

Conservative properties of finite difference schemes for incompressible flow

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1. Motivation and objectives

The purpose of this research is to construct accurate finite difference schemes for incompressible unsteady flow simulations such as LES (large-eddy simulation) or DNS (direct numerical simulation).

Experience has shown that kinetic energy conservation of the convective terms is required for stable incompressible unsteady flow simulations. Arakawa (1966) showed that a finite difference scheme that conserves the enstrophy in the absence of viscous dissipation is required for long-time integration in the two-dimensional vorticity-streamfunction formulation. The corresponding conserved variable is kinetic energy in velocity-pressure formulation, and some energy conservative finite difference schemes have been developed for the Navier-Stokes equations in three dimensions. Staggered grid systems are usually required to obtain physically correct pressure fields. The standard second order accurate finite difference scheme (Harlow & Welch 1965) in a staggered grid system conserves kinetic energy and this scheme has proven useful for LES and DNS. However, the accuracy of the second order finite difference scheme is low and fine meshes are required (Ghosal 1995). Spectral methods (Canuto *et al.* 1988) offer supreme accuracy, but these methods are limited to simple flow geometries. Existing fourth order accurate convective schemes (A-Domis 1981, Kajishima 1994) for staggered grid systems do not conserve kinetic energy. Higher order staggered grid schemes that conserve kinetic energy have not been presented in the literature.

The conservation of kinetic energy is a consequence of the Navier-Stokes equations for incompressible flow in the inviscid limit. In contrast, energy conservation in a discrete sense is not a consequence of momentum and mass conservation. It is possible to derive numerical schemes that conserve both mass and momentum but do not conserve kinetic energy. It is also possible to derive schemes that conserve kinetic energy even though mass or momentum conservation are violated.

In this report, conservation properties of the continuity, momentum, and kinetic energy equations for incompressible flow are specified as analytical requirements for a proper set of discretized equations. Existing finite difference schemes in staggered grid systems are checked for satisfaction of the requirements. Proper higher order accurate finite difference schemes in a staggered grid system are then proposed. Plane channel flow is simulated using the proposed fourth order accurate finite difference scheme and the results compared with those of the second order accurate Harlow and Welch (1965) algorithm.

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2. Accomplishments

2.1 Analytical requirements

The continuity and momentum equations describe the motion of incompressible flow. For convenience later in the analysis, these equations are written symbolically as

$$(Cont.) = 0 \quad (1)$$

$$\frac{\partial v_i}{\partial t} + (Conv.)_i + (Pres.)_i + (Visc.)_i = 0 \quad (2)$$

where

$$(Cont.) \equiv \frac{\partial v_i}{\partial x_i}, \quad (Pres.)_i \equiv \frac{\partial p}{\partial x_i}, \quad (Visc.)_i \equiv \frac{\partial \tau_{ij}}{\partial x_j} \quad (3), (4), (5)$$

Here, v_i is velocity component, p is pressure divided by density, and τ_{ij} is viscous stress. Henceforth, p will be referred to as pressure.

The conservation properties of Eqs. (1) and (2) will now be established. Note that Eq. (2) is in the following form.

$$\frac{\partial \phi}{\partial t} + {}^1 Q^\phi + {}^2 Q^\phi + {}^3 Q^\phi + \dots = 0 \quad (6)$$

The term ${}^k Q^\phi$ is *conservative* if it can be written in divergence form

$${}^k Q^\phi = \nabla \cdot ({}^k F_j^\phi) = \frac{\partial ({}^k F_j^\phi)}{\partial x_j} \quad (7)$$

To see that the divergence form is conservative, integrate Eq. (6) over the volume and make use of Gauss's theorem for the flux terms $k = 1, 2, \dots$, all of which are assumed to satisfy Eq. (7)

$$\frac{\partial}{\partial t} \int \int \int_V \phi \, dV = - \int \int_S ({}^1 F^\phi + {}^2 F^\phi + {}^3 F^\phi + \dots) \cdot dS \quad (8)$$

From Eq. (8), we notice that the time derivative of the sum of ϕ in a volume V equals the sum of the flux ${}^k F^\phi$ on the surface S of the volume. In particular, the sum of ϕ never changes in periodic field if ${}^k Q^\phi$ is conservative for all k .

Note that the pressure $(Pres.)_i$ and viscous terms $(Visc.)_i$ are conservative *a priori* in the momentum equation since they appear in divergence form. The convective term is also conservative *a priori* if it is cast in divergence form. This is not always the case, however, and we shall investigate alternative formulations. To perform the analysis, we regard $(Conv.)_i$ as a generic form of the convective term in the momentum equation. At least four types of convective forms have been used traditionally in analytical or numerical studies. These forms are defined as follows.

$$(Div.)_i \equiv \frac{\partial v_j v_i}{\partial x_j} \quad (9)$$

$$(Adv.)_i \equiv v_j \frac{\partial v_i}{\partial x_j} \quad (10)$$

$$(Skew.)_i \equiv \frac{1}{2} \frac{\partial v_j v_i}{\partial x_j} + \frac{1}{2} v_j \frac{\partial v_i}{\partial x_j} \quad (11)$$

$$(Rot.)_i \equiv v_j \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial v_j v_j}{\partial x_i} \quad (12)$$

As mentioned above, the *divergence* form, $(Div.)$, is conservative *a priori*. $(Adv.)_i$, $(Skew.)_i$, and $(Rot.)_i$ are referred to as *advective*, *skew-symmetric*, and *rotational* forms respectively. The four forms are connected with each other through following relations.

$$(Adv.)_i = (Div.)_i - v_i \cdot (Cont.) \quad (13)$$

$$(Skew.)_i = \frac{1}{2} (Div.)_i + \frac{1}{2} (Adv.)_i \quad (14)$$

$$(Rot.)_i = (Adv.)_i \quad (15)$$

We notice that there are only two independent convective forms, and the two are equivalent if $(Cont.) = 0$ is satisfied. It is also apparent that the advective, skew-symmetric, and rotational forms are conservative as long as the continuity equation is satisfied.

The transport equation of the square of a velocity component, $v_1^2/2$, is v_1 times $i = 1$ component of Eq. (2).

$$\frac{\partial v_1^2/2}{\partial t} + v_1 \cdot (Conv.)_1 + v_1 \cdot (Pres.)_1 + v_1 \cdot (Visc.)_1 = 0 \quad (16)$$

In the above equation, the convective term can be modified into the following forms corresponding to those in the momentum equation.

$$v_1 \cdot (Div.)_1 = \frac{\partial v_j v_1^2/2}{\partial x_j} + \frac{1}{2} v_1^2 \cdot (Cont.) \quad (17)$$

$$v_1 \cdot (Adv.)_1 = \frac{\partial v_j v_1^2/2}{\partial x_j} - \frac{1}{2} v_1^2 \cdot (Cont.) \quad (18)$$

$$v_1 \cdot (Skew.)_1 = \frac{\partial v_j v_1^2/2}{\partial x_j} \quad (19)$$

Note that the skew-symmetric form is conservative *a priori* in the velocity square equation. Since the rotational form is equivalent to advective form, the four convective forms are conservative if $(Cont.) = 0$ is satisfied.

The terms involving pressure and viscous stress in Eq. (16) can be modified into following forms.

$$v_1 \cdot (Pres.)_1 = \frac{\partial p v_1}{\partial x_1} - p \frac{\partial v_1}{\partial x_1} \quad (20)$$

Terms in Momentum Eq.	Transport Equations		
	v_i	$v_1^2/2$	K
(Div.)	⊙	○	○
(Adv.) = (Rot.)	○	○	○
(Skew.)	○	⊙	⊙
(Pres.)	⊙	×	○
(Visc.)	⊙	×	×

Table 1. Conservative properties of convective, pressure, and viscous terms in the v_i , $v_1^2/2$, and K equations. ⊙ is conservative *a priori*, ○ is conservative if $(Cont.) = 0$ is satisfied, and × is not conservative.

$$v_1 \cdot (Visc.)_1 = \frac{\partial \tau_{1j} v_1}{\partial x_j} - \tau_{1j} \frac{\partial v_1}{\partial x_j} \quad (21)$$

These terms are not conservative since they involve the pressure-strain and the viscous dissipation.

We can determine the conservative properties of $v_2^2/2$ and $v_3^2/2$ in the same manner as for $v_1^2/2$.

The transport equation of kinetic energy, $K \equiv v_i v_i / 2$, is v_i times i component of Eq. (2) with summation over i .

$$\frac{\partial K}{\partial t} + v_i \cdot (Conv.)_i + v_i \cdot (Pres.)_i + v_i \cdot (Visc.)_i = 0 \quad (22)$$

In Eq. (22), the conservation property of the convective term is determined in the same manner as for $v_1^2/2$. In addition, the terms involving pressure and viscous stress in Eq. (22) can be modified into following forms.

$$v_i \cdot (Pres.)_i = \frac{\partial p v_i}{\partial x_i} - p \cdot (Cont.) \quad (23)$$

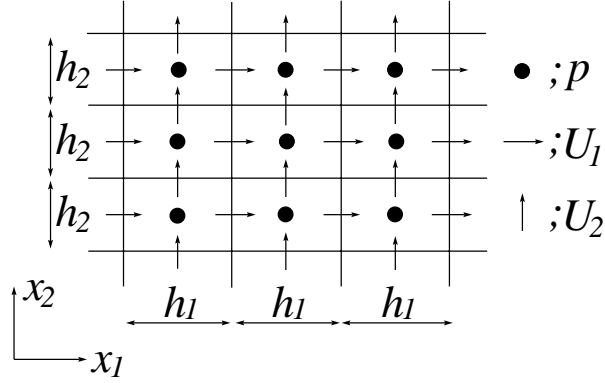
$$v_i \cdot (Visc.)_i = \frac{\partial \tau_{ij} v_i}{\partial x_j} - \tau_{ij} \frac{\partial v_i}{\partial x_j} \quad (24)$$

The pressure term in Eq. (22) is conservative if $(Cont.) = 0$ is satisfied. The viscous stress term in Eq. (22) is not conservative because the second term on the right-hand side of Eq. (24) is the energy dissipation.

Table 1 provides a summary of conservative properties of convective, pressure and viscous terms in the transport equations of v_i , $v_1^2/2$ and K for incompressible flow. The final goal of this work is to derive higher order accurate finite difference schemes that satisfy these conservative properties in a discretized sense.

2.2 Discretized operators

Before starting the main discussion, discretized operators need to be defined. In this report, the discussion of the discretized equations will be limited to uniform


 FIGURE 1. Staggered grid system in $x_1 - x_2$ plane.

grid systems. The widths of the numerical grid in each direction, h_1 , h_2 , h_3 , are constant. The grid system shown in Fig. 1 will be referred to as a *staggered* grid system. In the staggered grid system, the velocity components U_i ($i = 1, 2, 3$) are distributed around the pressure points. The continuity equation is discretized centered at pressure points. The momentum equation corresponding to each velocity component is centered at the respective velocity point.

Let the finite difference operator acting on ϕ with respect to x_1 and with stencil n be defined as follows.

$$\left. \frac{\delta_n \phi}{\delta_n x_1} \right|_{x_1, x_2, x_3} \equiv \frac{\phi(x_1 + nh_1/2, x_2, x_3) - \phi(x_1 - nh_1/2, x_2, x_3)}{nh_1} \quad (25)$$

Also, define an interpolation operator acting on ϕ in the x_1 direction with stencil n as follows.

$$\left. \overline{\phi}^{nx_1} \right|_{x_1, x_2, x_3} \equiv \frac{\phi(x_1 + nh_1/2, x_2, x_3) + \phi(x_1 - nh_1/2, x_2, x_3)}{2} \quad (26)$$

In addition, define a special interpolation operator of the product between ϕ and ψ in the x_1 direction with stencil n .

$$\begin{aligned} \left. \widetilde{\phi\psi}^{nx_1} \right|_{x_1, x_2, x_3} &\equiv \frac{1}{2} \phi(x_1 + nh_1/2, x_2, x_3) \psi(x_1 - nh_1/2, x_2, x_3) \\ &\quad + \frac{1}{2} \psi(x_1 + nh_1/2, x_2, x_3) \phi(x_1 - nh_1/2, x_2, x_3) \end{aligned} \quad (27)$$

Equations (25) and (26) are second order accurate approximations to first derivative and interpolation, respectively. Combinations of the discretized operators can be used to make higher order accurate approximations to the first derivative and interpolation. For example, fourth order accurate approximations are as follows.

$$\frac{9}{8} \frac{\delta_1 \phi}{\delta_1 x_1} - \frac{1}{8} \frac{\delta_3 \phi}{\delta_3 x_1} \simeq \frac{\partial \phi}{\partial x_1} - \frac{3}{640} \frac{\partial^5 \phi}{\partial x_1^5} h_1^4 + \dots \quad (28)$$

$$\frac{9}{8}\overline{\phi}^{-1x_1} - \frac{1}{8}\overline{\phi}^{-3x_1} \simeq \phi - \frac{3}{128}\frac{\partial^4\phi}{\partial x_1^4}h_1^4 + \dots \quad (29)$$

Discretized operators in the x_2 and x_3 directions are defined in the same way as for the x_1 direction.

We define two types of conservative forms in the discretized systems. ${}^kQ^\phi$ in Eq. (6) is (*locally conservative*) if the term can be written as

$${}^kQ^\phi = \frac{\delta_1({}^kF_j^1\phi)}{\delta_1x_j} + \frac{\delta_2({}^kF_j^2\phi)}{\delta_2x_j} + \frac{\delta_3({}^kF_j^3\phi)}{\delta_3x_j} + \dots \quad (30)$$

This definition corresponds to the analytical conservative form of Eq. (7). ${}^kQ^\phi$ is (*globally conservative*) if the following relation holds in a periodic field.

$$\sum_{x_1} \sum_{x_2} \sum_{x_3} {}^kQ^\phi \Delta V = 0 \quad (31)$$

The sum that appears in Eq. (31) is taken over the period of respective direction. $\Delta V (\equiv h_1h_2h_3)$ is a constant in a uniform grid system. The definition of global conservation corresponds to the conservation property of Eq. (8) in a periodic field. The condition for (local) conservation satisfies the condition for global conservation.

2.3 Continuity and pressure term in a staggered grid system

Now we are ready to consider our main problem. First of all, let's examine the conservative property of the pressure term. As we have observed, the pressure term should be conservative in the transport equations of momentum and kinetic energy.

In the staggered grid system, define the discretized continuity and pressure term as follows.

$$(Cont. - S2) \equiv \frac{\delta_1U_i}{\delta_1x_i} = 0 \quad (32)$$

$$(Pres. - S2)_i \equiv \frac{\delta_1p}{\delta_1x_i} \quad (33)$$

The $-S2$ denotes that the above approximations are second order accurate in space. Fourth order approximations for the continuity and pressure term in the staggered grid system are

$$(Cont. - S4) \equiv \frac{9}{8}\frac{\delta_1U_i}{\delta_1x_i} - \frac{1}{8}\frac{\delta_3U_i}{\delta_3x_i} = 0, \quad (34)$$

$$(Pres. - S4)_i \equiv \frac{9}{8}\frac{\delta_1p}{\delta_1x_i} - \frac{1}{8}\frac{\delta_3p}{\delta_3x_i}. \quad (35)$$

Local kinetic energy can not be defined uniquely in staggered grid systems since the velocity components are defined on staggered grid points. Some sort of interpolation must be used in order to obtain the three components of the kinetic energy at the same point. The required interpolations for the pressure terms in the v_1^2 and K equations are

$$\overline{U_i \frac{\delta_1p}{\delta_1x_i}}^{-1x_i} = \frac{\delta_1U_i \overline{p}^{-1x_i}}{\delta_1x_i} - p \cdot (Cont - S2), \quad (36)$$

FD Schemes for Momentum Eq.	Transport Equations		
	U_i	$U_1^2/2$	K
(Pres. - S2)	\odot	\times	\bigcirc_1
(Pres. - S4)	\odot	\times	\bigcirc_2

Table 2. Conservative properties of finite difference schemes for the pressure term in a staggered grid system. \odot is conservative *a priori*, \bigcirc_1 is globally conservative if $(Cont. - S2) = 0$ is satisfied, \bigcirc_2 is globally conservative if $(Cont. - S4) = 0$ is satisfied, and \times is not conservative.

$$\frac{9}{8} \overline{U_i \frac{\delta_1 p}{\delta_1 x_i}}^{1x_i} - \frac{1}{8} \overline{U_i \frac{\delta_3 p}{\delta_3 x_i}}^{3x_i} = \frac{9}{8} \frac{\delta_1 U_i \bar{p}^{1x_i}}{\delta_1 x_i} - \frac{1}{8} \frac{\delta_3 U_i \bar{p}^{3x_i}}{\delta_3 x_i} - p \cdot (Cont - S4). \quad (37)$$

The following relations can be used to show global conservation unambiguously.

$$\sum_{x_1} \sum_{x_2} \sum_{x_3} \overline{U_i \frac{\delta_1 p}{\delta_1 x_i}}^{1x_i} = \sum_{x_1} \sum_{x_2} \sum_{x_3} U_i \cdot (Pres. - S2)_i \quad (38)$$

$$\sum_{x_1} \sum_{x_2} \sum_{x_3} \left(\frac{9}{8} \overline{U_i \frac{\delta_1 p}{\delta_1 x_i}}^{1x_i} - \frac{1}{8} \overline{U_i \frac{\delta_3 p}{\delta_3 x_i}}^{3x_i} \right) = \sum_{x_1} \sum_{x_2} \sum_{x_3} U_i \cdot (Pres. - S4)_i \quad (39)$$

Therefore, Eqs. (33) and (35) are globally conservative if the corresponding discretized continuity equations are satisfied.

Table 2 shows the summary of the conservative property of the discretized pressure terms in a staggered grid system.

2.4 Convective schemes in a staggered grid system

As we have already mentioned, local kinetic energy K ($\equiv U_i U_i / 2$) can not be defined uniquely in a staggered grid system. Let us assume that a term is (locally) conservative in the transport equation of K if the term is (locally) conservative in the transport equations of $U_1^2/2$, $U_2^2/2$ and $U_3^2/2$. Since the conservative properties of $U_2^2/2$ and $U_3^2/2$ are estimated in the same manner as for $U_1^2/2$, only conservative properties of convective schemes in the momentum and $U_1^2/2$ equations need to be considered.

2.4.1 Proper second order accurate convective schemes

Define second order accurate convective schemes in a staggered grid system as follows.

$$(Div. - S2)_i \equiv \frac{\delta_1 \overline{U_j}^{1x_i} \overline{U_i}^{1x_j}}{\delta_1 x_j} \quad (40)$$

$$(Adv. - S2)_i \equiv \overline{\overline{U_j}^{1x_i} \frac{\delta_1 U_i}{\delta_1 x_j}}^{1x_j} \quad (41)$$

$$(Skew. - S2)_i \equiv \frac{1}{2} (Div. - S2)_i + \frac{1}{2} (Adv. - S2)_i \quad (42)$$

<i>FD Schemes for Momentum Eq.</i>	<i>Transport Equations</i>		
	U_i	$U_1^2/2$	K
$(Div. - S2)$	⊙	○	○
$(Adv. - S2)$	○	○	○
$(Skew. - S2)$	○	⊙	⊙

Table 3. Conservative properties of proper second order accurate convective schemes in a staggered grid system. ⊙ is conservative *a priori* and ○ is conservative if $(Cont. - S2) = 0$ is satisfied.

$(Adv. - S2)_i$ is connected with $(Div. - S2)_i$ through the following relation.

$$(Adv. - S2)_i = (Div. - S2)_i - U_i \cdot \overline{(Cont. - S2)}^{1x_i} \quad (43)$$

$(Div. - S2)_i$ is the standard divergence form in a staggered grid system (Harlow & Welch 1965). $(Adv. - S2)_i$ was proposed by Kajishima (1994). $(Skew. - S2)_i$ is equivalent to the scheme that was proposed by Piacsek & Williams (1970). $(Div. - S2)_i$ is conservative *a priori* in the momentum equation. The product between U_1 and $(Skew. - S2)_1$ can be rewritten as

$$U_1 \cdot (Skew. - S2)_1 = \frac{\delta_1 \overline{U_j}^{1x_1} \widetilde{U_1}^{1x_j} / 2}{\delta_1 x_j}. \quad (44)$$

Therefore, $(Skew. - S2)_1$ is conservative *a priori* in the transport equation of $U_1^2/2$. By using Eq. (43), conservative properties of the various schemes are determined. Table 3 shows the conservative properties of $(Div. - S2)_i$, $(Adv. - S2)_i$ and $(Skew. - S2)_i$. These schemes are seen to be conservative provided continuity is satisfied. In addition, the rotational form is also conservative in light of Eq. (15).

2.4.2 Proposal of proper higher order accurate convective schemes

It is of interest to derive a proper fourth order accurate convective scheme for a staggered grid system. Existing fourth order accurate convective schemes for staggered grid systems (A-Domis 1981, Kajishima 1994) do not conserve kinetic energy. Here, we propose the following set of fourth order accurate convective schemes in a staggered grid system.

$$(Div. - S4)_i \equiv \frac{9}{8} \frac{\delta_1}{\delta_1 x_j} \left[\left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \overline{U_i}^{1x_j} \right] - \frac{1}{8} \frac{\delta_3}{\delta_3 x_j} \left[\left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \overline{U_i}^{3x_j} \right] \quad (45)$$

$$(Adv. - S4)_i \equiv \frac{9}{8} \overline{\left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \frac{\delta_1 U_i}{\delta_1 x_j}}^{1x_j} - \frac{1}{8} \overline{\left(\frac{9}{8} \overline{U_j}^{1x_i} - \frac{1}{8} \overline{U_j}^{3x_i} \right) \frac{\delta_3 U_i}{\delta_3 x_j}}^{3x_j} \quad (46)$$

<i>FD Schemes for Momentum Eq.</i>	<i>Transport Equations</i>		
	U_i	$U_1^2/2$	K
<i>(Div. - S4)</i>	⊙	○	○
<i>(Adv. - S4)</i>	○	○	○
<i>(Skew. - S4)</i>	○	⊙	⊙

Table 4. Conservative properties of proper fourth order accurate convective schemes in a staggered grid system. ⊙ is conservative *a priori* and ○ is conservative if $(Cont. - S4) = 0$ is satisfied.

$$(Skew. - S4)_i \equiv \frac{1}{2}(Div. - S4)_i + \frac{1}{2}(Adv. - S4)_i \quad (47)$$

$(Div. - S4)_i$ is conservative *a priori* in the momentum equation. The product between U_1 and $(Skew. - S4)_1$ can be rewritten as follows.

$$U_1 \cdot (Skew. - S4)_1 = \frac{9}{8} \frac{\delta_1}{\delta_1 x_j} \left[\left(\frac{9}{8} \overline{U_j^{1x_1}} - \frac{1}{8} \overline{U_j^{3x_1}} \right) \frac{\widetilde{U_1 U_1}^{1x_j}}{2} \right] - \frac{1}{8} \frac{\delta_3}{\delta_3 x_j} \left[\left(\frac{9}{8} \overline{U_j^{1x_1}} - \frac{1}{8} \overline{U_j^{3x_1}} \right) \frac{\widetilde{U_1 U_1}^{3x_j}}{2} \right] \quad (48)$$

Thus, $(Skew. - S4)_i$ is conservative *a priori* in the transport equation of $U_1^2/2$. The relation between $(Adv. - S4)_i$ and $(Div. - S4)_i$ is the following.

$$(Adv. - S4)_i = (Div. - S4)_i - U_i \cdot \left[\frac{9}{8} \overline{(Cont. - S4)^{1x_i}} - \frac{1}{8} \overline{(Cont. - S4)^{3x_i}} \right] \quad (49)$$

This equation is a proper discrete analog Eq. (13), and $(Adv. - S4)_i$, $(Div. - S4)_i$, and $(Skew. - S4)_i$ are equivalent if $(Cont. - S4) = 0$ is satisfied. Using this relation, the conservative properties of the present schemes are determined. Table 4 shows the conservative properties of the present schemes. Comparing Table 4 with Table 1, we see that the present schemes are a proper set of convective schemes provided that the continuity equation is satisfied.

Proper higher order accurate finite difference schemes in a staggered grid system can be constructed in the same way as for the fourth order schemes.

2.5 Channel flow simulation

Numerical tests of the schemes described above are performed using plane channel flow. The continuity and momentum equations for incompressible viscous flow are solved using the proper second and fourth order accurate finite difference schemes in a staggered grid system using the dynamic subgrid scale model (Germano *et al.* 1991). The flow is driven by a streamwise pressure gradient. A semi-implicit time marching algorithm is used where the diffusion terms in the wall normal direction

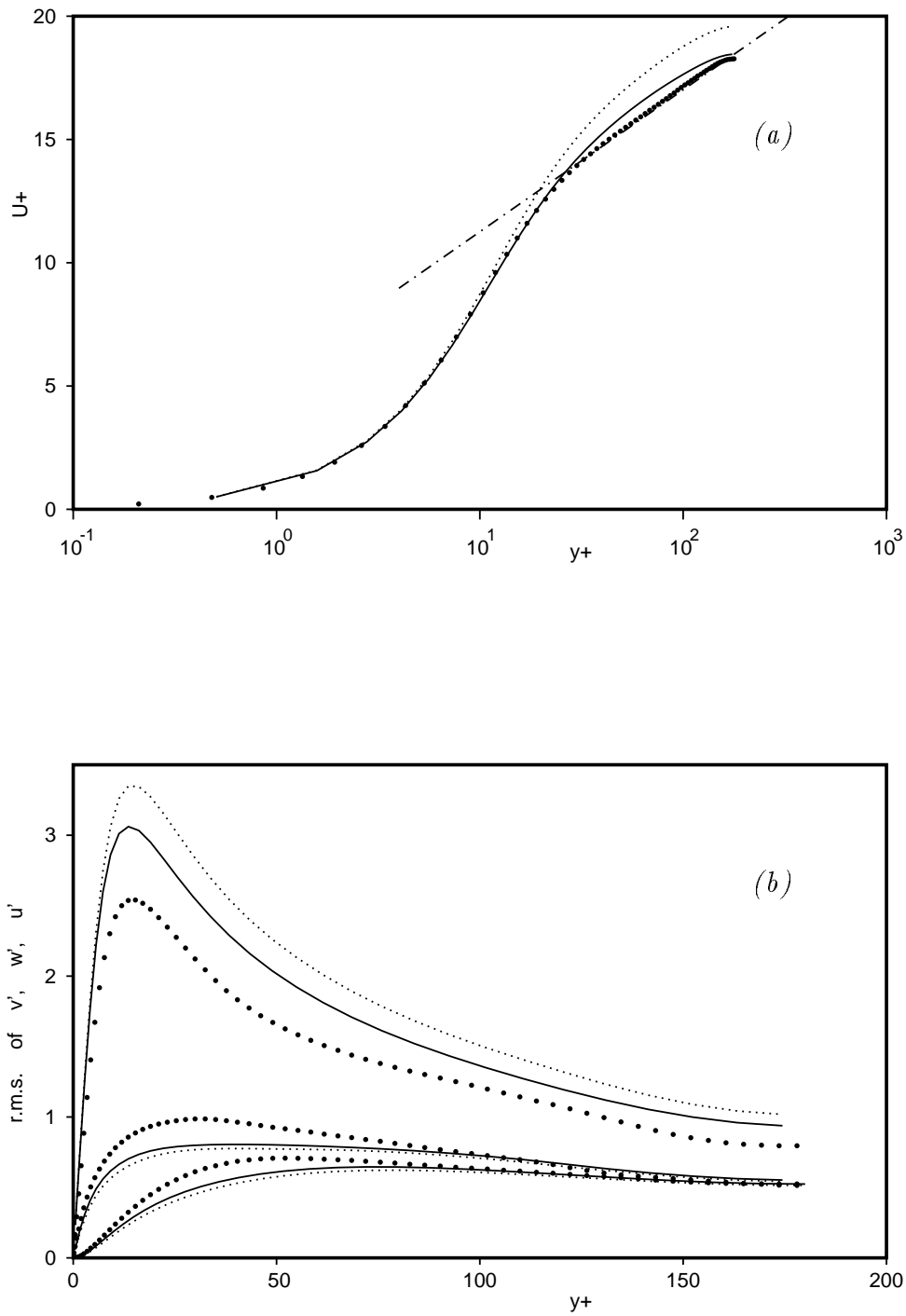


FIGURE 2. LES of plane channel flow at $Re=180$ by proper second and fourth order accurate finite difference. (a) Mean streamwise velocity; (b) Velocity fluctuations. Symbols: \cdots : 2nd order scheme; — : 4th order scheme; \bullet : DNS, Kim, *et al.* (1987); --- ; $U^+ = 5.5 = 2.5 \log y^+$.

are treated implicitly with the Crank-Nicolson scheme and a third order Runge-Kutta scheme (Wray 1986) is used for all other terms. The fractional step method (Dukowicz & Dvinsky 1992) is used in conjunction with the Van Kan (1986) type of pressure term and wall boundary treatment. Periodic boundary conditions are imposed in the streamwise and spanwise directions.

The subgrid-scale model is the dynamic model (Germano *et al.* 1991) with the least square technique (Lilly 1992). Averaging in homogeneous directions is used. Filtering is performed in the spanwise and streamwise directions.

The spatial discretization of the second order scheme is a usual one: (*Div.* – *S2*) for the convective term, (*Pres.* – *S2*) for the pressure term, and (*Cont.* – *S2*) for the continuity. The corresponding Poisson’s equation of pressure is solved using a tri-diagonal matrix algorithm in wall normal direction with fast Fourier transforms (FFT) in the periodic directions. The second order accurate control volume type discretization is used for the viscous term.

The spacial discretization of the fourth order scheme is as follows. The convective term, the pressure term, and the continuity are discretized by (*Div.* – *S4*), (*Pres.* – *S4*), and (*Cont.* – *S4*), respectively. The corresponding Poisson’s equation of pressure is solved using a septa-diagonal matrix algorithm in wall normal direction with FFT in the periodic directions. A fourth order accurate control volume type discretization is used for the viscous term. The subgrid scale terms are estimated with second order finite differences. The wall boundary condition of the fourth order scheme is designed to conserve mass and momentum in the wall normal direction in a discretized sense.

The Reynolds number based on channel half width and wall friction velocity, Re , is 180. The computational box is $4\pi \times 2 \times \frac{4}{3}\pi$, and the mesh contains $32 \times 65 \times 32$ points (streamwise, wall-normal, and spanwise respectively).

Figure 2 shows the profiles of mean streamwise velocity and velocity fluctuations from the proper second and fourth order schemes. Filtered DNS data (Kim *et al.* 1987) are plotted as a reference in the figures. The mean streamwise velocity profile from the second order scheme is shifted up in the logarithmic region. This defect of the second order scheme is usually observed in coarse LES (Cabot 1994). Another defect of the second order scheme in coarse LES is the peak value of streamwise velocity fluctuation is too high (Cabot 1994). These defects are improved by using the fourth order scheme. The computational cost of the fourth order method is about 1.9 times that for the second order method.

3. Future plans

The fourth order scheme will be tested in high Reynolds number channel flow to see if it has a greater advantage when the velocity fluctuations have a relatively larger fraction of energy near the cutoff wavenumber.

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