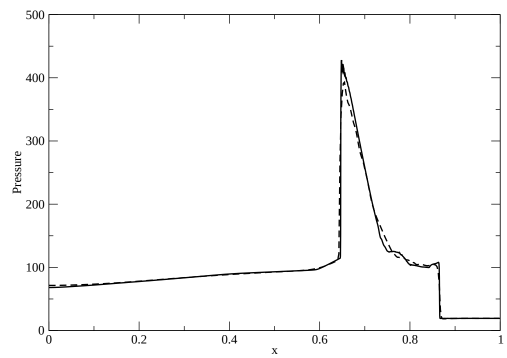

(c)

FIGURE 3. Colella-Woodward problem. (a) Density at time t=0.038, (b) Velocity at time t=0.038, (c) Pressure at time t=0.038. $N = 800$ (**- - -**) , $N = 5000$ (—).

resolution to effect the subgrid modeling. It is in the same spirit as some previous work by the author (Stinis (2003)). It would be interesting to see how the scheme performs in more realistic applications and how it compares with other approaches.

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