Discontinuous Galerkin method with WENO limiter for flows with discontinuity

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1. Motivation and objectives

High-order schemes, such as the discontinuous Galerkin (DG) method (Cockburn & Shu 1998) and the spectral difference (SD) method (Liang *et al.* 2009) are well suited for solving hyperbolic Euler equations. In the presence of discontinuity in flows, a high-order weighted essentially non-oscillatory (WENO) scheme has been proposed by Qiu & Shu (2005) as a nonlinear limiter to control spurious oscillation. The objectives of this work are to look at the performance of the DG method with WENO limiter for some flow problems with discontinuity, such as the Sod shock tube problem, 1D shallow water dam break problem and the problem of two mixing fluids with different densities.

The DG method has been applied by Fagherazzi *et al.* (2004) to solve the hyperbolic shallow water equations recently with low-order limiters. The first objective of this paper is to solve the shallow water equations using the DG method with high-order WENO limiter. The shallow water equations have been used as a kernel for both oceanic and atmospheric general circulation models and are useful in evaluating numerical methods for hydraulic flow, weather forecasting and climate modeling. In addition, when solute transport is considered, we need to solve an additional conservative equation

$$(\rho\phi)_t + (\rho u\phi)_x = 0. \tag{1.1}$$

Moreover, the above scalar transport equation together with Euler equations are often chosen as governing equations for two fluids, see Johnsen (2008); Cook (2009). Johnsen (2008) has characterized the errors generated by different formulations of the governing equations for interface-capturing methods. Recently, Ham & Johnsen (2009) have proposed a way to control the above spurious errors using conservative formulations of Euler equations and scalar transport equation together with consistent treatments of numerical fluxes of continuity and scalar transport equations. The second objective of this paper is to extend this idea to solve two-fluid problems using the DG method with a WENO limiter.

2. Numerical formulation

The DG method which is implemented for the shallow water equations is the same as the one published by Cockburn & Shu (1998) for Euler equations. For the third-order DG scheme, we choose four Gauss-Lobatto quadrature points within each cell for its volume integration which is also adopted by Qiu & Shu (2005). For the fourth-order DG scheme, five Gauss-Lobatto quadrature points are employed in each cell.

The DG scheme is continuous within each element where we use the Legendre polynomial basis, but discontinuous across element interfaces. The Rusanov solver (Rusanov 1961) is employed for computing interface fluxes. The Rusanov scheme takes the following form

$$\hat{F} = 1/2\left[(F_L + F_R)\cdot\mathbf{n} - \lambda(Q_R - Q_L)\right]; \tag{2.1}$$

where $\lambda = |V_n| + c$ is an upper bound for the absolute values of the characteristic speeds. V_n is the fluid velocity normal to edge interface and c is the sound speed.

In this paper, we consider only the conservation forms of the governing equations

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0; \tag{2.2}$$

where U is the vector of conserved variables; F is the flux in the flow direction.

The WENO scheme developed here is limited on uniform meshes. It is implemented according to the fifth-order upwind scheme published by Jiang & Shu (1996). Firstly, it is used to produce an 'exact' solution using very dense mesh for comparison. Secondly, in the presence of shock in the flow solution obtained by the DG scheme, we detect the 'troubled' cells associated with the shock and abandon the solution polynomials in such troubled cells and "reconstruct" moments u_i^l for l=1,...,k from the cell-averaged values of neighboring cells and the cell-averaged value of this troubled cell u_i^0 using the WENO procedure.

For the system cases, such as Euler equations and shallow water equations, in order to achieve better qualities at the price of more complicated computations, Qiu & Shu (2005) have pointed out that the WENO reconstruction limiter should be used with a local characteristic field decomposition.

2.1. *Euler equations*

The Euler equations can be transformed into uncoupled wave equations if they are expressed in characteristic variables instead of conserved variables. The right eigenvectors $R = (R_1, R_2, R_3)$ of the Jacobian matrix $J = \frac{\partial F}{\partial U}$ are

$$\begin{pmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & u^2/2 & H+ua \end{pmatrix};$$

where H is the total enthalpy.

Its left eigenvectors $L = (L_1, L_2, L_3)$ are

$$\begin{pmatrix} \frac{(\gamma-1)u^2}{4a^2} + \frac{u}{2a} & \frac{(1-\gamma)u-a}{2a^2} & \frac{\gamma-1}{2a^2} \\ 1 - \frac{(\gamma-1)u^2}{2a^2} & \frac{(\gamma-1)u}{a^2} & \frac{1-\gamma}{a^2} \\ \frac{(\gamma-1)u^2}{a^2} - \frac{u}{2a} & \frac{(1-\gamma)u+a}{2a^2} & \frac{\gamma-1}{2a^2} \end{pmatrix}.$$

The conservative equations in 2.2 are linearized as

$$\frac{\partial W}{\partial t} + \Lambda\frac{\partial W}{\partial x} = 0; \tag{2.3}$$

where $\Lambda = diag(u-c, u, u+c)$ is a diagonal matrix whose entries are eigenvalues of the Jacobian matrix.

We can then perform the WENO reconstruction for W. Finally, a conversion process $U = RW$ is made. Details of the local field decomposition can be seen in Shu (1998).

2.2. *Shallow water equations*

For the shallow water equations, we have

$$U = \begin{pmatrix} \phi \\ \phi u \end{pmatrix}$$

$$F = \begin{pmatrix} \phi u \\ \phi u^2 + 1/2\phi^2 \end{pmatrix}.$$

where $\phi = gh$ and u denotes fluid velocity; g refers to gravity acceleration and h refers to water depth.

Denote $c = \sqrt{gh}$, the jacobian matrix $J = \frac{\partial F}{\partial U}$ is

$$\begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}.$$

The right eigenvectors of J are

$$\begin{pmatrix} 1 & 1 \\ u - c & u + c \end{pmatrix}.$$

The left eigenvectors of J are

$$\begin{pmatrix} \frac{u+c}{2c} & \frac{-1}{2c} \\ \frac{c-u}{2c} & \frac{1}{2c} \end{pmatrix},$$

and $\Lambda = diag(u - c, u + c)$ is a diagonal matrix whose entries are eigenvalues of J.

2.3. *Binary mixing*

For the two-fluid mixing problem, we solve Euler equations and equation 1.1. ϕ in equation 1.1 refers to the mixture mass fraction. The mixture molecular weight M is defined as

$$\frac{1}{M} = \frac{\phi_A}{M_A} + \frac{1 - \phi_A}{M_B}. \tag{2.4}$$

The ratio of mixture specific heats is defined as

$$\frac{1}{\gamma - 1} = \frac{\phi_A}{\gamma_A - 1}\frac{M}{M_A} + \frac{1 - \phi_A}{\gamma_B - 1}\frac{M}{M_B}. \tag{2.5}$$

All computations utilize a three-stage Runge Kutta scheme employed by Cockburn & Shu (1998), and the time step size was kept sufficiently small.

3. Results

3.1. *One-dimensional Sod shock tube*

We consider 1D shock tube introduced by Sod (1981) as the first test case. The computational domain is [0,1]. We specify the initial condition as $\rho = 1$, $u = 0$ and $p = 1$ for [0,0.5] and $\rho = 0.125$, $u = 0$ and $p = 0.1$ for (0.5,1]. The end time is 0.15. For the purpose of comparison, 'exact' solutions for density, velocity and pressure are obtained using the fifth-order WENO scheme with a local characteristic field decomposition on a uniform mesh with 4000 grid points.

Figure 1 presents the density field predicted by the third-order DG scheme with the

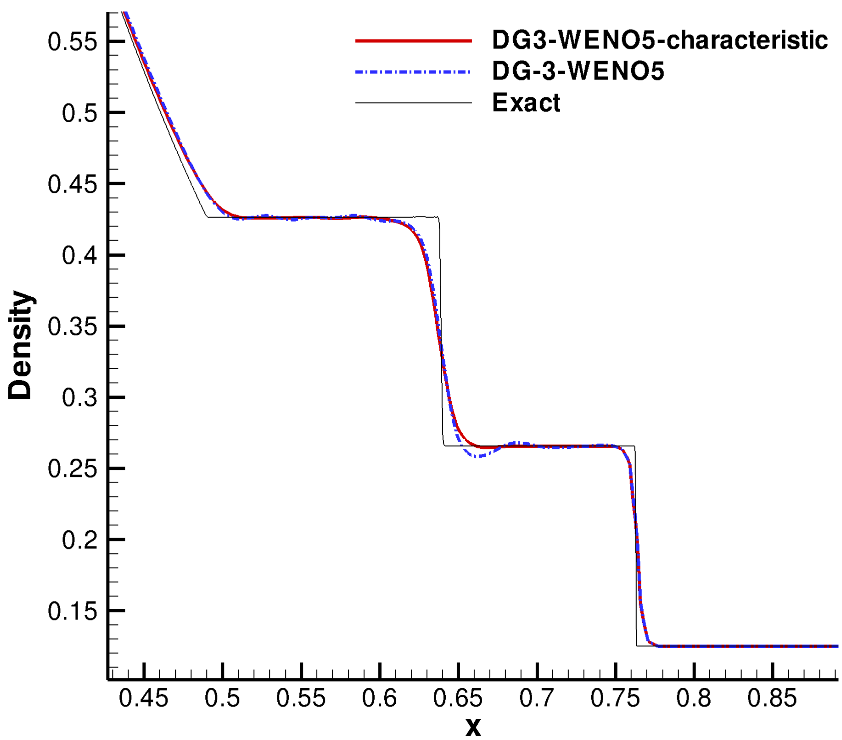

Figure 1. Density at time 0.15 for the Sod shock tube problem

WENO limiters using a mesh of 200 grid points. The WENO reconstruction limiter without the local characteristic field decomposition produces spurious oscillations between the positions of contact discontinuity and shock.

Figure 2 presents the velocity field predicted by the third-order DG scheme with the WENO limiters using a mesh of 200 grid points. The WENO reconstruction limiter without the local characteristic field decomposition produces spurious oscillations.

Figure 3 presents the pressure field predicted by the third-order DG scheme with WENO limiters using a mesh of 200 grid points. The WENO reconstruction limiters with or without the local characteristic field decomposition both obtained satisfactory results.

3.2. *One-Dimensional Dam Break*

Both third- and fourth-order DG schemes with WENO limiters are then applied to the 1D dam break problem. The equations are solved on a domain [0,2000]. At the beginning of the simulation, a dam divides the domain into two equal parts: the reservoir at the left and the tailwater at the right. The initial water height at the left is $h_L = 10$ and that at the right is zero. The initial velocity is zero everywhere. The dam is instantanaeously removed and the solution is collected at a time until $t = 50$.

The 'exact' solutions for water depth and flow velocity are obtained using the fifth

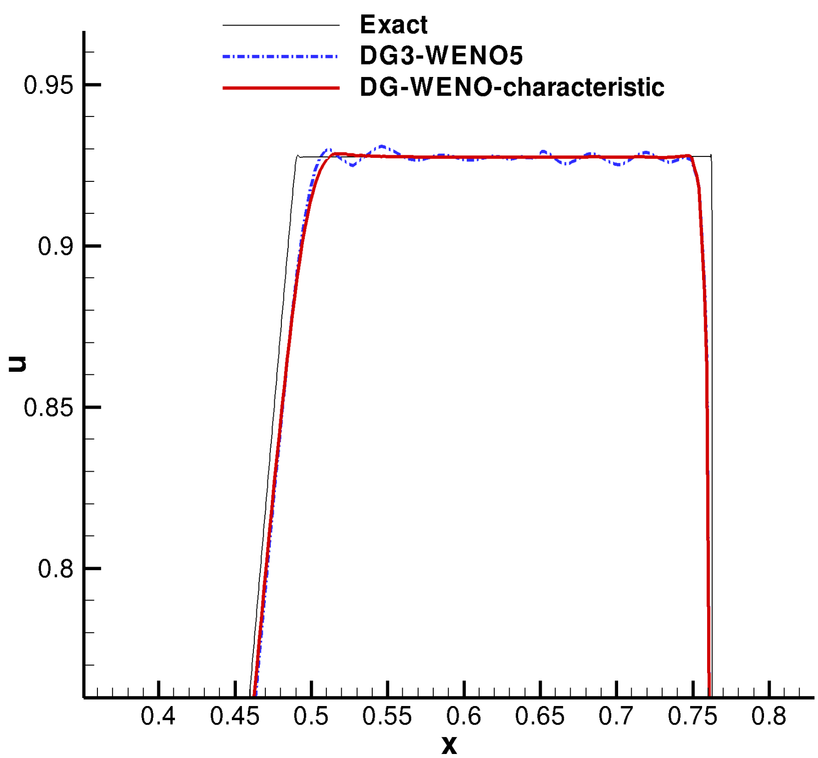

FIGURE 2. Velocity at time 0.15 for Sod shock tube problem

WENO scheme on a mesh of 4000 cells. As shown in figure 4, the fourth-order DG scheme with WENO limiter on a grid with 100 points produces results similar to those of the third-order DG scheme with WENO limiter. The WENO reconstructions for both DG4 and DG3 cases, are used with the local characteristic field decomposition.

Figure 5 shows that the third-order DG scheme with WENO limiter without the local characteristic decomposition produces spurious oscillations for both profiles of water depth and flow velocity.

3.3. *Binary mixing at constant temperature*

Both third- and fourth-order DG schemes are finally employed to study a two-fluid mixing problem which was considered by Ham & Johnsen (2009). This test problem is about helium and hydrogen mixing at a constant temperature of 300K and at a constant pressure level. The compuational domain is [0,1]. Initial conditions are helium locating at [0.3,0.7] and hydrogen elsewhere with $p = 1atm$, $u = 300$ and $T = 300$.

For this test case, solution discontinuity is detected whenever the gradient of density reaches a threshold. The DG scheme is replaced with the WENO reconstruction for the cells with such discontinuity. Figure 6 shows density and temperature profiles predicted by the pure DG scheme and the DG scheme with WENO limiter. The WENO limiter is not performed with the local characteristic field decomposition for this test problem. The

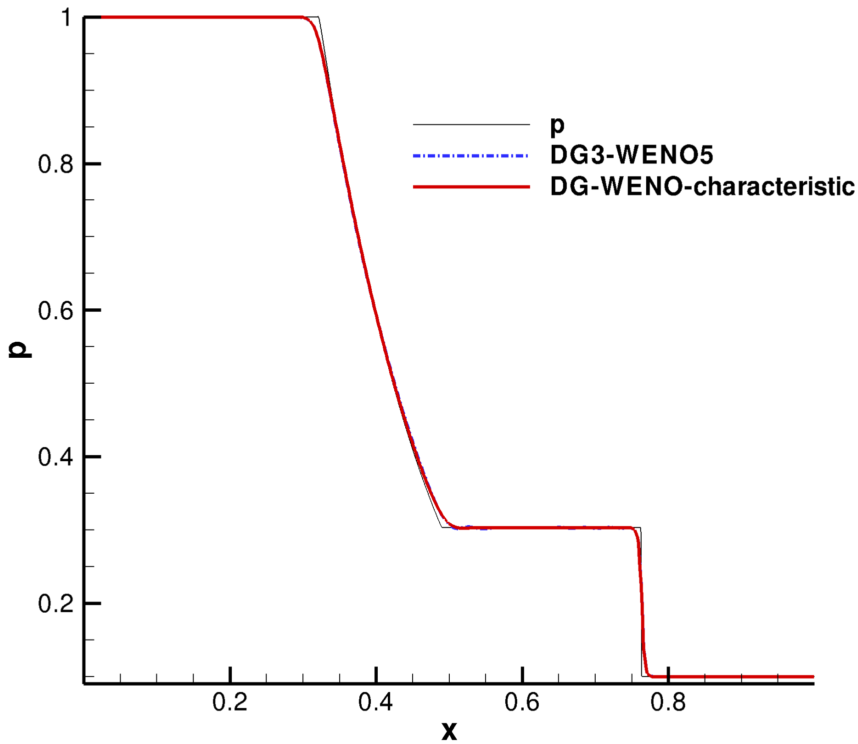

FIGURE 3. Pressure at time 0.15 for Sod shock tube problem

first-order DG scheme does not need any limiter but is too dissipative to predict accurate density field on a grid with 2000 points even though it produces an exact solution for temperature.

The third-order DG scheme with WENO limiter produces better results for the density field than the pure DG scheme which produces spurious oscillations near the material interfaces of density jump. The single DG schemes are able to predict exactly a constant temperature distribution. The WENO limiter produces a couple of very small-amplitude bulges near the material interfaces. The first possible reason may be that some accuracy is lost during the WENO reconstruction of the DG high-order moments. The second possible reason could be due to the fact that the WENO reconstruction limiter is not performed with the local characteristic field decomposition.

4. Conclusion and future work

The discontinuous Galerkin method with the WENO limiter is successfully implemented for 1D Euler and scalar transport equations. It is able to accurately simulate a few different types of problems defined in the paper. Incorporation of the local characteristic field decomposition is important for WENO reconstruction for both shock tube and dam break problems. Our next step is to investigate the effect of local characteris-

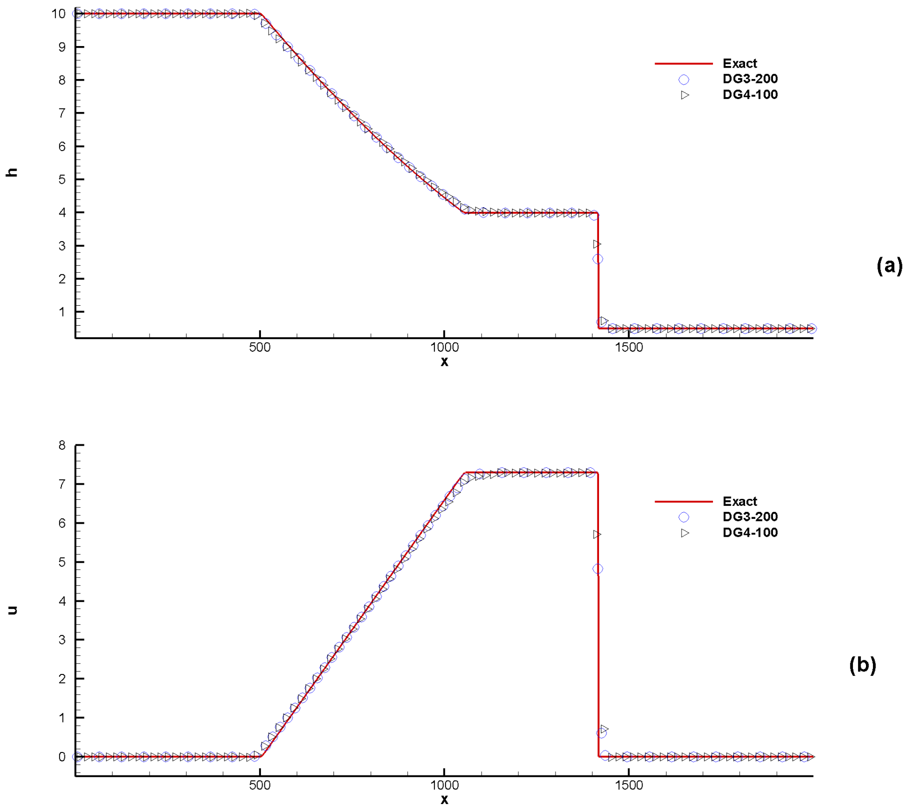

FIGURE 4. The water depth (a) and flow velocity (b) predicted for the 1D dam-break problem

tic field decomposition for the binary mixing test case. A future work plan will involve application of the DG method with the WENO limiter on multi-dimensional two-fluid mixing problems and shallow water flow with solute transport.

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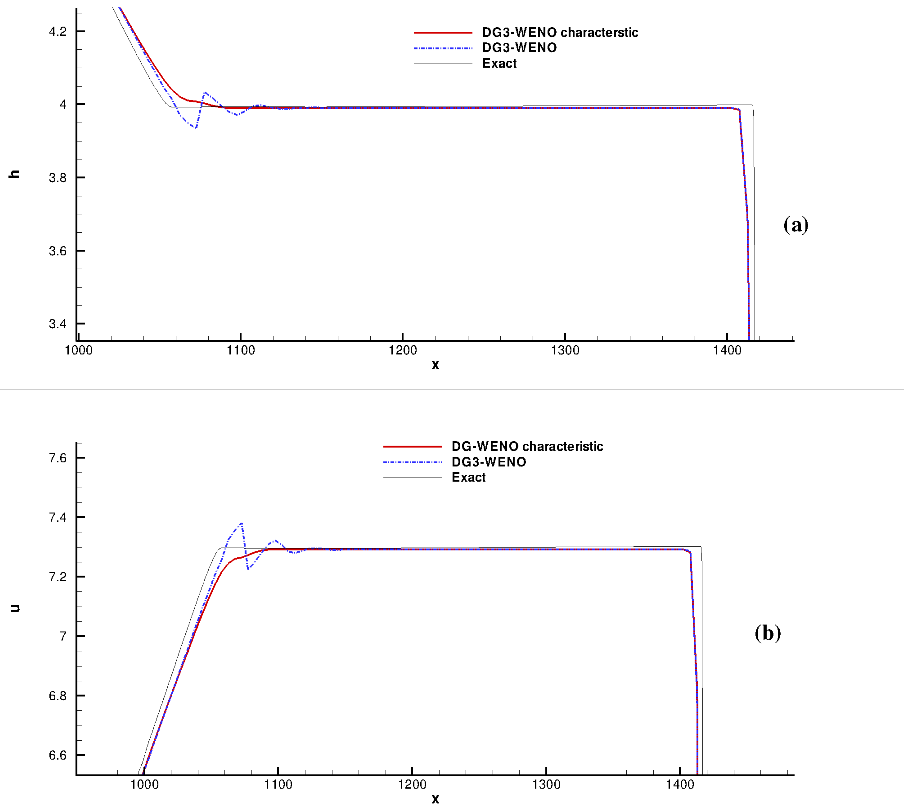

Figure 5. The water depth (a) and flow velocity (b) predicted for 1D dam-break problem (400 grid points)

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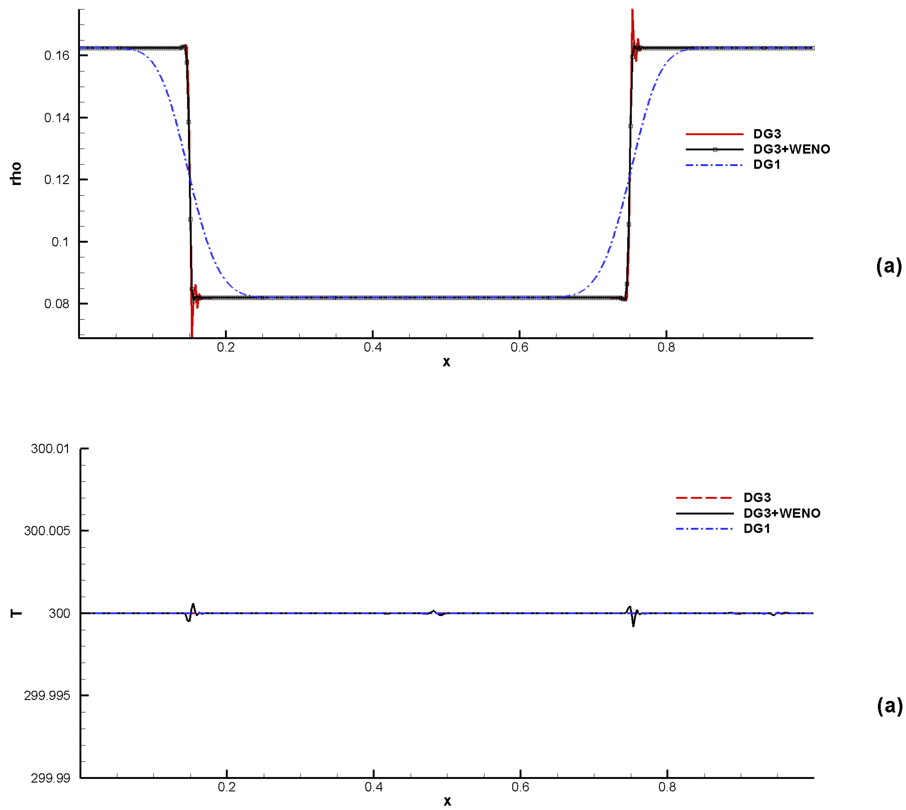

FIGURE 6. Density (a) and temperature (b) at time 0.0015s predicted for binary mixture of hydrogen and helium.

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