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Invariant algebraic wall-bounded turbulence modeling

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1. Motivation and objectives

Based on the recent work by Frewer (2009), the central idea of modeling turbulence on a four-dimensional non-Riemannian manifold is extended herein to flow configurations which include solid walls as boundaries. Special focus is directed to rotating wall-bounded flows. The model development itself will be based on the strategy of closing the one-point statistical incompressible and isothermal Navier-Stokes equations algebraically by proposing a nonlinear eddy viscosity model within the $k-\epsilon$ family. The aim here is to construct an algebraic wall-bounded turbulence model

(i) which applies to all geometries which are topologically equivalent to the three canonical wall-bounded flow geometries: the flat plate, the flat channel and the axisymmetric pipe,

(ii) which manifests itself in an overall local and tensorial set of differential equations,

(iii) which admits all continuous and discrete point-symmetries of the instantaneous Navier-Stokes equations,

(iv) which gives the correct asymptotic near-wall behavior for all statistical flow quantities being considered,

(v) which can cover the two extreme flow regimes occurring near and distant to the wall without making use of empirical wall-functions,

(vi) which stays bounded as the wall is approached,

(vii) where the gradient-order of the modeled terms do not exceed the differential order of the underlying unclosed transport equations,

(viii) and which finally produces a realizable flow.

To motivate these characteristics, each statement will be briefly clarified and commented upon. The difficulties arising in following these guidelines and how they are to be solved by applying novel modeling techniques will also be briefly discussed.

1.1. *Geometry*

The topological equivalent geometries mentioned in (i) are obtained by translating, rotating or smoothly deforming the three canonical geometries into any desirable configuration, where the change in geometry may depend on time. In mathematical terms the model should apply to all manifolds which differ only by a four-dimensional diffeomorphism relative to the three canonical wall-bounded manifolds mentioned above. To assure a model development within Newtonian physics, these four-dimensional manifolds have to be restricted to non-Riemannian manifolds whereas the diffeomorphism has to be restricted to invariant time transformations only (Frewer 2008, 2009).

Important to note here is that the topological equivalence only applies to the geometries and *not* to the solutions of the model it will produce in each of these cases. Any solution obtained by imposing initial and boundary conditions in a specific geometry is valid only in that particular geometry.

1.2. *Locality and tensoriality*

The locality requirement of (ii) has no rigorous mathematical justification. It rather has to be seen as a first intuitive model assumption. In this sense, the model equations will only contain differential operators of a positive integer order, and will thus avoid any integral or differential operators of fractional order (Samko *et al.* 2002) as well as any non-polynomial functions containing differential operators.

Unlike the locality requirement, the tensorial requirement of (ii) has a firm mathematical foundation. Using ordinary tensors is not the only possible way to formulate covariant dynamical equations. Alternatively, a covariant formulation can also be developed by using tensor densities of a fixed weight, which in principle transform as ordinary tensors but with the difference that they are weighted by a power of the Jacobian determinant of the transformation being considered (Synge & Schild 1978; Schrödinger 1985). By definition, a tensor density of weight zero is an ordinary tensor; an example for a tensor density is the well-known Levi-Civita tensor, alias ϵ-tensor, with a certain rank, which precisely is a fully antisymmetric numerical invariant tensor density of weight one. As pointed out by Schrödinger (1985), physical equations should be formulated either by using tensors only or by using tensor densities of a fixed wight; mixing the quantities can lead to inconsistencies in the transformation behavior of the equations. For obtaining general covariance in the case of the velocity-formulated Navier-Stokes equations only the ordinary tensor formulation may therefore be used.

1.3. *Symmetries*

A major challenge for turbulence modeling was, and still is, to obey the *nonlocal* continuous point-symmetry based on uniform rotations for spatially two-dimensional and two-velocity-componential flows (Ibragimov 1994; Cantwell 2002). This symmetry was coined by Speziale (1981, 1998) as 2DMFI (material frame-indifference in the limit of two-dimensional turbulence) due to its close analogy with Euclidean invariance of constitutive equations in continuum mechanics.

The challenge of using 2DMFI as a restriction in the context of algebraic modeling lies in the fact that it always places far too severe constraints onto the model being formulated (Speziale 1987). When using the standard techniques to develop nonlinear eddy-visocity models first proposed by Pope (1975) and then refined by Speziale (1987); Gatski & Speziale (1993); Shih *et al.* (1995); Craft *et al.* (1996); Apsley & Leschziner (1998) and Wallin & Johansson (2000), any method of incorporating 2DMFI into such models will eventually lead to the unwanted effect that the modeled terms will always be forced to a structure which then shows full three-dimensional frame-indifference under uniform rotations. In other words, the standard techniques for developing nonlinear eddy-viscosity models face the serious drawback that as soon as two-dimensional frame-indifference under uniform rotations in the unclosed terms is demanded, it automatically implies three-dimensional frame-indifference under uniform rotation about any axis, a property fully inconsistent with experimental and numerical findings of actual turbulent flows. For this very reason, the 2DMFI principle was always avoided as an algebraic modeling constraint with the result that all of the above-mentioned nonlinear eddy-viscosity models are inconsistent with 2DMFI. However, this problem can be solved by making use of a new general modeling technique by reformulating the statistical Navier-Stokes equations on a curved four-dimensional Newton-Cartan manifold (Havas 1964).

1.4. *Asymptotic near-wall behavior*

It is sufficient to assure a correct implementation only in the *general* near-wall solutions of the three canonical wall-bounded configurations; any topologically equivalent configuration will then show an identical near-wall behavior. For compactness we will choose for all further discussion the flat plate geometry located at $y = 0$ as a representative of all three canonical wall geometries — any near-wall results which will differ from the flat plate geometry will be stated.

To attain the correct near-wall behavior for every statistical quantity within the model we will use an appropriate scaling sequence of the four-dimensional invariant turbulent eddy viscosity $\nu_T = \mathcal{K}^2/\mathcal{E}$ (Frewer 2009), but not based on the pseudo-dissipation rate $\mathcal{E}$ but rather on the invariant dissipation rate $\mathcal{E}^* := \mathcal{E} - \nu h^{\alpha\beta}\nabla_\alpha\nabla_\beta\mathcal{K}$ (Pope 2000). It has the decisive advantage that at the wall one can allocate the simple boundary condition $\mathcal{E}^*|_{y=0} = 0$, since there the identity $\mathcal{E} = \nu h^{\alpha\beta}\nabla_\alpha\nabla_\beta\mathcal{K}$ is always strictly satisfied.

A dissipation-rate of such type was first used as a modeling variable by Hanjalić & Launder (1976), who actually used the invariant dissipation rate $\tilde{\mathcal{E}} := \mathcal{E} - 2\nu h^{\alpha\beta}\nabla_\alpha\mathcal{K}^{1/2}\cdot\nabla_\beta\mathcal{K}^{1/2}$ in order to keep terms proportional to $\tilde{\mathcal{E}}/\mathcal{K}$ in their model bounded as $y \to 0$. The major difference in using either $\mathcal{E}^*$ or $\tilde{\mathcal{E}}$ is that the former quantity varies linearly $\mathcal{E}^* \sim \mathcal{O}(y)$ as $y \to 0$ whereas the latter quantity varies quadratically $\tilde{\mathcal{E}} \sim \mathcal{O}(y^2)$, as it does for the invariant turbulent kinetic energy $\mathcal{K} \sim \mathcal{O}(y^2)$.

The reason for using $\mathcal{E}^*$ instead of $\tilde{\mathcal{E}}$, as we will see in more detail later, is that the transport equation for $\mathcal{E}^*$ induces a set of differential equations which is of fourth order in the turbulent kinetic energy $\mathcal{K}$, while the transport equation for $\tilde{\mathcal{E}}$ will only induce a set of differential equations which is of third order in $\mathcal{K}$. Hence, in particular for the flat channel geometry and its topological equivalences, the former choice is definitely the more appropriate one to account for the four natural boundary conditions which $\mathcal{K}$ exhibits on the wall: $\mathcal{K}|_{y=y_0} = \partial_y\mathcal{K}|_{y=y_0} = 0$ for $y_0 = 0$ and $y_0 = 1$.

1.5. *Bridging the distant- and near-wall regime*

To smoothly bridge the highly turbulent regime very far from the wall (measured relative to the turbulent Reynolds number $\mathrm{Re}_T = \nu_T/\nu$) and the strongly viscosity-affected damped turbulent regime very close to the wall without having to make use of wall-functions, as mentioned in (v), will be approached by proposing a power series in the molecular viscosity ν. For dimensional reasons the turbulent eddy viscosity ν_T then must scale similarly. To avoid conflict with requirement (vi), it will be shown that all powers in the expansion ν^n must be positive multiples of one-third $n = m/3$ where $m \in \mathbb{N}_0$, if the turbulent eddy-viscosity ν_T is based on the dissipation rate $\mathcal{E}^*$ as defined above (since it varies cubicly: $\nu_T = \mathcal{K}^2/\mathcal{E}^* \sim \mathcal{O}(y^3)$ as $y \to 0$).

1.6. *Analyticity at the wall*

As it is a basic property of all terms in the instantaneous Navier-Stokes equations to remain analytical and bounded as the wall is approached, this property should also hold for all terms in any near-wall turbulence model, not only for moderate global Reynolds numbers but also for the asymptotic regime $\mathrm{Re} \to \infty$ (or equivalently in the regime of vanishing molecular viscosity $\nu \to 0$ if one operates with the transport equations in dimensionalized form, as we will do herein in order to keep track of the physical dimensions as we model along). This requirement will always guarantee the existence of a Taylor series at the wall, particularly in the wall-normal coordinate y, which is necessary for modeling the asymptotic near-wall behavior as discussed in §1.4.

1.7. *Preserving equational structure*

An important guideline for the near-wall modeling process is to assure that the gradient-order of the modeled terms does not exceed the differential order of the underlying unclosed transport equation for each flow variable the model is to be solved for, particularly in the limit of $\nu \to 0$. Otherwise we would completely change both the mathematical and the physical character of these equations. Especially the type and amount of boundary-layers which will develop in the solutions will certainly be different, but also in order to assure unique physical solutions in such a system it will make it necessary to implement additional higher-order boundary conditions on the wall, with no guarantee that they are even a priori accessible within a physical flow.

1.8. *Realizability*

In order to satisfy all realizability constraints for the Reynolds-stress tensor (Schumann 1977) poses the strongest and severest modeling constraint, in particular that based on Schwarz's inequality which immediately elevates every local algebraic model to a highly complex nonlocal set of differential equations.

The modeling aim therefore is at first to satisfy only weak realizability in the sense of guaranteeing positiveness of the turbulent kinetic energy for any possible flow — the next step would then be to require positiveness for each component of the energy.

2. Results

The main results can be categorized into three parts: (i) Without changing the physical content of the theory, the statistical one-point Navier-Stokes equations can be equivalently reformulated on a *curved* four-dimensional non-Riemannian manifold, such that the mean pressure gradient is automatically and consistently incorporated into the modeling scheme and therefore need not be introduced as an ad hoc modeling variable as previously done in Frewer (2009). (ii) With the aid of this reformulation, a natural sequence of kinematic differential operators can be constructed showing full frame-indifference under two-dimensional but the necessary frame-dependence under three-dimensional uniform rotations, hence for the first time leading to a turbulence model which admits *all* Lie-point symmetries of the instantaneous Navier-Stokes equations without being frame-indifferent in its dynamic part under three-dimensional uniform rotations. (iii) To bridge the distant- and near-wall regime without wall functions, a series expansion in the molecular viscosity is proposed which allows the asymptotic near-wall behavior of all unclosed flow quantities to be exactly satisfied.

2.1. *The Navier-Stokes equations on a curved Newton-Cartan manifold*

Taking the generally covariant instantaneous Navier-Stokes equations on a flat four-dimensional Newtonian manifold $\mathcal{N}$ as they were derived in Frewer (2009),

$$\begin{aligned} \nabla_\alpha u^\alpha &= 0, \\ u^\beta \nabla_\beta u^\alpha &= -h^{\alpha\beta}\nabla_\beta p + \nu h^{\beta\gamma}\nabla_\beta\nabla_\gamma u^\alpha, \end{aligned} \tag{2.1}$$

where all Greek indices run from 0 to 3 and where x^0 represents the time coordinate, we can *equivalently* rewrite the momentum equations as follows:

$$\begin{aligned} \nu h^{\beta\gamma}\nabla_\beta\nabla_\gamma u^\alpha &= u^\beta\big(\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\lambda}u^\lambda\big) + h^{\alpha\beta}\partial_\beta p \\ &= u^\beta\partial_\beta u^\alpha + \big(\Gamma^\alpha_{\beta\lambda} + g_{\beta\lambda}h^{\alpha\sigma}\partial_\sigma p\big)u^\beta u^\lambda, \end{aligned} \tag{2.2}$$

where we have made use of the 4-velocity time component $u^0 = 1$ and of the time-like metrical tensor $g_{\alpha\beta}$ which has only the contribution $g_{00} = 1$. Since both quantities stay numerically invariant under any space-time coordinate transformations within Newtonian reference frames (Havas 1964), the above rewritten momentum equations represent a true equivalent reformulation from the aspect of general covariance. In other words, one can define a quantity

$$\Lambda^{\alpha}_{\beta\lambda} := \Gamma^{\alpha}_{\beta\lambda} + g_{\beta\lambda} h^{\alpha\sigma} \partial_\sigma p, \tag{2.3}$$

which (i) shows all properties of being an affine connection of a new manifold $\mathcal{M}$, and (ii) that this new manifold is Newtonian again.

Like $\mathcal{N}$, the new manifold $\mathcal{M}$ is an overall non-metrical (or non-Riemannian) manifold showing again the same four special singular tensors which in either a temporal or spatial subspace of $\mathcal{M}$ behave as metrical tensors (Frewer 2009) — in arbitrary coordinate systems they can again be characterized by their invariant property of a vanishing covariant derivative in those subspaces in which their metrical properties apply

$$\hat{\nabla}_\lambda h^{ij} = 0, \;\; \hat{\nabla}_\lambda k_{ij} = 0; \quad \hat{\nabla}_\lambda g_{00} = 0, \;\; \hat{\nabla}_\lambda m^{00} = 0, \tag{2.4}$$

where the Latin indices represent the spatial indices which run from 1 to 3, and where $\hat{\nabla}_\lambda$ now denotes the covariant derivative based on the new affine connection $\Lambda^{\rho}_{\mu\nu}$.

But unlike $\mathcal{N}$, the new manifold $\mathcal{M}$ shows the property of being curved. This expresses itself in the non-commutativity of the new covariant derivative: $\hat{\nabla}_\alpha \hat{\nabla}_\beta - \hat{\nabla}_\beta \hat{\nabla}_\alpha \neq 0$, a property of an affine space which is called curvature and described by the Riemann curvature tensor (Schrödinger 1985; Synge & Schild 1978), here for the manifold $\mathcal{M}$:

$$\hat{R}^{\kappa}_{\mu\lambda\nu} = \partial_\lambda \Lambda^{\kappa}_{\mu\nu} - \partial_\nu \Lambda^{\kappa}_{\mu\lambda} + \Lambda^{\kappa}_{\rho\lambda} \Lambda^{\rho}_{\mu\nu} - \Lambda^{\kappa}_{\rho\nu} \Lambda^{\rho}_{\mu\lambda}. \tag{2.5}$$

Since the Riemann curvature is a tensor, it will be zero in all coordinate systems if it is zero in one specified coordinate system. Globally this is the case for the flat manifold $\mathcal{N}$ where one can always transform to the Cartesian and inertial coordinate system in which the affine connection vanishes, thus implying a vanishing curvature tensor. However, this is not the case for the manifold $\mathcal{M}$ as long as the pressure field is non-zero. Only locally at an arbitrary but fixed space-time event x_0^α the pressure field can be transformed away, hence giving us the feasibility to always choose a locally flat reference frame in which the incompressible fluid is only influenced by viscous forces within a local neighborhood of x_0^α. After all, this covariant reformulation of the Navier-Stokes equation on a curved Newtonian manifold will lead us to a full analogy to the gravitational theory of general relativity (Einstein 1916), which is beyond the scope of this paper and not discussed any further.

Continuing the reformulation of the instantaneous Navier-Stokes equations, it can be straightforwardly shown that not only the kinematic part but also the viscous part of the momentum equations, as well as the continuity equation, can be equivalently reformulated such that all incompressible dynamics can be consistently imbedded into the new curved Newton-Cartan manifold $\mathcal{M}$, showing at the end the following covariant structure

$$\begin{aligned} \hat{\nabla}_\alpha u^\alpha &= 0, \\ u^\beta \hat{\nabla}_\beta u^\alpha &= \nu h^{\beta\gamma} \hat{\nabla}_\beta \hat{\nabla}_\gamma u^\alpha. \end{aligned} \tag{2.6}$$

The pressure field could be completely absorbed into the underlying geometry of $\mathcal{M}$. This is a key property for any kind of incompressible turbulence modeling, since now the differential operators $\hat{\nabla}_\alpha$ of the curved manifold $\mathcal{M}$ will inherently take care of how the pressure variable has to be taken along in any closure scheme, in contrast to the differential operators ∇_α of the flat manifold $\mathcal{N}$ which are fully devoid of this property. Hence, we now have a rigorous mathematical justification which states that (i) for a complete closure the pressure variable must be considered as a modeling variable and (ii) that the formalism automatically takes care as how the pressure variable has to appear within the model.

On the other hand, the dynamics of the pressure field will define the geometrical structure of the manifold $\mathcal{M}$. Hence, in a turbulent flow where the pressure is strongly fluctuating the underlying geometrical structure of $\mathcal{M}$ will fluctuate accordingly. To be consistent when setting up the ensemble-averaged Navier-Stokes equation and its statistical consequences, all covariant derivatives on the manifold $\mathcal{M}$ must also be equivalently decomposed into an averaged part and into a fluctuating part

$$\hat{\nabla}_\alpha = \langle\hat{\nabla}_\alpha\rangle + \hat{\nabla}'_\alpha, \tag{2.7}$$

which reflects the fact that the underlying geometry is defined by a turbulent flow quantity, here induced by the decomposition into an average and fluctuating pressure respectively.

2.2. *The governing near-wall model equations*

When reformulating the generally covariant statistically unclosed $\mathcal{K}$-$\mathcal{E}$ model equations (Frewer 2009) on the curved Newton-Cartan manifold $\mathcal{M}$, we end up with the non-trivial result that all equations can be consistently imbedded into the averaged part $\langle\mathcal{M}\rangle$ of the instantaneous manifold $\mathcal{M}$ as follows:

$$\begin{aligned}
\langle\hat{\nabla}_\alpha\rangle\langle u^\alpha\rangle &= 0,\\
\langle u^\beta\rangle\langle\hat{\nabla}_\beta\rangle\langle u^\alpha\rangle &= \nu h^{\beta\gamma}\langle\hat{\nabla}_\beta\rangle\langle\hat{\nabla}_\gamma\rangle\langle u^\alpha\rangle - \langle\hat{\nabla}_\beta\rangle\tau^{\alpha\beta},\\
\langle u^\alpha\rangle\langle\hat{\nabla}_\alpha\rangle\mathcal{K} &= -k^{\langle u\rangle}_{\beta\lambda}\tau^{\lambda\alpha}\langle\hat{\nabla}_\alpha\rangle\langle u^\beta\rangle - \mathcal{E}^* + \langle\hat{\nabla}_\lambda\rangle\mathcal{D}^\lambda_{(\mathcal{K})},\\
\langle u^\alpha\rangle\langle\hat{\nabla}_\alpha\rangle\mathcal{E}^* &= \underset{(1)}{\mathcal{P}}{}^\alpha{}_\beta\langle\hat{\nabla}_\alpha\rangle\langle u^\beta\rangle + \underset{(2)}{\mathcal{P}}{}^{\alpha\beta}{}_\lambda\langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\langle u^\lambda\rangle + \underset{(3)}{\mathcal{P}} - \Upsilon + \langle\hat{\nabla}_\lambda\rangle\mathcal{D}^\lambda_{(\mathcal{E}^*)}\\
&\quad + \nu h^{\alpha\beta}\langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\mathcal{E}^* - \nu h^{\rho\sigma}\langle u^\alpha\rangle\langle\hat{\nabla}_\alpha\rangle\Big[\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\mathcal{K}\Big]\\
&\quad + \nu^2 h^{\rho\sigma}h^{\alpha\beta}\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\Big[\langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\mathcal{K}\Big],
\end{aligned} \tag{2.8}$$

where instead of the pseudo-dissipation rate $\mathcal{E}$ we based the two-equational model on the dissipation rate $\mathcal{E}^*$ as defined in §1.4. Up to the changes which originate from using $\mathcal{E}^*$ instead of $\mathcal{E}$, we see that the above equations are structurally identical to those given on the flat manifold $\mathcal{N}$ (Frewer 2009), but with the decisive difference that now the above equations do not show explicit dependence on the averaged pressure field, and that in general one must keep track of how the covariant derivatives are ordered with respect to each other since they do no longer commute.

The above equations represent a fully determined set of six strongly coupled differential equations for the three spatial averaged velocity fields $\langle u^i\rangle$, the turbulent kinetic $\mathcal{K}$ and the dissipation rate $\mathcal{E}^*$, as well as for the averaged pressure field $\langle p\rangle$ which determines the geometrical structure of the underlying manifold $\langle\mathcal{M}\rangle$ for these set of equations. For the velocity fields and the dissipation rate the set of equations are of second differential order, while for the turbulent kinetic energy it is of fourth order. Now since these flow quantities vary as $\langle u^i\rangle \sim \mathcal{O}(y)$, $\langle\mathcal{E}^*\rangle \sim \mathcal{O}(y)$ and $\langle\mathcal{K}\rangle \sim \mathcal{O}(y^2)$ as the solid wall at $y = 0$ is approached, we face with the given $\mathcal{K}$-$\mathcal{E}^*$ model equations an appropriate system to account for the natural wall boundary conditions of all these flow variables.

This set of equations is a priori unclosed and needs to be modeled. Next to the symmetric and space-like Reynolds-stress tensor $\tau^{\alpha\beta}$ we have the two space-like turbulent diffusion vectors $\mathcal{D}^{\lambda}_{(\mathcal{K})}$ and $\mathcal{D}^{\lambda}_{(\mathcal{E}^*)}$ for $\mathcal{K}$ and $\mathcal{E}^*$ respectively; then the two turbulent production tensors $\mathcal{P}^{\alpha}_{(1)\,\beta}$ and $\mathcal{P}^{\alpha\beta}_{(2)\,\lambda}$ for $\mathcal{E}^*$, which in the upper (contravariant) indices are space-like while the lower (covariant) index will give no time-like contribution in the transport equation due to the incompressibility constraint of the flow; and finally the turbulent scalar production $\mathcal{P}_{(3)}$ and destruction rate Υ for $\mathcal{E}^*$ which are merged into one scalar rate $\Psi := \mathcal{P}_{(3)} - \Upsilon$ as it will be impossible to distinguish between these two quantities during any invariant modeling process (Frewer 2009).

If the motivation in algebraically closing the $\mathcal{K}$-$\mathcal{E}^*$ equations is only to make use of the sole information these unclosed equations can supply, and not using additional turbulence-structure parameters (Reynolds & Kassinos 1995), the most basic ansatz for a local algebraic closure is then to use only those variables for which the equations are being solved, along with their gradients. As was discussed in §1.7, the order of the gradients for each flow variable must be limited in order to preserve the equational structure of the unclosed system (2.8). For the averaged velocity variables $\langle u^i\rangle$ as well as for the dissipation rate $\mathcal{E}^*$, we cannot therefore go beyond the second order in its gradients, while for the turbulent kinetic energy $\mathcal{K}$ we may not go beyond fourth order. In particular for the Reynolds-stress tensor $\tau^{\alpha\beta}$ and the two diffusion vectors $\mathcal{D}^{\lambda}_{(\mathcal{K})}$ and $\mathcal{D}^{\lambda}_{(\mathcal{E}^*)}$, the gradients for the modeling variables $\langle u^i\rangle$, $\mathcal{K}$ and $\mathcal{E}^*$ are restricted to be even one order less than mentioned before, due to the appearance of a first-order gradient in front of those unclosed quantities.

It is clear from the construction of (2.8) that the averaged pressure variable $\langle p\rangle$ need no longer be considered as an explicit modeling variable on the manifold $\langle\mathcal{M}\rangle$, since it is always automatically and consistently taken along throughout the covariant modeling process as soon as the gradients of the five modeling variables $\langle u^i\rangle$, $\mathcal{K}$ and $\mathcal{E}^*$ get evaluated — a key reason why incompressible turbulence modeling should be performed on a curved Newton-Cartan manifold.

In this sense the *complete* covariant algebraic closure set $\mathcal{V}$ for the $\mathcal{K}$-$\mathcal{E}^*$ equations on the curved averaged manifold $\langle\mathcal{M}\rangle$ will therefore consist of the following elements:

$$\begin{gathered}\langle u^\alpha\rangle,\quad \langle\hat{\nabla}_\rho\rangle\langle u^\alpha\rangle,\quad \langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\langle u^\alpha\rangle,\\ \mathcal{K},\quad \langle\hat{\nabla}_\alpha\rangle\mathcal{K},\quad \langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\mathcal{K},\quad \langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\langle\hat{\nabla}_\rho\rangle\mathcal{K},\quad \langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\mathcal{K},\\ \mathcal{E}^*,\quad \langle\hat{\nabla}_\alpha\rangle\mathcal{E}^*,\quad \langle\hat{\nabla}_\alpha\rangle\langle\hat{\nabla}_\beta\rangle\mathcal{E}^*.\end{gathered} \tag{2.9}$$

2.3. *Statistical symmetry analysis*

As was discussed in §1.3, any closure which can be constructed from the set $\mathcal{V}$ (2.9) for the $\mathcal{K}$-$\mathcal{E}^*$ equations (2.8) should admit all continuous and discrete symmetries of the instantaneous Navier-Stokes equations. Part of this restriction are also all equivalence transformations (Ibragimov 1994) of the instantaneous Navier-Stokes equations when the molecular viscosity ν (or equivalently the global Reynolds number Re) is being varied. The complete list of *all* point-symmetry and equivalence transformations which the instantaneous Navier-Stokes equations admit are nicely listed in Oberlack (2000).

However, a discrete symmetry transformation only mentioned for the inviscid instantaneous Euler equations is the time-reflection symmetry

$$\tilde{t} = -t,\quad \tilde{x}^i = x^i,\quad \tilde{u}^i = -u^i,\quad \tilde{p} = p. \tag{2.10}$$

Due to the viscous term appearing in the Navier-Stokes equations, it is obvious that this transformation is not a symmetry transformation for the Navier-Stokes equations. Also from a physical point of view, it is clear that any diffusive or dissipative system cannot allow for a time-reflection symmetry due to an increase of entropy in that system as it evolves forward in time, which simply cannot be reversed. Nevertheless, if we look at the transformation

$$\tilde{t} = -t, \quad \tilde{x}^i = x^i, \quad \tilde{u}^i = -u^i, \quad \tilde{p} = p, \quad \tilde{\nu} = -\nu, \tag{2.11}$$

it turns out to be a discrete equivalence transformation of the instantaneous Navier-Stokes equations. Although this transformation is not physically realizable, since it is impossible to construct a fluid having negative molecular viscosity, it represents a mathematical property which is characteristic for the Navier-Stokes equations. In particular, this equivalence transformation will justify the molecular viscosity expansion to be developed in the next section to smoothly bridge the distant- and near-wall regimes in the $\mathcal{K}$-$\mathcal{E}^*$ equations, as discussed in §1.5.

To determine the explicit structure of all continuous and discrete point-symmetries which the $\mathcal{K}$-$\mathcal{E}^*$ equations (2.8) should admit, the instantaneous symmetry transformations are equivalently Reynolds-decomposed (via ensemble-averaging) into their average and fluctuating parts. For each symmetry the fluctuating part will then determine the transformation rule for the scalar variables $\mathcal{K}$ and $\mathcal{E}^*$. Important to note here is that for a symmetry analysis it is irrelevant which coordinate system or reference frame is being used since the symmetry property itself stays invariant under any coordinate transformation. Hence, it is fully sufficient to do this symmetry analysis only within the Cartesian and inertial system.

This investigation reveals that when modeling dimensional-consistently and covariantly on the curved Newton-Cartan manifold $\langle\mathcal{M}\rangle$ the only transformations able to impose restrictions on the modeling procedure are the two-dimensional continuous symmetry $\mathsf{R}_{2\mathsf{D}}$, alias 2DMFI, as discussed in detail in §1.3, and the discrete equivalence transformation D_2, as already discussed before:

$$\begin{aligned}
\mathsf{R}_{2\mathsf{D}}:\ & \tilde{t} = t,\ \ \tilde{x}^i = Q^i_{\ j}x^j,\ \ \langle\tilde{u}^i\rangle = Q^i_{\ j}\langle u^j\rangle + \dot{Q}^i_{\ j}x^j,\ \ \text{with } i,j = 1,2,\\
& \langle\tilde{p}\rangle = \langle p\rangle + \tfrac{1}{2}\omega_z^2\delta_{ij}x^i x^j + 2\omega_z\langle\psi\rangle,\ \ \tilde{\mathcal{K}} = \mathcal{K},\ \ \tilde{\mathcal{E}}^* = \mathcal{E}^*,\\
& \text{with } \langle\psi\rangle := -\int_C \Big(\langle u^2\rangle dx^1 - \langle u^1\rangle dx^2\Big),\\
\mathsf{D}_2:\ & \tilde{t} = -t,\ \ \tilde{x}^i = x^i,\ \ \langle\tilde{u}^i\rangle = -\langle u^i\rangle,\ \ \langle\tilde{p}\rangle = \langle p\rangle,\ \ \tilde{\nu} = -\nu,\\
& \tilde{\mathcal{K}} = \mathcal{K},\ \ \tilde{\mathcal{E}}^* = -\mathcal{E}^*.
\end{aligned} \tag{2.12}$$

Here $Q^i_{\ j}$ represents a two-dimensional rotation matrix for *uniform* rotations, with ω_z being the constant angular velocity and $\langle\psi\rangle$ the corresponding stream function defined as a planar curve integral over the two-componential velocity field $\langle u^i\rangle$. Imposing the constraint $\mathsf{R}_{2\mathsf{D}}$ onto the $\mathcal{K}$-$\mathcal{E}^*$ model equations based on the closure set $\mathcal{V}$ (2.9) will force its elements to change or to behave to the effect

(i) that $\langle\hat{\nabla}_\rho\rangle\langle u^\alpha\rangle$ must be discarded as a modeling variable from the set $\mathcal{V}$ and be replaced by the sequence of objective tensors (relative to uniform rotations) which are generated by a Lie-derivative iteration (Frewer 2008) of the objective space-like metric $h^{\alpha\beta}$ (here up to second order only) which, since our manifold $\langle\mathcal{M}\rangle$ is free of torsion, take on the following explicit form on $\langle\mathcal{M}\rangle$

$$\hat{S}^{\alpha\beta} := -\tfrac{1}{2}\mathcal{L}_{\langle u\rangle}h^{\alpha\beta} = \tfrac{1}{2}\big(h^{\alpha\lambda}\langle\hat{\nabla}_\lambda\rangle\langle u^\beta\rangle + h^{\beta\lambda}\langle\hat{\nabla}_\lambda\rangle\langle u^\alpha\rangle\big),$$
$$\hat{T}^{\alpha\beta} := \mathcal{L}_{\langle u\rangle}\hat{S}^{\alpha\beta} = \langle u^\lambda\rangle\langle\hat{\nabla}_\lambda\rangle\hat{S}^{\alpha\beta} - \hat{S}^{\alpha\lambda}\langle\hat{\nabla}_\lambda\rangle\langle u^\beta\rangle - \hat{S}^{\beta\lambda}\langle\hat{\nabla}_\lambda\rangle\langle u^\alpha\rangle, \tag{2.13}$$

where $\hat{S}^{\alpha\beta}$ is the covariant strain-rate tensor and $\hat{T}^{\alpha\beta}$ the covariant Oldroyd derivative tensor of $\hat{S}^{\alpha\beta}$, a result which is shared by the investigations of Speziale (1987),

(ii) that for all remaining elements in $\mathcal{V}$, except for the three elements $\langle u^\alpha\rangle$, $\langle\hat{\nabla}_\alpha\rangle\mathcal{K}$ and $\langle\hat{\nabla}_\alpha\rangle\mathcal{E}^*$, only the pure spatial tensor components are allowed to contribute in the final $\mathcal{K}$-$\mathcal{E}^*$ transport equations regardless of how and to which polynomial order these elements are later combined in order to finally model the unclosed terms. For this very reason the time-like element $\langle u^\alpha\rangle$ may only be contracted with the two elements $\langle\hat{\nabla}_\alpha\rangle\mathcal{K}$ and $\langle\hat{\nabla}_\alpha\rangle\mathcal{E}^*$,

(iii) that the time-like element $\langle u^\alpha\rangle$ may not appear in an uncontracted form within the final $\mathcal{K}$-$\mathcal{E}^*$ transport equations, i.e. due to reason (ii) it may only appear in a contracted form with $\langle\hat{\nabla}_\alpha\rangle\mathcal{K}$ and/or $\langle\hat{\nabla}_\alpha\rangle\mathcal{E}^*$.

The unwanted side-effect of this restriction is that all modeled terms in the transport equations now exhibit full three-dimensional frame-indifference under any uniform rotation, evidently an unacceptable outcome. Hence, this restricted subset of $\mathcal{V}$, which now guarantees two-dimensional frame-indifference under uniform rotations, must obviously be extended by elements which are objective in two, but non-objective in three dimensions relative to any uniform rotation. On the curved Newton-Cartan manifold $\langle\mathcal{M}\rangle$ such elements can be naturally determined by making use of the following two scalar operators

$$\hat{\mathcal{D}}_1 := \langle u^\lambda\rangle\langle\hat{\nabla}_\lambda\rangle, \quad \hat{\mathcal{D}}_2 := \langle u^\rho\rangle\langle u^\sigma\rangle\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle. \tag{2.14}$$

If these two operators act on any of the flow variables $\langle u^\alpha\rangle$, $\mathcal{K}$, $\mathcal{E}^*$ or in general on any spatially 2DMFI-invariant tensor field $A^{\alpha\beta\cdots}_{\mu\nu\cdots}$ the spatial components of the resulting tensor will be frame-*indifferent* under two-dimensional but frame-*dependent* under three-dimensional uniform rotations.

That operator $\hat{\mathcal{D}}_1$ exhibits such behavior under uniform rotations is a rather trivial insight from the viewpoint of the curved manifold $\langle\mathcal{M}\rangle$ since it represents the kinematic operator of all $\mathcal{K}$-$\mathcal{E}^*$ transport equations (2.8) on that manifold. However, that operator $\hat{\mathcal{D}}_2$ also exhibits such behavior under uniform rotations is highly non-trivial and in a certain sense a surprising result, since it is able to generate second-order time derivatives on any flow quantity that it acts on. Interestingly to note is that the covariant derivatives in $\hat{\mathcal{D}}_2$ commute, also a non-trivial property.

It is straightforward now to construct higher-order differential operators which possess the same transformation behavior under uniform rotations as $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$. One must only follow the composition rule of operators, e.g. $\hat{\mathcal{D}}_1\circ\hat{\mathcal{D}}_1$ is next to $\hat{\mathcal{D}}_2$ an additional second-order differential operator of such kind, while e.g. $\hat{\mathcal{D}}_1\circ\hat{\mathcal{D}}_2$ will be a third-order differential operator of such kind, and so on.

2.4. *Algorithm for algebraic near-wall modeling*

The new relevant algebraic closure set $\mathcal{W}$ to model the unclosed $\mathcal{K}$-$\mathcal{E}^*$ transport equations (2.8), which is (i) consistent with *all* symmetry and equivalence transformations of the instantaneous Navier-Stokes equations, and (ii) which preserves the equational structure of the unclosed system, can be written as the union of two distinct sets

$$\mathcal{W} = \mathcal{W}_I \cup \mathcal{W}_R, \tag{2.15}$$

where the elements of set $\mathcal{W}_I$ give rise to spatially fully three-dimensional objective tensor expressions (relative to uniform rotations) within the transport equations, irrespective of how and in which polynomial order these elements are combined to model the unclosed terms, whereas the elements of set $\mathcal{W}_R$ only show frame-indifference under two-dimensional uniform rotations, thus in general inducing frame-dependency under three-dimensional uniform rotations in the unclosed terms.

The dimensionless and covariant elements of $\mathcal{W}_I$ on the curved averaged Newton-Cartan manifold $\langle\mathcal{M}\rangle$ are

$$\begin{aligned} &\frac{\mathcal{K}}{\mathcal{E}^*}\hat{S}^{\alpha\beta}, \quad \frac{\mathcal{K}^2}{\mathcal{E}^{*2}}\hat{T}^{\alpha\beta}, \quad \frac{\mathcal{K}^{5/2}}{\mathcal{E}^{*2}}\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\langle u^\alpha\rangle, \\ &\frac{1}{\mathcal{E}^*}\hat{\mathcal{D}}_1\mathcal{K}, \quad \frac{\mathcal{K}^{1/2}}{\mathcal{E}^*}\langle\hat{\nabla}_\alpha\rangle\mathcal{K}, \quad \frac{\mathcal{K}^2}{\mathcal{E}^{*2}}\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\mathcal{K}, \\ &\frac{\mathcal{K}}{\mathcal{E}^{*2}}\hat{\mathcal{D}}_1\mathcal{E}^*, \quad \frac{\mathcal{K}^{3/2}}{\mathcal{E}^{*2}}\langle\hat{\nabla}_\alpha\rangle\mathcal{E}^*, \quad \frac{\mathcal{K}^3}{\mathcal{E}^{*3}}\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\mathcal{E}^*, \end{aligned} \tag{2.16}$$

while the elements of $\mathcal{W}_R$ are

$$\begin{aligned} &\frac{\mathcal{K}^{1/2}}{\mathcal{E}^*}\hat{\mathcal{D}}_1\langle u^\alpha\rangle, \quad \frac{\mathcal{K}^{3/2}}{\mathcal{E}^{*2}}\hat{\mathcal{D}}_1\big(\hat{\mathcal{D}}_1\langle u^\alpha\rangle\big), \quad \frac{\mathcal{K}^{3/2}}{\mathcal{E}^{*2}}\hat{\mathcal{D}}_2\langle u^\alpha\rangle, \\ &\frac{\mathcal{K}}{\mathcal{E}^{*2}}\hat{\mathcal{D}}_1\big(\hat{\mathcal{D}}_1\mathcal{K}\big), \quad \frac{\mathcal{K}}{\mathcal{E}^{*2}}\hat{\mathcal{D}}_2\mathcal{K}; \qquad \frac{\mathcal{K}^2}{\mathcal{E}^{*3}}\hat{\mathcal{D}}_1\big(\hat{\mathcal{D}}_1\mathcal{E}^*\big), \quad \frac{\mathcal{K}^2}{\mathcal{E}^{*3}}\hat{\mathcal{D}}_2\mathcal{E}^*, \end{aligned} \tag{2.17}$$

where for reasons of simplicity we have only considered differential orders up to two in the turbulent kinetic energy variable $\mathcal{K}$. In the following, the analytical algorithm for modeling incompressible, isothermal wall-bounded turbulent flow algebraically within one-point statistics will be demonstrated explicitly only by modeling the Reynolds-stress tensor $\tau^{\alpha\beta}$ — modeling the remaining five unclosed terms in the $\mathcal{K}$-$\mathcal{E}^*$ transport equations (2.8) can be carried out analogously.

Based on the algebraic theory of tensor invariants (Spencer & Rivlin 1958) as used in the work of Pope (1975), the Reynolds-stress tensor $\tau^{\alpha\beta} = \tau^{\alpha\beta}(\mathcal{W})$ as a function of the closure set $\mathcal{W}$ will be modeled by formulating a polynomial tensor expansion in $\mathcal{W}$. Fortunately, owing to the Cayley-Hamilton theorem, the polynomial tensor expansion naturally truncates at a fixed order (Pope 1975). Although the expansion itself is finite the number of terms involved will be huge. Therefore, it is reasonable to simplify the exact expansion and to truncate it further to a level which will result in the first lowest nonlinear model. For the Reynolds-stress tensor this would be at quadratic level. The corresponding expansion

$$\tau^{\alpha\beta}(\mathcal{W}) = \sum_n \Xi_{(n)}\tau^{\alpha\beta}_{(n)} \tag{2.18}$$

consists of 46 terms, where $\Xi_{(n)}$ are the scalar dimensionalized expansion coefficients to be discussed later and $\tau^{\alpha\beta}_{(n)}$ the corresponding expansion terms. For modeling the three canonical flow configurations and all topological equivalences, three quadratic terms clearly emerge as highly relevant within the expansion

$$\begin{aligned} \tau^{\alpha\beta}_{(1)} = \hat{S}^{\alpha\rho}k^{\langle u\rangle}_{\rho\sigma}\hat{S}^{\sigma\beta}, \quad \tau^{\alpha\beta}_{(2)} &= \big(h^{\alpha\rho}\hat{S}^{\beta\sigma} + h^{\beta\rho}\hat{S}^{\alpha\sigma}\big)\langle\hat{\nabla}_\rho\rangle\langle\hat{\nabla}_\sigma\rangle\mathcal{K}, \\ \tau^{\alpha\beta}_{(3)} &= \hat{\mathcal{D}}_1\langle u^\alpha\rangle \cdot \hat{\mathcal{D}}_1\langle u^\beta\rangle. \end{aligned} \tag{2.19}$$

In channel (or pipe) flow $\tau^{\alpha\beta}_{(3)}$ reduces for the only non-zero off-diagonal Reynolds-stress component in the fully-developed regime to the interesting form of $\tau^{12}_{(3)} = -G \cdot \partial_y\langle p\rangle$, where $G := -\partial_x \langle p \rangle$ is the constant streamwise pressure gradient which drives the turbulent flow. In the case of streamwise rotation, an initial non-zero cross mean velocity $\langle u^3 \rangle$ will lead to secondary flow in the fully-developed regime, due to the forcing term $\tau^{\alpha\beta}_{(2)}$ which is non-zero in the relevant Reynolds-stress components τ^{23}. Frame-dependency will be generated by the term $\tau^{\alpha\beta}_{(3)}$, while $\tau^{\alpha\beta}_{(1)}$ ensures that in the fully-developed regime all Reynolds-stress components are non-zero.

All turbulence models face limitations when applied to flows for which the model was not originally designed. This rule also holds in the present case. The algebraic $\mathcal{K}$-$\mathcal{E}^*$ model proposed herein completely fails to predict the effects of rotation in the canonical case of uniform rotating homogeneous isotropic turbulence. This flow has the 'singular' property of having zero mean velocities. Now, since frame-dependency in any *covariantly* developed statistical one-point model can only enter via the mean Coriolis forces, being directly proportional to the mean velocities, a zero mean velocity field in all its components leads to the severe consequence of vanishing Coriolis forces and thus to a completely frame-indifferent model for this case. Hence, single-point modeling, which is developed on a covariant basis, is unable to describe the dynamics of rotating homogeneous isotropic turbulence. For such a 'singular' flow a covariant two-point approach will be better adapted.

The final issue to be discussed is the functional dependence of the tensor expansion coefficients $\Xi_{(n)}$ in (2.18). In general they can depend on all scalar invariants which can be constructed from the elements of the underlying closure set $\mathcal{W}$. However, again for reasons of simplicity, we will only consider a dependence on the two functionally independent scalar invariants $\mathcal{K}$ and $\mathcal{E}^*$.

Each tensor expansion coefficient carries a certain physical dimension which can be dynamically expressed by $\mathcal{K}$ and $\mathcal{E}^*$ (Pope 1975) as

$$\left[\Xi_{(n)}\right] = \mathcal{K}^{q_n} \mathcal{E}^{*r_n}, \quad q_n, r_n \in \mathbb{Q}. \tag{2.20}$$

In order to smoothly bridge between the two extremely different turbulent flow scales occurring far from the wall and those occurring close to the wall, a power series in the molecular viscosity ν for the physical dimension of each tensor-coefficient $\Xi_{(n)}$ (2.18) will be performed. This idea will now only be explicitly demonstrated by expanding the physical dimension of the tensor-coefficient belonging to $\tau^{\alpha\beta}_{(3)}$ — the expansion of all other tensor-coefficients can be carried out analogously. The dynamical dimension of $\Xi_{(3)}$, where

$$\left[\Xi_{(3)}\right] = \frac{\mathcal{K}^2}{\mathcal{E}^{*2}}, \tag{2.21}$$

will be identified as the zeroth order term of the power series in ν and accordingly as the high-Re_T term very far from the wall. To expand in a dimensionally consistent manner the invariant turbulent eddy viscosity $\nu_T = \mathcal{K}^2/\mathcal{E}^*$ has to scale along. For the functional dependence of the tensor-coefficient $\Xi_{(3)}$ we therefore make the following reasonable ansatz:

$$\Xi_{(3)}(\mathcal{K}, \mathcal{E}^*, \nu) = \sum_{n=0}^{N} a_n \left(\frac{\nu}{\nu_T}\right)^n \frac{\mathcal{K}^2}{\mathcal{E}^{*2}} = \sum_{n=0}^{N} a_n \nu^n \frac{\mathcal{K}^{2-2n}}{\mathcal{E}^{*2-n}}, \tag{2.22}$$

where the a_n denote arbitrary numbers. Now, since the turbulent kinetic energy varies as $\mathcal{K} \sim \mathcal{O}(y^2)$ and the scalar dissipation rate as $\mathcal{E}^* \sim \mathcal{O}(y)$ when the wall $y \to 0$ is approached one has to guarantee analyticity at the wall $y = 0$. This is achieved by allowing the relative powers Z_n in the wall-normal coordinate y of all series terms to be only positive integers

$$Z_n = 2 \cdot (2 - 2n) - (2 - n) = 2 - 3n \in \mathbb{N}_0. \tag{2.23}$$

This equation has three solutions: $n_0 = 0$, $n_1 = 1/3$ and $n_2 = 2/3$; hence the series truncates naturally at $N = 2$

$$\begin{aligned} \Xi_{(3)}(\mathcal{K}, \mathcal{E}^*, \nu) = &\ a_0 \frac{\mathcal{K}^2}{\mathcal{E}^{*2}} + a_1 \nu^{1/3} \frac{\mathcal{K}^{4/3}}{\mathcal{E}^{*5/3}} + a_2 \nu^{2/3} \frac{K^{2/3}}{\mathcal{E}^{*4/3}} \\ \sim &\ \mathcal{O}(y^2) + \quad \mathcal{O}(y) \quad + \quad \mathcal{O}(1), \end{aligned} \tag{2.24}$$

where the line below indicates the asymptotic near-wall behavior of each term when $y \to 0$. Interestingly to note is that the exponents q_n and r_n in (2.20) are integers for all tensorial expansion coefficients $\Xi_{(n)}$ of the Reynolds-stress tensor. The same is true for the tensorial expansion coefficients of the other five unclosed terms in (2.8), meaning that Z_n always has the structure $Z_n = m - 3n$ where m is an integer, thus implying for all unclosed terms power expansions in the molecular viscosity ν which are multiples of one-third, a result concordant with the discrete time-reflection symmetry D_2 given in (2.12), since its transformation stays within the set of real numbers $\mathbb{R}$.

Looking back on (2.24), we clearly see that as the molecular viscosity ν gains in power, the dominance of the turbulent kinetic energy $\mathcal{K}$ decreases and the dominance of the dissipation rate $\mathcal{E}^*$ increases. Therefore, it is reasonable to identify the $\mathcal{O}(1)$-term as the low-Re_T term very close to the wall. Hence, this power series now gives us the ability to satisfy the exact near-wall behavior of the Reynolds-stress tensor. For example, in the fully-developed regime of a non-rotating flat channel the near-wall behavior for all non-zero components of the full tensor expansion term $\Xi_{(3)}\tau_{(3)}^{\alpha\beta}$ in this flow regime is

$$\begin{gathered} \Xi_{(3)} \cdot \tau_{(3)}^{12} \sim \mathcal{O}(1) \cdot G \partial_y \langle p \rangle \sim \mathcal{O}(y^3), \\ \Xi_{(3)} \cdot \tau_{(3)}^{11} \sim \mathcal{O}(1) \cdot G^2 \sim \mathcal{O}(1), \quad \Xi_{(3)} \cdot \tau_{(3)}^{22} \sim \mathcal{O}(1) \cdot (\partial_y \langle p \rangle)^2 \sim \mathcal{O}(y^6), \end{gathered} \tag{2.25}$$

where only the relevant, i.e. the lowest order terms were taken along. Since the near-wall behavior of the Reynolds-stress tensor $\tau^{\alpha\beta}$ always is

$$\tau^{12} \sim \tau^{23} \sim \mathcal{O}(y^3), \quad \tau^{11} \sim \tau^{13} \sim \mathcal{O}(y^2), \quad \tau^{22} \sim \mathcal{O}(y^4), \tag{2.26}$$

the term $\Xi_{(3)}\tau_{(3)}^{\alpha\beta}$ violates the near-wall behavior only in the normal Reynolds-stress component τ^{11}. This can be solved by calibrating the constant coefficients a_2 and a_1 in (2.24) to zero, i.e. the tensor-coefficient has to vary as $\Xi_{(3)} \sim \mathcal{O}(y^2)$. However, since we have not calibrated to the general mean solution of the channel flow but only to a reduced solution space of the fully-developed flow regime, this calibrated near-wall behavior will certainly not stay invariant under *any* topological equivalent deformation. But nevertheless, one can show that in the case of fully-developed rotating channel flow about any axis the near-wall behavior for the Reynolds-stress tensor will not be violated as long as $\Xi_{(3)} \sim \mathcal{O}(y^2)$. The same is true for fully-developed axially rotating and non-rotating pipe flow. Hence, the tensorial expansion term $\Xi_{(3)}\tau_{(3)}^{\alpha\beta}$ definitely is an important quantity for modeling the rotating flat channel as well as the axially rotating pipe flow.

Finally, any analytically calibrated Reynolds-stress tensor must be constrained to produce real flow solutions. Within the framework presented here, it is reasonable to first aim at weak realizability in which the turbulent kinetic energy $\mathcal{K}$ yields positive values only. This can be achieved by using the consistency constraint that $\mathcal{K}$ must be one-half the contraction of the Reynolds-stress tensor

$$\mathcal{K} = \tfrac{1}{2} k^{\langle u \rangle}_{\alpha\beta} \tau^{\alpha\beta}, \tag{2.27}$$

as a differential constraint-equation next to the $\mathcal{K}$-$\mathcal{E}^*$ transport equation (2.8), thus leading to an overdetermined system of seven strongly coupled differential equations for the six flow variables $\langle u^i \rangle$, $\langle p \rangle$, $\mathcal{K}$ and $\mathcal{E}^*$. The model parameters in the Reynolds-stress tensor then have to be adjusted such that the right-hand side of the constraint (2.27) can only take on positive values during a numerical computation.

3. Future work

To get a first quality judgment on the newly proposed covariant $\mathcal{K}$-$\mathcal{E}^*$ model for wall-bounded turbulent flows, it is reasonable to first study its numerical behavior for the simple canonical flow of a rotating channel in the stationary regime. Particularly to study the effect of secondary flow in a streamwise rotating channel (Oberlack *et al.* 2006) as well as the effect of re-laminarization at moderate rotation rates in the spanwise rotating channel (Grundestam *et al.* 2008).

The $\mathcal{K}$-$\mathcal{E}^*$ model will first be reduced to only those terms which from an analytical point of view appear to be the most promising to capture these rotational effects. This helps on the one side to easily implement the model into a currently used numerical flow solver, and on the other side to easily keep track of usual numerical issues as convergence and stability. Once the numerical code runs at a satisfactory level, more and more relevant modeling terms can be added to increase the accuracy of predicting the effects of rotating turbulent channel flow.

In the fully-developed stationary and spatially one-dimensional regime the numerical code to be chosen must be able to handle an overdetermined set of at most six strongly-coupled, highly nonlinear ordinary differential equations (ODEs) for five one-dimensional flow variables, the mean streamwise velocity $U(y)$, the mean cross-stream (secondary flow) velocity $W(y)$, the mean pressure $P(y)$, the turbulent kinetic energy $\mathcal{K}(y)$, and the dissipation rate $\mathcal{E}^*(y)$. The differential equation for $\mathcal{K}$ is of fourth order, while of second order for the remaining variables. To avoid the difficult inherent stiffness problems which nonlinear ODE systems generally face, it is preferable to run a pseudo-time stepping method by studying the solution then in the stationary regime $t \to \infty$.

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