# Boundary procedures for the time-dependent stochastic Burgers' equation

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## 1. Motivation and objective

In many physical problems our knowledge is limited by our ability to measure, by our bias in the observations and, in general, by an incomplete understanding of the physical processes. When we attempt to simulate the problem numerically, we must account for those limitations, and in addition, we must identify the possible limitations of the numerical techniques and phenomenological models that we employ.

Numerical simulations are subject to uncertainty in boundary or initial conditions, model parameter values, and even in the geometry of the physical domain of the problem; this results in uncertainty in the output data that must be clearly identified and quantified. Fields of application of uncertainty quantification include, but are not limited to, turbulence, climatology (Poroseva *et al.* 2005), combustion (Reagan *et al.* 2003), flow in porous media (Christie *et al.* 2006), fluid mixing (Yu *et al.* 2006), and computational electromagnetics (Chauvière *et al.* 2006).

The problem considered in this paper (the uncertain position of the shock in Burgers' equation) models the more important one of the shock position on a wing. We perform an uncertainty quantification analysis using the stochastic Burgers' equation and employ a spectral representation of the solution in the form of polynomial chaos expansion. The equation is stochastic as a result of uncertainty in the initial and boundary values. Stochastic Galerkin projection of the stochastic Burgers' equation results in a deterministic system of equations from which the expected values and variance of the solution can be determined.

Uncertainty quantification of scalar hyperbolic problems with the polynomial chaos approach gives rise to new hyperbolic systems with multiple discontinuities. The weak resemblance to the corresponding deterministic problems would typically give a hint on how to specify boundary conditions to the mean, but gives no real clue to the treatment of higher order moments.

Due to the lack of boundary data as well as to the computational cost of higher order polynomial chaos simulations, low-order approximation with appropriate utilization of available data is a viable option. In this paper we investigate to what extent low-order approximations can be used when appropriate high-order boundary data are missing. Because of the hyperbolic nature of the problem, information is traveling with finite but unknown speed through the domain and will eventually affect the boundary.

By the convergence properties of the polynomial chaos series expansion, higher-order boundary terms are expected to decrease rapidly. On the other hand, although small, these coefficients have a relatively large impact on the system eigenvalues and might thus

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be crucial for accurate boundary treatment. In addition to this, there are discontinuities in the stochastic dimension (we assume only one stochastic dimension), which deteriorate the convergence. The net effect of the higher-order boundary coefficients is not clear and motivates the investigation of this paper.

## 2. Polynomial chaos expansion of Burgers' equation

The polynomial chaos representation $u(x,t,\xi) = \sum_{i=0}^{\infty} u_i \Psi_i(\xi)$ is inserted into the Burgers' equation,

$$u_t + uu_x = 0, \ 0 \le x \le 1, \tag{2.1}$$

which yields

$$\sum_{i=0}^{\infty} \frac{\partial u_i}{\partial t} \Psi_i(\xi) + \left( \sum_{j=0}^{\infty} u_j \Psi_j(\xi) \right) \left( \sum_{i=0}^{\infty} \frac{\partial u_i}{\partial x} \Psi_i(\xi) \right) = 0, \tag{2.2}$$

where $\{\Psi_i\}_{i=0}^{\infty}$ is the set of Hermite polynomials and $\xi \in N(0,1)$. A stochastic Galerkin projection is performed by multiplying (2.2) by $\Psi_k(\xi)$ for non-negative integers $k$ and integrating over the probability domain $\Omega$. The orthogonality of the basis polynomials then yields a system of deterministic equations. By truncating the number of polynomial chaos coefficients to a finite number $M$, the solution is projected onto a finite dimensional deterministic space. The result is a symmetric system of equations.

$$\frac{\partial u_k}{\partial t} \langle \Psi_k^2 \rangle + \sum_{i=0}^{M} \sum_{j=0}^{M} u_i \frac{\partial u_j}{\partial x} \langle \Psi_i \Psi_j \Psi_k \rangle = 0 \qquad \text{for } k = 0, 1, ..., M. \tag{2.3}$$

For simplicity of notation, equation (2.3) can be written in matrix form as

$$Bu_t + A(u)u_x = 0 \qquad \text{or} \qquad Bu_t + \frac{1}{2} \frac{\partial}{\partial x} (A(u)u) = 0, \tag{2.4}$$

which is the form that will be used in the section about well-posedness.

As an illustration, the $3 \times 3$ system given by truncating the expansion to $M = 2$ with a Hermite polynomial basis for Burgers' equation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} u_0 & u_1 & 2u_2 \\ u_1 & u_0 + 2u_2 & 2u_1 \\ 2u_2 & 2u_1 & 2u_0 + 8u_2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}_x = 0.$$

Note that the matrix $A(u)$ is symmetric and that the mass matrix $B$ is diagonal.

## 3. Problem setup

In order to quantify how accurate the results are, we need an analytical solution to our problem. Consider the stochastic Riemann problem with an initial shock location $x_0 \in \Omega$

$$\begin{aligned} &u(x,0,\xi) = \begin{cases} u_L = a + P(\xi) & \text{if } x < x_0 \\ u_R = -a + P(\xi) & \text{if } x > x_0 \end{cases} \\ &u(0,t,\xi) = u_L, \ u(1,t,\xi) = u_R, \\ &\xi \in N(0,1). \end{aligned} \tag{3.1}$$

As the most intuitive choice of polynomial basis with regard to the boundary uncertainty, the set of Hermite polynomials will be used. Here we only consider $P(\xi) = b\xi$ as a first order stochastic polynomial and $a$ is a constant. By the Rankine-Hugoniot condition, the shock speed is given by $s = P(\xi)$, so for any bounded $\xi$ the shock location $x_s$ is

$$x_s = x_0 + tP(\xi).$$

The solution (for any bounded $\xi$) is given by

$$u(x,t,\xi) = \begin{cases} u_L & \text{if } x < x_0 + tP(\xi) \\ u_R & \text{if } x > x_0 + tP(\xi) \end{cases}.$$

The solution is uniquely given by the countable set of polynomial chaos coefficients $\{u_0, u_1, ...\}$ where

$$u_i(x,t) = \frac{1}{\langle \Psi_i^2 \rangle} \int_{-\infty}^{\infty} u(x,t,\xi)\Psi_i(\xi)f(\xi)d\xi. \tag{3.2}$$

An analytical solution to the stochastic problem can be derived by the use of deterministic techniques for Riemann problems (Pettersson *et al.* 2009). We consider polynomial chaos of order $M = 1$, and the analytical solution to the corresponding $2 \times 2$-system (section 5).

Expectation and variance can be expressed in terms of the polynomial chaos coefficients, as

$$\mathbf{E}(u) = u_0 \tag{3.3}$$

and

$$\text{Var}(u) = \sum_{i=1}^{\infty} u_i \langle \Psi_i^2 \rangle, \tag{3.4}$$

respectively. Clearly, $\mathbf{E}(u)$ (i.e., $u_0$) will be available (however distorted) no matter the order of truncation of the system, whereas only the first few coefficients are used to approximate the variance. Clearly, the order of truncation strongly affects the accuracy of the variance approximation.

## 4. Well-posedness and stability

In order to ensure stability of the discretized system of equations, summation by parts operators and weak imposition of boundary conditions (Carpenter *et al.* 1994; Nordström & Carpenter 1999, 2001; Carpenter *et al.* 1999) are used to obtain energy estimates. The system is expressed in a split form that combines the conservative and non-conservative formulation (Nordström 2006). A particular set of artificial dissipation operators (Mattsson *et al.* 2004) is used to enhance the stability close to the shock. Burgers' equation has been discretized with a fourth-order central difference operator in space and the fourth-order Runge-Kutta method in time. For stability, artificial dissipation is added based on the local system eigenvalues. The order of accuracy is not affected by the addition of artificial dissipation.

The dominating error is instead due to truncation of the polynomial chaos expansion. General difficulties related to solving hyperbolic problems and nonlinear conservation laws with spectral methods are discussed in (Gottlieb & Hesthaven 2001; Pettersson *et al.* 2009).

For the truncated system of arbitrary order that is used to approximate the original stochastic problem, we investigate well-posedness. A problem is well-posed (Gustafsson *et al.* 1995; Nordström 2006) if the solution that exists is unique and depends continuously on the problem data.

The system is written in split form as

$$Hu_t + \beta\frac{\partial}{\partial x}(\frac{A}{2}u) + (1-\beta)Au_x = 0,\ 0 \leq x \leq 1,$$

The solution is assumed to be smooth. After multiplication by $u^T$ and integration by parts, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u\right\|_H^2 = -\frac{\beta}{2}[u^T Au]_{x=0}^{x=1} + \frac{\beta}{2}\int_0^1 u_x^T Au dx - (1-\beta)\int_0^1 u^T Au_x dx, \quad (4.1)$$

where we choose $\beta = 2/3$. The energy method yields an energy estimate of the form

$$\left\|u\right\|_\Omega^2 + \frac{2}{3}\int_0^t \left\|w_0\right\|_\Gamma^2 + \left\|w_1\right\|_\Gamma^2 d\tau \leq \left\|f\right\|_\Omega^2 + \frac{4}{3}\int_0^t \left\|g_0\right\|_\Gamma^2 + \left\|g_1\right\|_\Gamma^2 d\tau, \quad (4.2)$$

where $f(x)$ is the initial function and $g_0(t), g_1(t)$ boundary conditions corresponding to $x = 0$ and $x = 1$, respectively. Because $w = V^{-1}u$ and $\left\|w\right\| \leq \left\|V^{-1}\right\|\left\|u\right\| \leq C\left\|u\right\|$ for some $C < \infty$, the estimate (4.2) leads to strong well-posedness. To obtain stability, we use the so-called penalty technique (Mattsson *et al.* 2004) to impose boundary conditions for the discrete problem (Pettersson *et al.* 2009). Let $E_0 = (e_{ij})$ where $e_{11} = 1, e_{ij} = 0, \forall i,j \neq 1$, and $E_n = (e_{ij})$, and where $e_{nn} = 1, e_{ij} = 0, i,j \neq n$. Define the block diagonal matrix $A_g$ where the diagonal blocks are the symmetric matrices $A(u(x))$. With penalty matrices $\Sigma_0$ and $\Sigma_1$ corresponding to the left and right boundaries, respectively, the discretized system can be expressed as

$$(I\otimes H)u_t + A_g(P^{-1}Q\otimes I)u = (P^{-1}\otimes I)(E_0\otimes\Sigma_0)(u-g_0) + (P^{-1}\otimes I)(E_n\otimes\Sigma_1)(u-g_1). \quad (4.3)$$

Similarly, the conservative system in (2.4) can be discretized as

$$(I\otimes B)u_t + \frac{1}{2}(P^{-1}Q\otimes I)A_g u = (P^{-1}\otimes I)(E_0\otimes\Sigma_0)(u-g_0) + (P^{-1}\otimes I)(E_n\otimes\Sigma_1)(u-g_1). \quad (4.4)$$

A linear combination of the conservative and the non-conservative form is used for the energy estimates, just as in the continuous case. The split form is given by

$$\begin{aligned}(I\otimes H)u_t + \beta\frac{1}{2}(P^{-1}Q\otimes I)A_g u + (1-\beta)A_g(P^{-1}Q\otimes I)u = \\ = (P^{-1}\otimes I)\left[(E_0\otimes\Sigma_0)(u-g_0) + (E_n\otimes\Sigma_1)(u-g_1)\right]. \quad (4.5)\end{aligned}$$

Multiplication by $u^T(P\otimes I)$ and then addition of the transpose of the resulting equation yields

$$\begin{aligned}\frac{\partial}{\partial t}\left\|u\right\|_{(P\otimes H)}^2 + \frac{\beta}{2}u^T\left((Q\otimes I)A_g + A_g(Q^T\otimes I)\right)u + \\ + (1-\beta)u^T\left(A_g(Q\otimes I) + (Q^T\otimes I)A_g\right)u = \\ = 2u^T(E_0\otimes\Sigma_0)(u-g_0) + 2u^T(E_n\otimes\Sigma_1)(u-g_1). \quad (4.6)\end{aligned}$$

With the choice $\beta = 2/3$, the energy methods yields

$$\frac{\partial}{\partial t}\left\|u\right\|_{(P\otimes B)}^2 = \frac{2}{3}\left(u_{x=0}^T Au_{x=0} - u_{x=1}^T Au_{x=1}\right) + 2u_{x=0}^T\Sigma_0(u_{x=0}-g_0) + 2u_{x=1}^T\Sigma_1(u_{x=1}-g_1). \quad (4.7)$$

Restructuring (4.7) yields

$$\frac{\partial}{\partial t}\|u\|^2_{(P\otimes B)} = u^T_{x=0}(\frac{2}{3}A+2\Sigma_0)u_{x=0} - 2u^T_{x=0}\Sigma_0 g_0 - u^T_{x=1}(\frac{2}{3}A-2\Sigma_1)u_{x=1} - 2u^T_{x=1}\Sigma_1 g_1. \tag{4.8}$$

Stability is achieved by a proper choice of the penalty matrices $\Sigma_0$ and $\Sigma_1$. For that purpose $A$ is split according to the sign of its eigenvalues as

$$A = A^+ + A^- \text{ where } A^+ = x^T\Lambda^+ x \text{ and } A^- = x^T\Lambda^- x. \tag{4.9}$$

Choose $\Sigma_0$ and $\Sigma_1$ such that $\frac{2}{3}A^+ + 2\Sigma_0 = -\frac{2}{3}A^+ \Leftrightarrow \Sigma_0 = -\frac{2}{3}A^+$ and $\frac{2}{3}A^- - 2\Sigma_1 = \frac{2}{3}A^- \Leftrightarrow \Sigma_1 = \frac{2}{3}A^-$. We now get the energy estimate

$$\begin{aligned}\frac{\partial}{\partial t}\|u\|^2_{(P\otimes B)} &= -\frac{2}{3}(u_{x=0}-g_0)^T A^+(u_{x=0}-g_0) + \frac{2}{3}\left[u^T_{x=0}A^-u_{x=0} + g_0^T A^+ g_0\right] \\ &\quad - \frac{2}{3}\left[u^T_{(x=1)}A^+u_{(x=1)} + g_1^T A^- g_1\right] + \frac{2}{3}(u_{(x=1)}-g_1)^T A^-(u_{(x=1)}-g_1),\end{aligned} \tag{4.10}$$

which shows that the system is stable.

In the short summary of well-posedness and stability analysis above we have assumed that we have perfect knowledge of boundary data, but in practice this is rarely true. In practical calculations lack of data makes such analysis impossible and one has to rely on estimates to assign boundary data. We investigate the effect of that problem next.

## 5. Dependence on available data

For $M = 1$, the system (2.4) can be diagonalized with constant eigenvectors and we get an exact solution to the truncated problem. With $a$ and $b$ as in the problem setup (section 3), the analytical solution for the $2\times 2$-system ($x \in [0,1]$) is given by

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{cases} \left.\begin{array}{ll} (a,b)^T & \text{if } x < x_0 - bt \\ (0,a+b)^T & \text{if } x_0 - bt < x < x_0 + bt \\ (-a,b)^T & \text{if } x > x_0 + bt \end{array}\right\} & \text{for } 0 \le t < \frac{x_0}{b} \\ (0,a+b)^T & \text{for } t > \frac{x_0}{b} \end{cases}. \tag{5.1}$$

We expect different numerical solutions depending on the amount of available boundary data. We assume that the boundary data are known on the boundary $x = 1$ and investigate three different cases for the left boundary $x = 0$ corresponding to a complete set of data, partial information about boundary data, and no data available, respectively. For all cases, we solve a system of the form

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}_t + \frac{1}{2}\left[\begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right]_x = 0, \tag{5.2}$$

with boundary data

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}_{x=0} = \begin{pmatrix} g_0(t) \\ g_1(t) \end{pmatrix};\quad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}_{x=1} = \begin{pmatrix} h_0(t) \\ h_1(t) \end{pmatrix}.$$

### 5.1. *Complete set of data*

The boundary conditions are

$$u(0,t) = \begin{cases} (a,b)^T & 0 \le t < \frac{x_0}{b} \\ (0,a+b)^T & t > \frac{x}{b} \end{cases}. \tag{5.3}$$

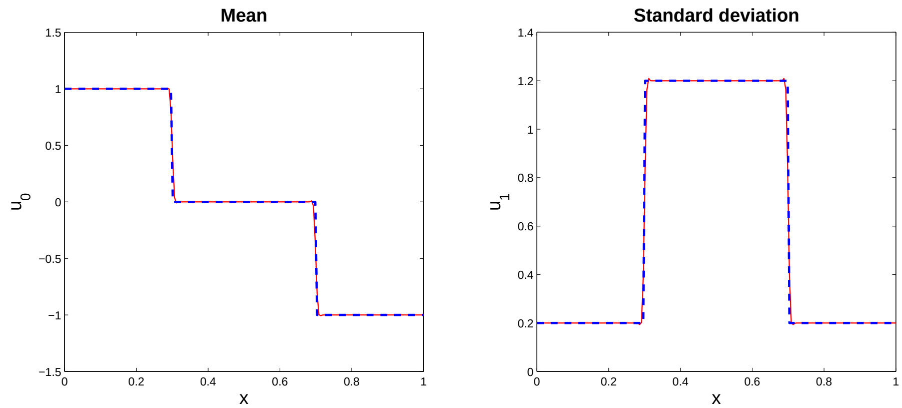


FIGURE 1. Analytical solution: - - - - ; Numerical solution: —— . $u_0$ (left) and $u_1$ (right). $t = 1$. Complete set of data.

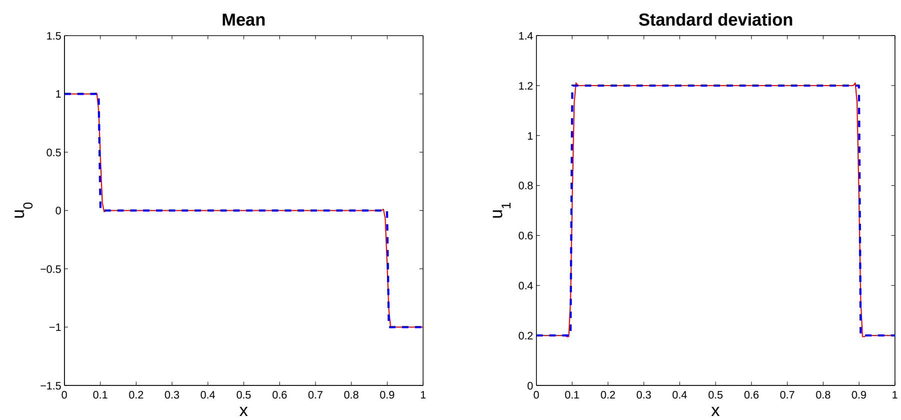


FIGURE 2. Analytical solution: - - - - ; Numerical solution: —— . $u_0$ (left) and $u_1$ (right). $t = 2$. Complete set of data.

Consider $a = 1$, $b = 0.2$. Both $u_0$ and $u_1$ are known at $x = 0$ and the two ingoing characteristics are assigned the analytical values. The system satisfies the energy estimate (4.10) and is stable. Figs. 1, 2, and 3 show the solution at time $t = 1$, $t = 2$, and $t = 3$, respectively.

### 5.2. *Incomplete set of boundary data*

Without a complete set of boundary data, the time-dependent behavior of the solution is hard to predict. Here we assume that the boundary condition at $x = 1$ is $u = (-a, b)$ as in (5.3) and consider different ways of dealing with unknown data at $x = 0$. The initial function is the same as in the analytical problem above, i.e.,

$$(u_0(x,0), u_1(x,0))^T = \begin{cases} (a,b)^T & \text{if } x < x_0 \\ (-a,b)^T & \text{if } x > x_0 \end{cases} .$$

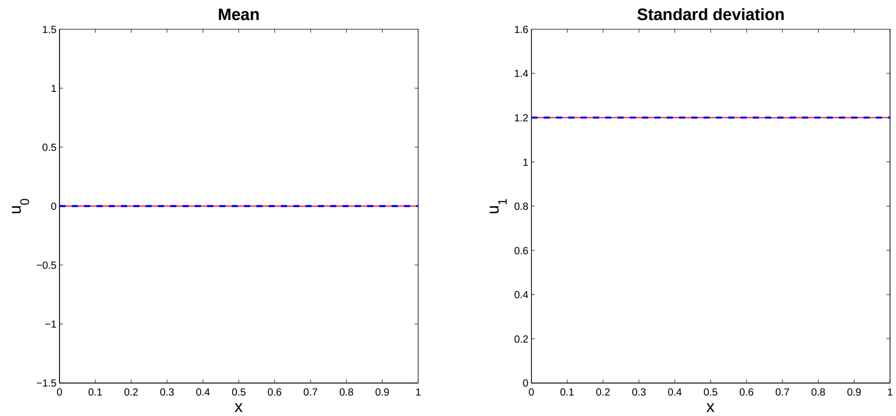


FIGURE 3. Analytical solution: - - - - ; Numerical solution ——— . $u_0$ (left) and $u_1$ (right). $t = 3$. Complete set of data.

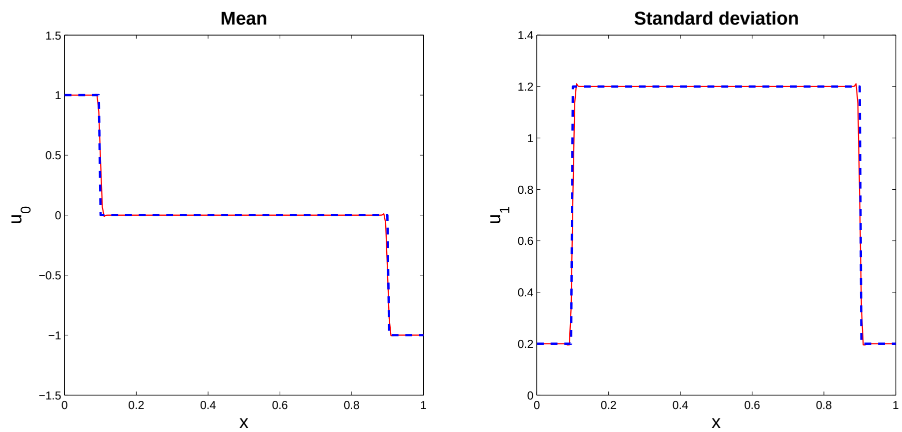


FIGURE 4. Analytical solution: - - - - ; Numerical solution: ——— . $u_1$ kept fixed at 0.2. $t = 2$.

5.2.1. $u_1$ *unknown at* $x = 0$, *guess* $u_1$

First assume that $u_0$ is known and $u_1$ is unknown and put $u_1 = 0.2$ at the boundary for all time. This problem setup leads to an energy estimate and stability. There are two ingoing characteristics at $t = 0$. $u_0$ at $x = 0$ changes with the boundary conditions of the analytical solution as given by (5.3). The time development follows the analytical solution at first (Fig. 4) but eventually becomes inconsistent with the boundary conditions (Figs. 5 and 6).

5.2.2. $u_1$ *unknown at* $x = 0$, *extrapolate* $u_1$

Now, the extrapolation $g_1 = (u_1)_1$ is used to assign boundary data to the presumably unknown coefficient $u_1$. This case does not lead to stability using the energy method. As long as the analytical boundary conditions do not change, the numerical solution follows the analytical solution as before (Fig. 7). After $t = 2.5$ the characteristics have reached the opposite boundaries and the error grows (Fig. 8) before reaching the steady state (Fig. 9).

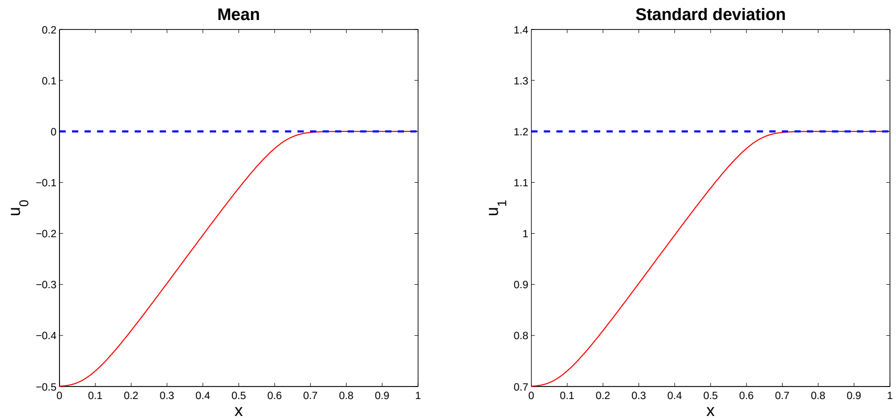


FIGURE 5. Analytical solution: - - - - ; Numerical solution: ——— . $u_1$ kept fixed at 0.2. $t = 3$.

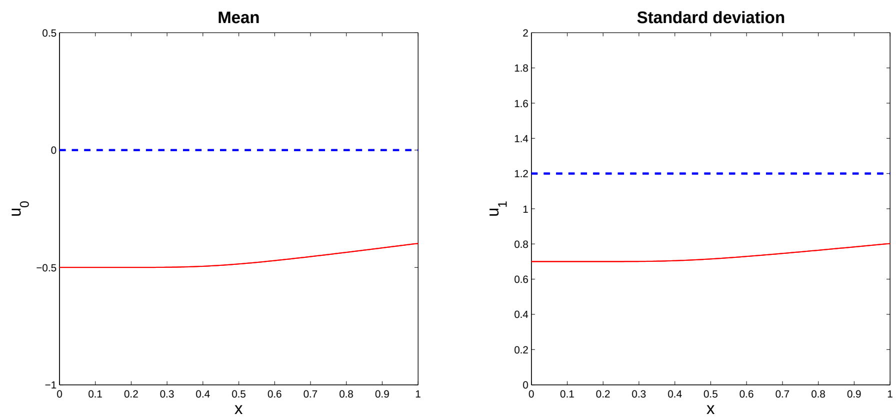


FIGURE 6. Analytical solution: - - - - ; Numerical solution: ——— . $u_1$ kept fixed at 0.2. $t = 5$.

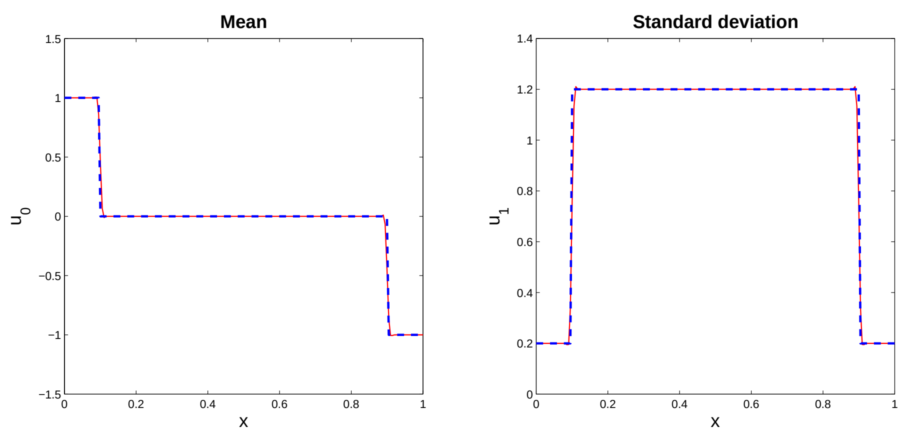


FIGURE 7. Analytical solution: - - - - ; Numerical solution: ——— . $u_1$ extrapolated from the interior. $t = 2$.

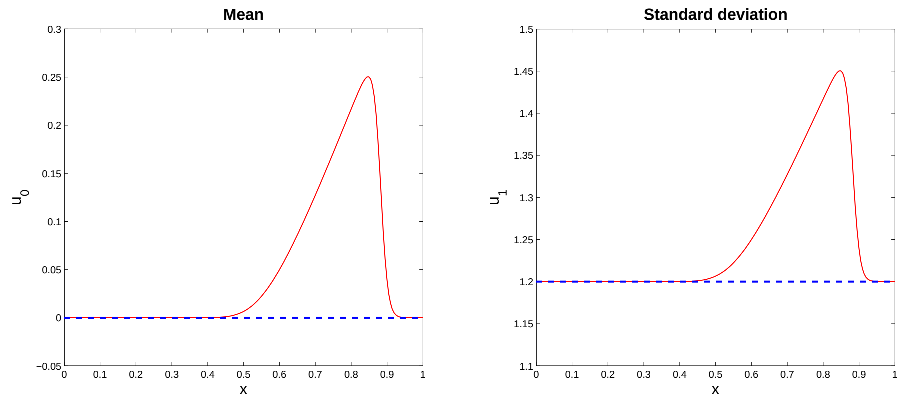


FIGURE 8. Analytical solution: - - - - ; Numerical solution: —— . $u_1$ extrapolated from the interior. $t = 3$.

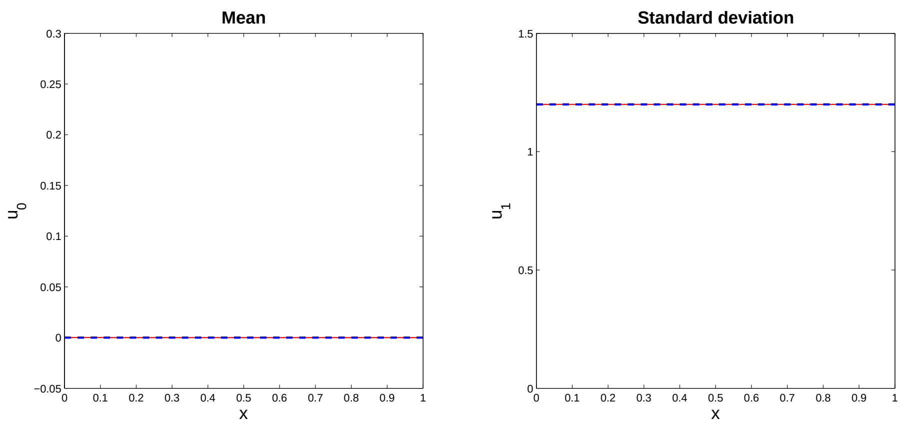


FIGURE 9. Analytical solution: - - - - ; Numerical solution: —— . $u_1$ extrapolated from the interior. $t = 5$. The discrete norm of the error, $\|\epsilon\|_2 = (\Delta x \sum_{j=1}^{m} (u_i - u_i^{ref})_j)^{1/2}$, is of the order $10^{-15}$ for $u_0$ and $u_1$.

### 5.2.3. *$u_0$ unknown at $x = 0$, guess $u_0$*

Next we assume that the boundary data for $u_0$ is unknown. This case leads to an energy estimate and stability. The same analysis is carried out for $u_0$ as was done for $u_1$ in the preceding section. First $u_0$ at $x = 0$ is held fixed for all times. Figs. 10 and 11 show the solution before and after the true characteristics reach the boundaries. Note that the solution after a long time is not coincident with the analytical solution and that the boundary conditions are not satisfied (Fig. 12).

### 5.2.4. *$u_0$ unknown at $x = 0$, extrapolate $u_0$*

The data for $u_0$ can instead be extrapolated from the interior of the domain. The extrapolation $g_0 = (u_0)_1$ is used (see Figs. 13, 14, and 15). This case does not lead to stability using the energy method. Note that the solution after a long time is very close to the analytical solution (Fig. 15).

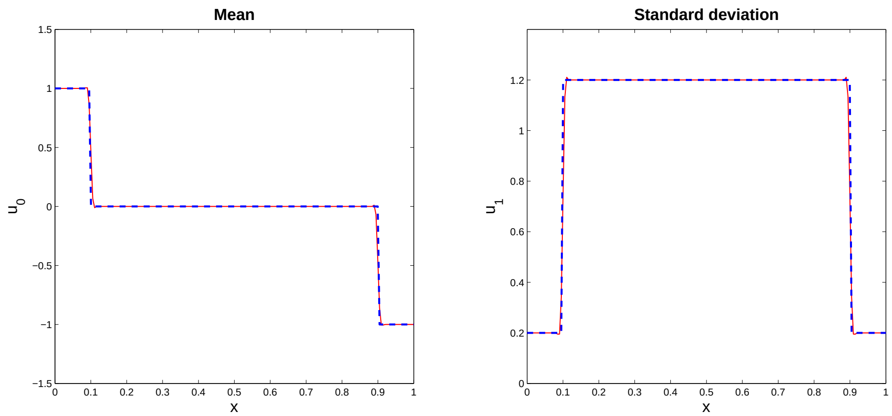


FIGURE 10. Analytical solution: ---- ; Numerical solution: —— . $u_0$ is held fixed. $t = 2$.

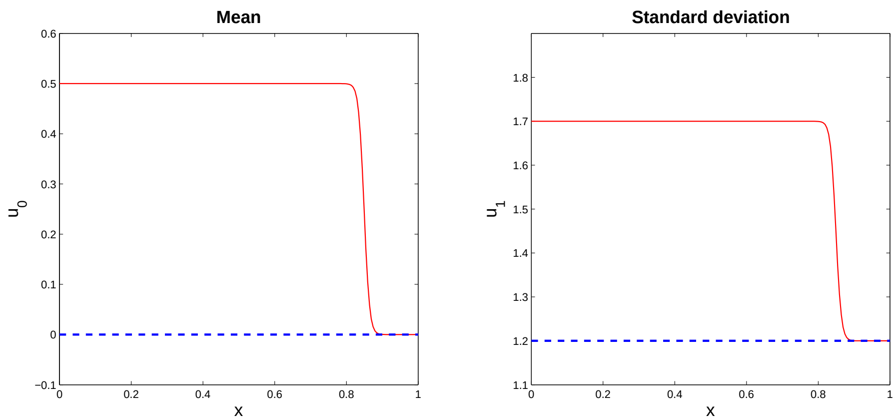


FIGURE 11. Analytical solution: ---- ; Numerical solution —— . $u_0$ is held fixed. $t = 3$.

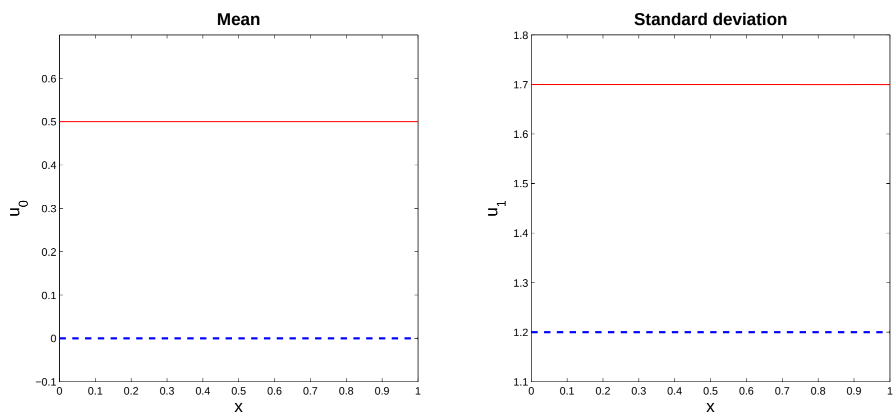


FIGURE 12. Analytical solution: ---- ; Numerical solution: —— . $u_0$ is held fixed. $t = 5$.

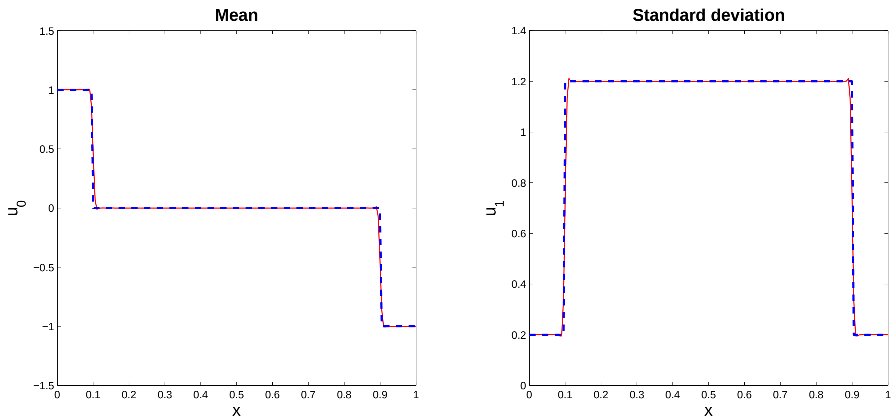


FIGURE 13. Analytical solution: - - - - ; Numerical solution: —— . $u_0$ extrapolated from the interior. $t = 2$.

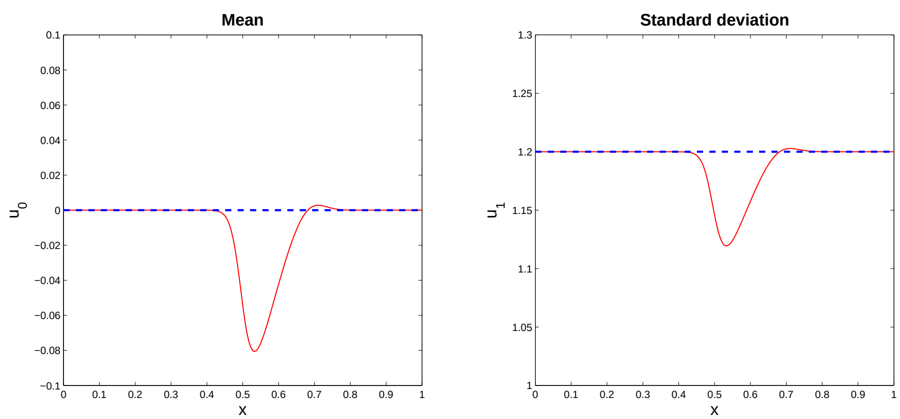


FIGURE 14. Analytical solution: - - - - ; Numerical solution: —— . $u_0$ extrapolated from the interior. $t = 3$.

### 5.3. *Discussion of the results with incomplete set of data*

The results in the preceding section are interesting and surprising. First of all, excellent results at steady state (for long time) are obtained using the extrapolation technique. This is probably due to the fact that only one bounadry condition is needed at the left boundary for $t > 2.5$. Also, by guessing data of the mean value and the variance, equally poor results are obtained. The higher-order modes might be very important. The order of the error obtained here indicates that appropriate approximation of the higher-order terms is as important as guessing the expectation to get accurate results.

## 6. Future research

In a general hyperbolic problem, the imposition of correct time-dependent boundary conditions will probably prove to be one of the more significant problems with the PCE method. A detailed investigation is necessary to find ways around the lack of time-

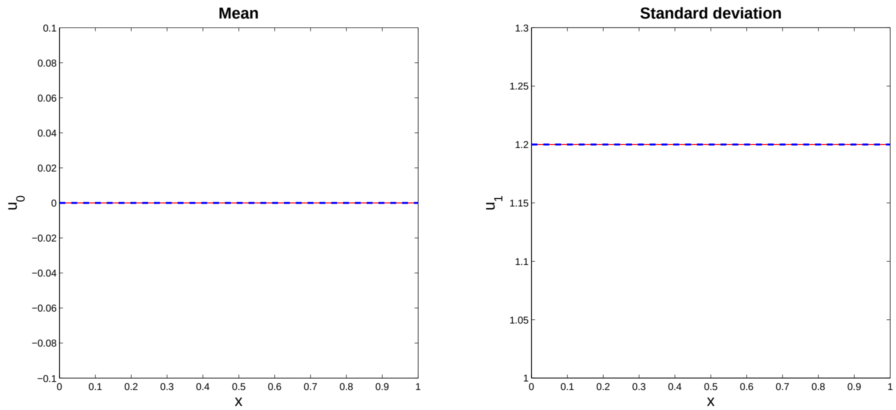


FIGURE 15. Analytical solution: - - - - ; Numerical solution: —— . $u_0$ extrapolated from the interior. $t = 5$.

dependent stochastic boundary data, especially for the higher moments. Most likely, special non-reflecting boundary conditions must be developed.

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