Well-posedness and stability of a coupled fluid flow and heat transfer problem

By J. Lindström† AND J. Nordström†‡

1. Introduction

The coupling of fluid and heat equations is an area that has many interesting scientific and engineering applications. From the scientific side it is interesting to mathematically derive conditions to make the coupled system well posed and compare with actual physics. The applications for conjugate heat transfer range between cooling of turbine blades, electronic components, nuclear reactors or spacecraft re-entry, to name just a few. The particular application we are working towards here is a microscale satellite cold gas propulsion system with heat sources that will be used for controlling the flow rate, see Lindström *et al.* (2009).

This paper is the first step in understanding the coupling procedure within our framework. The computational method that we are using is a finite difference method on Summation-By-Parts (SBP) form with the Simultaneous Approximation Term (SAT). This method has been developed in many papers e.g. Strand (1994); Mattsson & Nordström (2004); Carpenter *et al.* (1999); Gong & Nordström (2006) and used for many difficult problems where it has proven to be robust, Svärd *et al.* (2007); Svärd & Nordström (2008); Mattsson *et al.* (2007). The extension to multiple dimensions is relatively straightforward once the one-dimensional case has been investigated. The difficulty in extending to multiple dimensions lies rather in a high performance implementation than in the theory.

2. The continuous problem

The equations we studied in this paper are motivated by a gas flow in a long channel with heat sources. The channel is long compared to the height and hence the changes in the tangential direction are small in comparison to the changes in the normal direction. The equations are an incompletely parabolic system of equations for the flow and the scalar heat equation for the heat transfer,

$$w_t + Aw_x = \varepsilon Bw_{xx}, \qquad -1 \leq x \leq 0, \tag{2.1}$$

and

$$T_t = kT_{xx}, \qquad 0 \leq x \leq 1, \tag{2.2}$$

† Department of Information Technology, Uppsala University, SE-75105 Uppsala, Sweden.
‡ Senior Research Fellow, CTR; School of Mechanical, Industrial and Aeronautical Engineering, University of the Witvatersrand, PO WITS 2050, Johannesburg, South Africa; Department of Aeronautics and Systems Integration, FOI, The Swedish Defense Research Agency, SE-16490 Stockholm, Sweden

where

$$w = \begin{bmatrix} \rho \\ u \\ \mathcal{T} \end{bmatrix}, \qquad A = \begin{bmatrix} a & b & 0 \\ b & a & c \\ 0 & c & a \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}. \tag{2.3}$$

We can view Eq. (2.1) as the Navier-Stokes equations linearized and symmetrized around a state with some mean velocity. In that case we would have

$$a = \bar{u}, \quad b = \frac{\bar{c}}{\sqrt{\gamma}}, \quad c = \bar{c}\sqrt{\frac{\gamma - 1}{\gamma}}, \quad \alpha = \frac{\lambda + 2\mu}{\bar{\rho}}, \quad \beta = \frac{\gamma\mu}{Pr\bar{\rho}}, \tag{2.4}$$

where $\bar{u}$, $\bar{\rho}$ and $\bar{c}$ are the mean velocity, density and speed of sound respectively. γ is the ratio of specific heats, Pr the Prandtl number and λ and μ are the second and dynamic viscosities, Svärd *et al.* (2007). At this point the only assumption on the coefficients is that $a, \alpha, \beta > 0$ to keep the discussion general. Our objective is to couple Eq. (2.1) and Eq. (2.2) at $x = 0$ and investigate which interface conditions that will lead to a well-posed coupled system.

2.1. *Boundary and interface conditions*

Since we are concerned with the interface between the equations in this report, the boundary conditions will not be analyzed. At the left boundary $x = -1$ we have the semi characteristic boundary conditions

$$\frac{1}{\sqrt{2}d}(-\sqrt{2}c\rho + \sqrt{2}b\mathcal{T}) = f_1(t), \tag{2.5}$$

$$\frac{1}{\sqrt{2}d}(b\rho + du + c\mathcal{T}) = f_2(t) \tag{2.6}$$

$$\alpha d u_x - \beta c \mathcal{T}_x = f_3(t), \tag{2.7}$$

which is a well-posed set of boundary conditions. At the right boundary $x = 1$ we use a Dirichlet boundary condition on the temperature.

The energy method is used to derive the interface conditions. Define the energy norm of w as

$$||w||^2 = \int_\Omega w^T w d\Omega. \tag{2.8}$$

By multiplying Eq. (2.1) with w^T, Eq. (2.2) with T, integrating them over their respective domains and adding them together, we get (when ignoring boundary terms)

$$\frac{d}{dt}(||w||^2 + ||T||^2) = -w^T A w + 2\varepsilon w^T B w_x - 2kTT_x - 2\varepsilon \int_{-1}^{0} w_x^T B w_x dx - 2k \int_0^1 T_x^2 dx. \tag{2.9}$$

Since we are considering the interface as a solid wall which separates the fluid from the solid and since we want a continuous heat transfer we impose

$$u = 0, \qquad \mathcal{T} = T. \tag{2.10}$$

Using the interface conditions in Eq. (2.10), Eq. (2.9) reduces to

$$\frac{d}{dt}(||w||^2 + ||T||^2) = 2\mathcal{T}(\beta\varepsilon\mathcal{T}_x - kT_x) - 2\varepsilon \int_{-1}^{0} w_x^T B w_x dx - 2k \int_0^1 T_x^2 dx \tag{2.11}$$

and we can easily see that if we impose

$$\beta\varepsilon\mathcal{T}_x - kT_x = 0 \tag{2.12}$$

as the final interface condition we get an energy estimate. Without Eq. (2.12), the interface can act as an unphysical heat source or sink.

3. The semidiscrete problem

Eq. (2.1) is discretized on the single domain $[-1, 0]$ on a uniform grid of $M + 1$ grid points. The vector $\mathbf{w} = [w_0, w_1, \ldots, w_M]^T = [\rho_0, u_0, \mathcal{T}_0, \rho_1, u_1, \mathcal{T}_1, \ldots, \rho_M, u_M, \mathcal{T}_M]^T$ is the discrete approximation of w. The derivatives are approximated by the operators on SBP form†

$$\mathbf{w}_x \approx (D_1^L \otimes I_3)\mathbf{w} = (P_L^{-1}Q_L \otimes I_3)\mathbf{w} \tag{3.1}$$
$$\mathbf{w}_{xx} \approx (D_2^L \otimes I_3)\mathbf{w} = (P_L^{-1}Q_L \otimes I_3)^2\mathbf{w} \tag{3.2}$$

where P_L is a symmetric positive definite matrix and Q_L is an almost skew symmetric matrix satisfying $Q_L + Q_L^T = B_L = \mathrm{diag}(-1, 0, \ldots, 0, 1)$, Strand (1994); Mattsson & Nordström (2004). I_3 is the 3×3 identity matrix. Eq. (2.2) is similarly discretized on a uniform grid of $N + 1$ grid points.

The discretizations of Eq. (2.1) and Eq. (2.2) with the boundary and interface conditions using the SAT method are

$$\begin{aligned}
\mathbf{w}_t = &-(D_1^L \otimes A)\mathbf{w} + \varepsilon(D_2^L \otimes B)\mathbf{w} \\
&+ (P_L^{-1}E_0^L \otimes \Sigma_1^0)(X^T w_0 - g_1^0) \\
&+ (P_L^{-1}E_0^L \otimes \Sigma_3^0)(\alpha d(D_1^L u)_0 - \beta c(D_1^L \mathcal{T})_0 - g_3^0) \\
&+ (P_L^{-1}(D_1^L)^T E_0^L \otimes \Sigma_5^0)(c u_0 + d\mathcal{T}_0 - g_5^0) \\
&+ (P_L^{-1}E_M^L \otimes \Sigma_1^M)(w_M - g_1^M) \\
&+ (P_L^{-1}E_M^L \otimes \Sigma_2^M)(w_M - g_1^M) \\
&+ (P_L^{-1}E_M^L \otimes \Sigma_3^M)(\mathcal{T}_M - T_0) \\
&+ (P_L^{-1}(D_1^L)^T E_M^L \otimes \Sigma_4^M)(\mathcal{T}_M - T_0) \\
&+ (P_L^{-1}E_M^L \otimes \Sigma_5^M)(\beta\varepsilon(D_1^L \mathcal{T})_M - k(D_1^R T)_0) \\
&- \mathrm{DI}_L
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\mathbf{T}_t = &\, kD_2^R \mathbf{T} \\
&+ \tau_1^0 P_R^{-1} E_0^R (T_0 - \mathcal{T}_M) \\
&+ \tau_2^0 P_R^{-1} (D_1^R)^T E_0^R (T_0 - \mathcal{T}_M) \\
&+ \tau_3^0 P_R^{-1} E_0^R (k(D_1^R T)_0 - \beta\varepsilon(D_1^L \mathcal{T})_M) \\
&+ \tau_1^N P_R^{-1} E_N^R (T_N - h_1^N) \\
&- \mathrm{DI}_R.
\end{aligned} \tag{3.4}$$

The matrices $E_0^L = \mathrm{diag}(1, 0, \ldots, 0)$, $E_M^L = \mathrm{diag}(0, \ldots, 0, 1)$ and $E_{0,N}^R$ similarly defined are used to select boundary elements. The 3×3 matrices $\Sigma_i^{0,M}$ and coefficients $\tau_j^{0,N}$ are called penalty matrices and penalty coefficients which have to be determined for stability,

† The drawback to the approximation in Eq (3.2) is that the computational stencil is wide. Compact formulations that use minimal bandwidth do, however, exist. See Mattsson & Nordström (2004).

Strand (1994); Mattsson & Nordström (2004). The last term in Eq. (3.3) and Eq. (3.4) are artificial dissipation operators which reduce spurious oscillations. An extensive study of these operators can be found in Mattsson *et al.* (2004). The details are beyond the scope of this paper. They do not cause stability or accuracy problems and will be assumed to be treated correctly in the rest of the paper, and hence omitted in the stability derivations to keep the notation as simple as possible.

3.1. *Stability conditions at $x = 0$*

At $x = 0$ we have the two interface schemes

$$\begin{aligned}\mathbf{w}_t &= -(D_1^L \otimes A)\mathbf{w} + \varepsilon(D_2^L \otimes B)\mathbf{w} \\ &+ (P^{-1}E_M^L \otimes \Sigma_1^M)(w_M - g_1^M) \\ &+ (P^{-1}E_M^L \otimes \Sigma_2^M)(w_M - g_1^M) \\ &+ (P^{-1}E_M^L \otimes \Sigma_3^M)(\mathcal{T}_M - T_0) \\ &+ (P^{-1}(D_1^L)^T E_M^L \otimes \Sigma_4^M)(\mathcal{T}_M - T_0) \\ &+ (P^{-1}E_M^L \otimes \Sigma_5^M)(\beta\varepsilon(D_1^L \mathcal{T})_M - k(D_1^R T)_0)\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathbf{T}_t = kD_2^R\mathbf{T} & \\ &+ \tau_1^0 P_R^{-1}E_0^R(T_0 - \mathcal{T}_M) \\ &+ \tau_2^0 P_R^{-1}(D_1^R)^T E_0^R(T_0 - \mathcal{T}_M) \\ &+ \tau_3^0 P_R^{-1}E_0^R(k(D_1^R T)_0 - \beta\varepsilon(D_1^L\mathcal{T})_M).\end{aligned} \tag{3.6}$$

We let $g_1^M = 0$ and multiply $\mathbf{w}$, $\mathbf{T}$ from the left with $(\mathbf{w^T} \otimes I_3)$ and $\mathbf{T^T}$, respectively. By using the SBP property of the operators we obtain

$$\begin{aligned}\frac{d}{dt}(||\mathbf{w}||_{P_L}^2 + ||\mathbf{T}||_{P_R}^2) = &-w_M^T A w_M + 2\varepsilon w_M^T B(D_1^L w)_M \\ &- 2\varepsilon(D_1^L\mathbf{w})^T(I_N \otimes B)(D_1^L\mathbf{w}) \\ &+ 2w_M^T\Sigma_1^M w_M + 2w_M^T\Sigma_2^M w_M \\ &+ 2w_M^T\Sigma_3^M(\mathcal{T}_M - T_0) + (D_1^L w)_N^T\Sigma_4^M(\mathcal{T}_M - T_0) \\ &+ 2w_M^T\Sigma_5^M(\beta\varepsilon(D_1^L w)_M - k(D_1^R T)_0) \\ &- 2kT_0(D_1^R T)_0 - 2k(D_1^R\mathbf{T})^T P_R(D_1^R\mathbf{T}) \\ &+ 2\tau_1^0 T_0(T_0 - \mathcal{T}_M) + 2\tau_2^0(D_1^R T)_0(T_0 - \mathcal{T}_M) \\ &+ 2\tau_3^0 T_0(k(D_1^R T)_0 - \beta\varepsilon(D_1^L\mathcal{T})_M).\end{aligned} \tag{3.7}$$

By definition, Gustafsson *et al.* (1995), the interface scheme will be stable if we can prove that $\frac{d}{dt}(||\mathbf{w}||_{P_L}^2 + ||\mathbf{T}||_{P_R}^2) \leq 0$. Hence we need to choose appropriate penalty matrices and coefficients such that this condition is satisfied.

We choose the penalty matrices as

$$\Sigma_1^M = \begin{bmatrix} 0 & \sigma_1^H & 0 \\ 0 & \sigma_2^H & 0 \\ 0 & \sigma_3^H & 0 \end{bmatrix}, \qquad \Sigma_2^M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_2^M & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

$$\Sigma_3^M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_3^M \end{bmatrix}, \quad \Sigma_4^M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_4^M \end{bmatrix}, \quad \Sigma_5^M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_5^M \end{bmatrix}, \tag{3.9}$$

where Σ_1^M is the penalty matrix for the hyperbolic part of u, Σ_2^M for the parabolic part of u and $\Sigma^M_{3,4,5}$ for the coupling terms. With these choices of penalty matrices, Eq. (3.7) can be expanded and all coefficients determined as

$$\sigma_1^H = \frac{b}{2}, \quad \sigma_2^H \leq 0, \quad \sigma_3^H = \frac{c}{2}, \quad \sigma_2^M \leq \frac{-\alpha\varepsilon}{4p_M^L}, \quad \sigma_3^M = \tau_1^0 \leq 0 \tag{3.10}$$

where $p_M^L > 0$ such that $P_L^{(M,M)} - p_M^L \geq 0$. See Carpenter *et al.* (1999); Gong & Nordström (2006) for more details. Moreover, we have

$$s \in \mathcal{R}, \quad \sigma_4^M = -\beta\varepsilon(1+s), \quad \sigma_5^M = s, \quad \tau_2^0 = -ks, \quad \tau_3^0 = 1+s. \tag{3.11}$$

With these coefficients we have the energy estimate $\frac{d}{dt}(||\mathbf{w}||^2_{P_L} + ||\mathbf{T}||^2_{P_R}) \leq 0$ and hence the interface treatment is stable. Details on the technique used in this derivation can be found in Carpenter *et al.* (1999) and Gong & Nordström (2006).

4. Numerical results

The order of convergence is studied by the method of manufactured solutions. A small enough time step has been chosen in order to minimize the errors from the time discretization, which in this case is done by the classical 4th-order explicit Runge-Kutta method. We use the functions

$$\rho = xe^{-\kappa t}, \quad u = \sin(x)e^{-\kappa t}, \quad \mathcal{T} = \frac{1}{\varepsilon}\sin(x)e^{-\kappa t}, \quad T = \frac{1}{k}\sin(x)e^{-\kappa t}, \quad \kappa = 0.1, \tag{4.1}$$

which, when inserted into Eq. (2.1) and Eq. (2.2), gives a modified system of equations with additional forcing functions. The functions Eq. (4.1) have been chosen since they satisfy the interface conditions in a non-trivial way. Using Eq. (4.1) we create exact initial and time-dependent boundary conditions while no data are created at the interface. The rate of convergence is obtained as

$$q_j^i = \log_{10}\left(\frac{||u^i_{j-1} - v^i_{j-1}||}{||u^i_j - v^i_j||}\right) / \log_{10}\left(\frac{h_j}{h_{j-1}}\right), \tag{4.2}$$

where q_j^i denotes the convergence rate for either of the variables $i = \rho, u, \mathcal{T}, T$ at mesh refinement level j. u_j^i is the exact analytic solution for either of the variables i at mesh refinement level j and v_j^i is the discrete solution. The ratio h_j/h_{j-1} is the ratio between the number of grid points at each refinement level. The results can be seen in Table 1. The rates of convergence in Table 1 agree with the theoretically expected results, cf. Mattsson & Nordström (2004) and Gong & Nordström (2006).

An example of a solution, where the coefficients are chosen as

$$a = 0.5, \quad b = \frac{1}{\sqrt{\gamma}}, \quad c = \sqrt{\frac{\gamma-1}{\gamma}}, \quad \gamma = 1.4, \quad \alpha = \beta = 1, \quad \varepsilon = 0.1, \quad k = 1, \tag{4.3}$$

is given in Figure 1. We start with zero initial data and at time $t = 0$ we let $\rho = 0$, $u = 0.5$ and $\mathcal{T} = 1$ at the left boundary while $T = 0$ at the right boundary. The values at the left boundary are transformed into data for the boundary conditions.

5. Summary and conclusions

An incompletely parabolic system of equations has been coupled with the heat equation in one space dimension. The energy method has been used to derive well-posed boundary

Table 1. Order of convergence

$M = N$	2nd order	3rd order	4th order	2nd order	3rd order	4th order
	ρ	ρ	ρ	u	u	u
32	0.2895	2.0197	2.5359	1.6700	2.7938	3.7233
64	1.0769	3.0137	3.7153	2.0652	3.3314	3.9939
128	1.7340	3.5255	4.1774	2.1487	3.1518	4.3242
256	2.0922	3.3945	4.1646	2.1047	3.0587	4.1851
512	2.2167	3.1591	4.1140	2.0588	3.0229	4.0531
	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	T	T	T
32	0.9780	2.7634	3.8021	2.3601	3.1699	4.0291
64	1.7613	2.7542	3.3286	2.1627	3.2639	3.9000
128	2.0164	2.9310	3.5881	2.0824	3.1205	3.9133
256	2.0277	2.9789	3.7798	2.0420	3.0492	3.9476
512	2.0212	2.9928	3.8895	2.0213	3.0226	3.9711

and interface conditions and the resulting numerical scheme has been proven stable using finite differences on SBP form and the SAT boundary and interface treatment. The rate of convergence is verified by the method of manufactured solutions and the result is consistent with the theory within the SBP framework.

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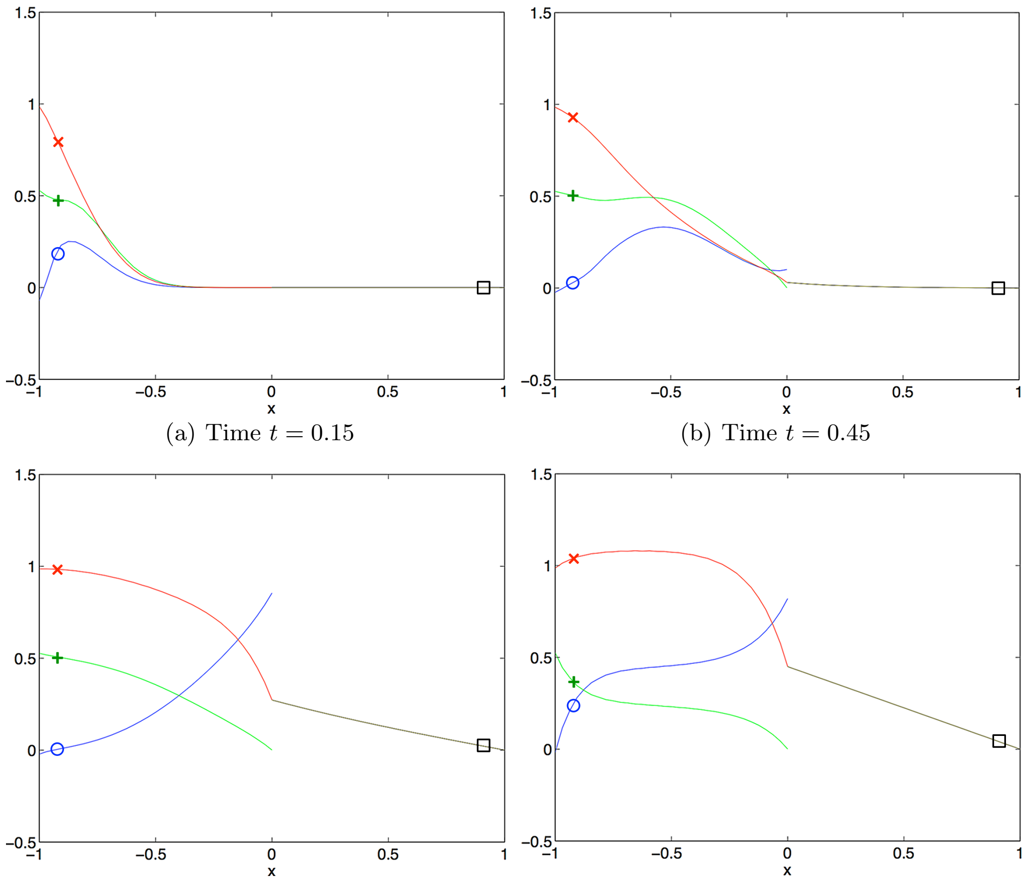

(a) Time $t = 0.15$ (b) Time $t = 0.45$

(c) Time $t = 1.25$ (d) Time $t = 5.0$

FIGURE 1. ρ (lines with $\circ$), u (lines with $+$), $\mathcal{T}$ (lines with $\times$), T (lines with $\square$). A sequence of solutions for different times using $M = N = 32$ grid points and 3rd-order operators. The last figure shows the steady-state solution where the ratio $\mathcal{T}_x/T_x$ of the heat fluxes at the interface is exactly the ratio k/ε of the diffusion coefficients.

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