Modeling of inhomogeneous compressible turbulence using a two-scale statistical theory

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1. Motivation and objectives

Turbulence modeling plays an important role in the study of high-speed flows in engineering and aerodynamic problems; they include flows in supersonic combustion engines and over hypersonic transport aircraft. The enhancement of the kinetic energy dissipation by the dilatational terms is one of the typical compressibility effects. Zeman (1990) and Sarkar *et al.* (1991) proposed that the dilatation dissipation is proportional to the solenoidal dissipation and is a function of the turbulent Mach number. Sarkar (1992) also modeled the pressure-dilatation correlation using the turbulent Mach number. Zeman (1991) related the correlation to the rate of change of the pressure variance.

Using a statistical theory Yoshizawa (1990) pointed out that compressibility effects are tightly linked with density fluctuations. He proposed a three-equation model that consists of transport equations for the kinetic energy, its dissipation, and the density variance (Yoshizawa 1992). Taulbee & VanOsdol (1991) also modeled transport equations for the density variance and the mass flux. Fujiwara & Arakawa (1993) proposed another type of three-equation model involving the sum of the normalized compressible turbulent kinetic energy and the density variance.

Yoshizawa (1990) used a statistical theory called the two-scale direct-interaction approximation (TSDIA) to derive compressible turbulence models. This method was originally developed for incompressible turbulence (Yoshizawa 1984). The TSDIA consists of two main procedures. First, two-scale variables are introduced and the direct-interaction approximation (DIA) is applied to express statistical quantities in terms of two-time velocity correlations in wavenumber space. Second, by using inertial-range spectra, expressions are simplified to derive one-point closure models. However, the second procedure has not been carried out for compressible turbulence because detailed inertial-range spectra are not available. Instead, Yoshizawa (1992) applied dimensional analysis to results of the first procedure. He also proposed an alternative simplified approach that treats the governing equations in physical space (Yoshizawa 1995). Several model expressions were obtained, and an important effect of density fluctuations was clarified by these methods. Some ambiguity still remains; since several nondimensional parameters are involved in compressible turbulence, statistical quantities cannot be uniquely modeled only by dimensional analysis.

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The energy spectrum for compressible turbulence has been examined both theoretically and numerically to some extent. Moiseev *et al.* (1981) theoretically obtained a spectral form that depends on the turbulent Mach number. Kida & Orszag (1990) showed that the spectrum of the solenoidal component in their DNS is very close to that for incompressible flows whereas the spectrum of the compressible component depends strongly on the turbulent Mach number. Bataille & Bertoglio (1993) used eddy-damped quasi-normal Markovian theory to examine inertial-range spectra of weakly compressible turbulence. Although more study needs to be done to understand inertial-range behavior, these findings help us to assume some spectral forms for compressible turbulence.

In this work, we introduce inertial-range spectra of density and velocity variances to simplify results of the first procedure of TSDIA. A deviation from the Kolmogorov spectrum is assumed for the spectrum of the compressible velocity variance. The dependence on nondimensional parameters is systematically obtained by the simplification. We apply the TSDIA to several correlations included in the mean-field equations to propose a three-equation model. We examine models for the dilatation dissipation using DNS of isotropic and homogeneous shear turbulence.

2. Accomplishments

2.1 Fundamental equations and $K-\varepsilon-K_\rho$ model

The motion of a viscous compressible fluid is described by the equations for the density ρ, the velocity u_i, and the internal energy e:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_j u_i)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(\mu s_{ji}), \tag{2}$$

$$\frac{\partial}{\partial t}(\rho e)+\frac{\partial}{\partial x_i}(\rho e u_i)=-p\frac{\partial u_i}{\partial x_i}+\mu s_{ji}\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_i}\left(\lambda\frac{\partial \theta}{\partial x_i}\right), \tag{3}$$

where μ is the viscosity, λ is the thermal conductivity, and θ is the temperature. The deviatoric part of the strain rate tensor, s_{ij}, is given by

$$s_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}. \tag{4}$$

For perfect gas, the pressure p and the internal energy e are written as

$$p=\rho R\theta=(\gamma-1)\rho e, \qquad e=c_v\theta, \tag{5}$$

where $\gamma=c_p/c_v$. Here, R is the specific gas constant, and c_v and c_p are the specific heats at constant volume and pressure, respectively.

We divide a physical quantity f into the mean F and the fluctuation f':

$$f=F+f', \quad F=\langle f\rangle, \tag{6}$$

where f denotes ρ, u_i, e, p, s_{ij}, and θ. Some mean quantities are denoted by an overbar as $\bar{\rho}$. By taking the ensemble average of (1)–(3), we obtain the equations for the mean quantities $\bar{\rho}$, U_i, and E. Those equations contain several correlations such as the mass flux $\langle \rho' u_i' \rangle$ and the Reynolds stress $\langle u_i' u_j' \rangle$. The correlations need to be modeled to close the mean-field equations.

Yoshizawa (1990) pointed out that compressibility effects are tightly linked with the density fluctuations; he proposed a three-equation model that consists of the equations for the turbulent kinetic energy $K(= \langle u_i'^2 \rangle /2)$, its dissipation rate ε, and the density variance $K_\rho (= \langle \rho'^2 \rangle)$. The equations for K and K_ρ can be written as

$$\begin{aligned}\frac{DK}{Dt} = &-\langle u_i' u_j' \rangle \frac{\partial U_i}{\partial x_j} - \varepsilon + \frac{1}{\bar{\rho}} \left\langle p' \frac{\partial u_i'}{\partial x_i} \right\rangle - \frac{1}{2} \frac{\partial}{\partial x_j} \langle u_i'^2 u_j' \rangle - \frac{1}{\bar{\rho}} \frac{\partial}{\partial x_i} \langle p' u_i' \rangle \\ &+ \frac{1}{\bar{\rho}^2} \langle \rho' u_i' \rangle \frac{\partial P}{\partial x_i},\end{aligned} \tag{7}$$

$$\frac{DK_\rho}{Dt} = -2K_\rho \frac{\partial U_i}{\partial x_i} - 2\langle \rho' u_i' \rangle \frac{\partial \bar{\rho}}{\partial x_i} - 2\bar{\rho} \left\langle \rho' \frac{\partial u_i'}{\partial x_i} \right\rangle - \frac{\partial}{\partial x_i} \langle \rho'^2 u_i' \rangle - \left\langle \rho'^2 \frac{\partial u_i'}{\partial x_i} \right\rangle. \tag{8}$$

The correlations included in (7) and (8) as well as the ε equation itself need to be modeled in terms of the mean quantities and the three variables.

Model expressions shown later contain two nondimensional parameters: the turbulent Mach number M_t [$= \sqrt{2K}/\bar{c}$ where $\bar{c}$ is the mean sound speed] and the normalized density variance ρ_n^2 [$= (K_\rho / \bar{\rho}^2)$]. By adopting K_ρ as one of the basic quantities, we can use ρ_n^2 as a parameter independent of M_t. Modeling with the two parameters is expected to be more flexible than that with M_t only.

2.2 Two-scale statistical theory

Here, we give a brief summary of the procedure of the TSDIA. Its mathematical details were given in Yoshizawa (1992).

We first introduce two time and space variables using a small-scale parameter δ as

$$\boldsymbol{\xi}(\equiv \boldsymbol{x}), \quad \boldsymbol{X}(\equiv \delta \boldsymbol{x}), \quad \tau(\equiv t), \quad T(\equiv \delta t). \tag{9}$$

Here, the fast variables $\boldsymbol{\xi}$ and τ describe the rapid variations of the fluctuating field whereas the slow variables $\boldsymbol{X}$ and T describe the slow variations of the mean field. A quantity f can be written as

$$f = F(\boldsymbol{X}, T) + f'(\boldsymbol{\xi}, \boldsymbol{X}, \tau, T). \tag{10}$$

Using the Fourier transform with respect to $\boldsymbol{\xi}$, we express f' as

$$f'(\boldsymbol{\xi}, \boldsymbol{X}, \tau, T) = \int d\boldsymbol{k} f(\boldsymbol{k}, \boldsymbol{X}, \tau, T) \exp[-i\boldsymbol{k} \cdot (\boldsymbol{\xi} - \boldsymbol{U}\tau)]. \tag{11}$$

This representation is equivalent to the viewpoint that the fluctuating motion consists of many small eddies moving with the mean velocity $\boldsymbol{U}$. Hereafter, the dependence of $f(\boldsymbol{k}, \boldsymbol{X}, \tau, T)$ on $\boldsymbol{X}$ and T is not written explicitly.

Applying (9)–(11) to the equations for ρ', u_i', and p' (or e'), we obtain a system of equations for the fluctuating field in wavenumber space. We expand the fluctuation $f(\boldsymbol{k}, \tau)$ in powers of δ:

$$f(\boldsymbol{k}, \tau) = \sum_{n=0}^{\infty} \delta^n f_n(\boldsymbol{k}, \tau). \tag{12}$$

Substituting (12) into the system of equations and equating quantities in each order of δ, we have an equation for each quantity $f_n(\boldsymbol{k}, \tau)$. By introducing the Green's functions for ρ_0, u_{0i}, and p_0 we can formally solve the equations for f_n $(n \geq 1)$ in terms of the lower-order quantities.

A correlation included in the mean-field equations can be written as

$$\begin{aligned} \langle f'(\boldsymbol{x}, t) g'(\boldsymbol{x}, t) \rangle &= \int d\boldsymbol{k} \langle f(\boldsymbol{k}, \tau) g(-\boldsymbol{k}, \tau) \rangle / \delta(0) \\ &= \int d\boldsymbol{k} \left(\langle f_0 g_0 \rangle + \langle f_1 g_0 \rangle + \langle f_0 g_1 \rangle + \cdots \right) / \delta(0). \end{aligned} \tag{13}$$

Here, $\delta(0)$ denotes the delta function $\delta(k)$ where the one-dimensional wavenumber k equals 0. Substituting the formal solution for f_n and g_n $(n \geq 1)$ and applying the DIA, we obtain a model expression for the correlation. It is written in terms of the mean field as well as the basic correlations and the Green's functions defined by

$$Q_\rho(\boldsymbol{k}, \tau, \tau') = \langle \rho_0(\boldsymbol{k}, \tau) \rho_0(-\boldsymbol{k}, \tau') \rangle / \delta(0) = Q_\rho(k, \tau, \tau'), \tag{14}$$

$$\begin{aligned} Q_{ij}(\boldsymbol{k}, \tau, \tau') &= \langle u_{0i}(\boldsymbol{k}, \tau) u_{0j}(-\boldsymbol{k}, \tau') \rangle / \delta(0) \\ &= D_{ij}(\boldsymbol{k}) Q_s(k, \tau, \tau') + \Pi_{ij}(\boldsymbol{k}) Q_c(k, \tau, \tau'), \end{aligned} \tag{15}$$

$$G_\rho(\boldsymbol{k}, \tau, \tau') = \langle \hat{G}_\rho(\boldsymbol{k}, \tau, \tau') \rangle = G_\rho(k, \tau, \tau'), \tag{16}$$

$$G_{ij}(\boldsymbol{k}, \tau, \tau') = \langle \hat{G}_{ij}(\boldsymbol{k}, \tau, \tau') \rangle = D_{ij}(\boldsymbol{k}) G_s(k, \tau, \tau') + \Pi_{ij}(\boldsymbol{k}) G_c(k, \tau, \tau'), \tag{17}$$

$$G_e(\boldsymbol{k}, \tau, \tau') = \langle \hat{G}_e(\boldsymbol{k}, \tau, \tau') \rangle = G_e(k, \tau, \tau'), \tag{18}$$

where

$$D_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \qquad \Pi_{ij}(\boldsymbol{k}) = \frac{k_i k_j}{k^2}. \tag{19}$$

For example, the expression for the eddy viscosity can be written as

$$\nu_e \sim \int d\boldsymbol{k} \int^\tau d\tau' G_s(k, \tau, \tau') Q_s(k, \tau, \tau') + \cdots, \tag{20}$$

The expression includes wavenumber and time integrals of two-time correlations and Green's functions. It is too complicated to be a practical model; some simplifications are necessary.

Following the TSDIA for incompressible turbulence, we assume inertial-range forms for the fundamental statistical quantities as

$$Q_a(k,\tau,\tau') = \sigma_a(k)\exp[-\omega_a(k)|\tau-\tau'|], \quad a=(\rho,s,c), \tag{21}$$

$$G_b(k,\tau,\tau') = H(\tau-\tau')\exp[-\omega'_b(k)(\tau-\tau')], \quad b=(\rho,s,c,e), \tag{22}$$

where

$$\sigma_\rho(k) = C_{\sigma\rho}M_t^2\bar{\rho}^2\varepsilon_d\varepsilon^{-1}k^{-3-\alpha-2\beta}k_m^{\alpha+2\beta}H(k-k_m), \tag{23}$$

$$\sigma_s(k) = C_{\sigma s}\varepsilon^{2/3}k^{-11/3}H(k-k_m), \tag{24}$$

$$\sigma_c(k) = C_{\sigma c}\varepsilon_d\varepsilon^{-1/3}k^{-(11/3)-\alpha}k_m^{\alpha}H(k-k_m), \tag{25}$$

$$[\omega_s(k),\omega'_s(k)] = [C_{\omega s},C'_{\omega s}]\varepsilon^{1/3}k^{2/3}, \tag{26}$$

$$\begin{aligned}[\omega_\rho(k),\omega'_\rho(k),&\omega_c(k),\omega'_c(k),\omega'_e(k)]\\ &= [C_{\omega\rho},C'_{\omega\rho},C_{\omega c},C'_{\omega c},C'_{\omega e}]M_t^{-1}\varepsilon^{1/3}k^{(2/3)+\beta}k_m^{-\beta}.\end{aligned} \tag{27}$$

Here, $C_{\sigma a}$, $C_{\omega a}$, and $C'_{\omega b}$ are model constants, $H(k)$ and $H(\tau)$ are the unit step functions, k_m is the wavenumber of the energy-containing range, and ε, ε_d, and M_t are the dissipation, the dilatation dissipation, and the turbulent Mach number defined by

$$\varepsilon = \bar{\nu}\left\langle s'_{ji}\frac{\partial u'_i}{\partial x_j}\right\rangle, \qquad \varepsilon_d = \frac{4}{3}\bar{\nu}\left\langle\left(\frac{\partial u'_i}{\partial x_i}\right)^2\right\rangle, \qquad M_t = \frac{\sqrt{2K}}{\bar{c}} = \left(\frac{2\bar{\rho}K}{\gamma P}\right)^{1/2}, \tag{28}$$

respectively. For the solenoidal quantities σ_s, ω_s, and ω'_s, the spectra are the same as those for incompressible turbulence. The compressible part of energy spectrum, σ_c, is set proportional to ε_d. This is because the ratio of the compressible to solenoidal parts of turbulent kinetic energy is shown to be proportional to the ratio of the dilatational to solenoidal dissipations. The spectrum is steeper than the Kolmogorov one by α. Moiseev *et al.* (1981) showed that the deviation α is a function of M_t. Here, we do not include such M_t dependence, but consider α as an unknown numerical parameter. The deviation from the incompressible inertial-range form is also introduced into $\omega(k)$ for compressible quantities. We assume that time scales for compressible quantities are shorter than those for incompressible ones; the ratio is of the order of M_t.

For example, substituting the above spectral forms into (20), we obtain a one-point closure model for the eddy viscosity as a function of k_m and ε. By converting k_m into K and ε, we have a usual expression proportional to K^2/ε.

2.3 Dilatation dissipation

We applied the procedure of the previous section to $\langle \rho' \rho' \rangle$ to obtain an expression for the density variance; it is a function of the mean field $\bar{\rho}$, U_i, and P as well as the quantities K, ε, ε_d, and M_t. Since the transport equation for K_ρ is solved in the $K - \varepsilon - K_\rho$ model, the modeling of K_ρ itself is not necessary. Instead, the expression can be considered a model for ε_d. Expanding ε_d in terms of the other quantities we have

$$\varepsilon_d = C_{\varepsilon d1} \frac{\rho_n^2}{M_t^2} \varepsilon \left[1 + C_{\varepsilon d7} M_t \left(2 \frac{K}{\varepsilon} \frac{\partial U_i}{\partial x_i} + \frac{1}{4} \frac{K}{\varepsilon \bar{\rho}} \frac{D\bar{\rho}}{Dt} - \frac{1}{4} \frac{K}{\varepsilon P} \frac{DP}{Dt} + \frac{3}{2\varepsilon} \frac{DK}{Dt} - \frac{K}{\varepsilon^2} \frac{D\varepsilon}{Dt} + \frac{K}{\varepsilon K_\rho} \frac{DK_\rho}{Dt} \right) \right], \tag{29}$$

where ρ_n^2 is the normalized density variance defined by

$$\rho_n^2 = \frac{K_\rho}{\bar{\rho}^2}, \tag{30}$$

and $C_{\varepsilon d1}$ and $C_{\varepsilon d7}$ are model constants. Hereafter, C_{an} denotes a model constant where a represents a physical quantity and n is the number of the term.

The factor before the square bracket in (29) shows that the ratio $\varepsilon_d/\varepsilon$ is proportional to ρ_n^2/M_t^2. Yoshizawa (1992) pointed out that this quantity is important in characterizing the compressibility effect and introduced a parameter $\chi (= \rho_n^2/M_t^2)$. Yoshizawa (1995) paid attention to the importance of the parameter χ and proposed the model:

$$\varepsilon_d/\varepsilon_s = C_{\varepsilon dY} \chi, \tag{31}$$

where $\varepsilon_s = \varepsilon - \varepsilon_d$ and $C_{\varepsilon dY}$ is a model constant. This model is the same as (29) to first order.

The modeling of ε_d was originally investigated by Sarkar *et al.* (1991) and Zeman (1990). Sarkar *et al.* (1991) used asymptotic analysis and DNS to model ε_d as follows

$$\varepsilon_d/\varepsilon_s = C_{\varepsilon dS} M_t^2. \tag{32}$$

Zeman (1990) assumed the existence of shock-like structure in flow fields to derive the model

$$\varepsilon_d/\varepsilon_s = C_{\varepsilon dZ} F(M_t, K_{Mt}), \tag{33}$$

where K_{Mt} is the flatness factor of M_t and $F(M_t, K_{Mt})$ is a complicated integral. He also derived a simple algebraic expression for use in practice (Blaisdell & Zeman 1992).

Blaisdell *et al.* (1991) used DNS of decaying isotropic turbulence to examine the above two models. They carried out two simulations that had the same initial values of M_t but different initial ratios of compressible to solenoidal velocity variances. In spite of the same turbulent Mach number, the two simulations showed different values of $\varepsilon_d/\varepsilon$. They concluded that the development of $\varepsilon_d/\varepsilon$ in isotropic turbulence depends more on its initial values than on the turbulent Mach number and that simulations of isotropic turbulence cannot be used to validate the proposed models. However, Yoshizawa's model as well as the present model show that $\varepsilon_d/\varepsilon$ depends not only on M_t but also on ρ_n^2. As was pointed out by Yoshizawa (1995), the difference in $\varepsilon_d/\varepsilon$ in the two simulations can be attributed to the difference in ρ_n^2. The assumption that $\varepsilon_d/\varepsilon$ depends only on M_t seems too restrictive to capture the behavior of decaying isotropic turbulence. In the $K-\varepsilon-K_\rho$ model, we use the two parameters M_t and ρ_n; the development of ρ_n^2 is obtained from the transport equation for K_ρ.

2.4 Mass flux

Since ensemble averaging is used in this work, the mean-velocity equation contains the mass flux; its modeling is necessary. Taulbee & VanOsdol (1991) examined the transport equation for the mass fluctuating velocity $\langle\rho' u_i'\rangle/\bar{\rho}$ and modeled terms included in the equation. Instead of the transport equation we model the mass flux itself. It can be modeled as

$$\langle\rho' u_i'\rangle = -C_{\rho u1} M_t \frac{K^2}{\varepsilon}\frac{\partial\bar{\rho}}{\partial x_i}\left[1 - 2\frac{\rho_n^2}{M_t^2} + C_{\rho u3}\left(\frac{K}{\varepsilon}\frac{\partial U_i}{\partial x_i} + \frac{3}{\varepsilon}\frac{DK}{Dt} - \frac{5}{4}\frac{K}{\varepsilon^2}\frac{D\varepsilon}{Dt}\right)\right]$$
$$-\frac{10+15\alpha}{10+6\alpha}C_{\rho u1}\frac{\rho_n^2}{M_t}\frac{K^2}{\varepsilon}\left[-\frac{17}{8}\frac{\partial\bar{\rho}}{\partial x_i} + \left(\frac{17}{8} - \frac{5}{2\gamma}\right)\frac{\bar{\rho}}{P}\frac{\partial P}{\partial x_i} + \frac{9}{4}\frac{\bar{\rho}}{K}\frac{\partial K}{\partial x_i}\right.$$
$$\left. -\frac{3}{2}\frac{\bar{\rho}}{\varepsilon}\frac{\partial\varepsilon}{\partial x_i} + \frac{3}{2}\frac{\bar{\rho}}{K_\rho}\frac{\partial K_\rho}{\partial x_i}\right]. \tag{34}$$

The term with the first square bracket depends on the gradient of mean density; it corresponds to the gradient-diffusion approximation. The eddy diffusivity is proportional to $M_t K^2/\varepsilon$. It is smaller than the eddy diffusivity in incompressible flows by a factor of M_t. The eddy diffusivity for the mass flux includes nonequilibrium effects due to DK/Dt and $D\varepsilon/Dt$ as well as compressibility effects due to ρ_n^2/M_t^2 and $\partial U_i/\partial x_i$.

On the other hand, the term with the second square bracket also depends on the gradients of mean quantities other than $\bar{\rho}$; this effect is called cross diffusion. For example, when the gradients of $\bar{\rho}$ and P are small and the isentropic relations hold, the profile of P is proportional to that of $\bar{\rho}$; the pressure gradient term simply represents the modification of the eddy diffusivity. However, when the temperature changes rapidly due to heat release, the profiles of density and pressure may be different; in such a case the cross diffusion effect due to the pressure gradient can be important in the mass flux model.

Using the simplified approach Yoshizawa (1995) derived a model for the mass flux as follows

$$\langle \rho' u_i' \rangle = -\left[1 + \frac{3(\gamma-1)}{\gamma}\frac{\sigma_\rho^2}{\sigma_e}\chi\right]\frac{\nu_T}{\sigma_\rho}\frac{\partial \bar{\rho}}{\partial x_i} - \frac{3}{\gamma}\chi\nu_T\bar{\rho}\frac{1}{E}\frac{\partial E}{\partial x_i} - \frac{3}{2}\frac{K_\rho}{\bar{\rho}}\frac{\nu_T}{K}\frac{DU_i}{Dt}, \qquad (35)$$

where $\nu_T = (2/3)C_u(K^2/\varepsilon)$ and σ_ρ, σ_e, and C_u are model constants. If we assume that $P = (\gamma - 1)\bar{\rho}E$ and $DU_i/Dt = -(1/\bar{\rho})\partial P/\partial x_i$, we can see that the second and third terms on the right-hand side correspond to the cross-diffusion term due to the mean pressure in (34). The major difference between (34) and (35) lies in the dependence of the eddy-diffusivity on M_t; the diffusivity of the former is of $O(M_t)$ whereas that of the latter is of $O(1)$. This difference stems from the different dependence of the time scale for density fluctuations on M_t.

2.5 Reynolds stress

Yoshizawa (1995) pointed out that compressibility effects are not incorporated into the Reynolds stress up to the order of δ; this order corresponds to the eddy-viscosity approximation. We calculated the Reynolds stress up to the order of δ^2 to obtain

$$\begin{aligned}\langle u_i' u_j' \rangle &= \frac{2}{3}K\delta_{ij} \\ &-C_{uu1}\frac{K^2}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)^*\left[1 - 2\frac{\rho_n^2}{M_t^2} + C_{uu4}\left(\frac{23}{49}\frac{K}{\varepsilon}\frac{\partial U_i}{\partial x_i} + \frac{1}{\varepsilon}\frac{DK}{Dt} - \frac{5}{12}\frac{K}{\varepsilon^2}\frac{D\varepsilon}{Dt}\right)\right] \\ &+C_{uuA}\frac{K^3}{\varepsilon^2}\left[\frac{62}{105}\left(\frac{\partial U_i}{\partial x_k}\frac{\partial U_j}{\partial x_k}\right)^* + \frac{2}{35}\left(\frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j}\right)^* + \frac{34}{105}\left(\frac{\partial U_i}{\partial x_k}\frac{\partial U_k}{\partial x_j} + \frac{\partial U_j}{\partial x_k}\frac{\partial U_k}{\partial x_i}\right)^*\right. \\ &\left.+\frac{7}{15}\frac{D}{Dt}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)^*\right] + C_{uu10}M_t\frac{K^3}{\varepsilon^2}\left[\frac{\partial}{\partial x_i}\left(\frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_i}\right)\right]^*, \qquad (36)\end{aligned}$$

where

$$(f_{ij})^* \equiv f_{ij} - \frac{1}{3}f_{kk}\delta_{ij}. \qquad (37)$$

Except for the isotropic part, $(2/3)K\delta_{ij}$, the expression consists of three parts. The first part represents the modification of the eddy viscosity due to compressibility and nonequilibrium effects. The second part corresponds to nonlinear models that have already been investigated for incompressible flows (Speziale 1987). The third part represents the compressibility effect due to a mean pressure gradient.

The modification of the eddy viscosity due to DK/Dt and $D\varepsilon/Dt$ has already been proposed for incompressible flows (Yoshizawa & Nisizima 1993). Yoshizawa (1995) also mentioned its importance for compressible flows. Expression (36) suggests that we should take into account not only the nonequilibrium effect but also the compressibility effects due to the density variance and mean-velocity divergence.

Sarkar (1995) showed that the reduced growth rate of turbulence energy in homogeneous shear flows is primarily due to the decrease in turbulence production. Since the production term includes the Reynolds stress, compressibility effects on the Reynolds stress need to be modeled appropriately. In the present model the direct effect of compressibility on the eddy viscosity is expressed by ρ_n^2/M_t^2 in (36) because the mean-velocity divergence vanishes for homogeneous shear flows. For inhomogeneous turbulence the mean-velocity divergence can play an important role when the flow speed rapidly changes in the streamwise direction as in a shock wave. If the flow speed decreases and the divergence is negative, the eddy viscosity becomes smaller than the usual estimate, K^2/ε.

Although the third part is smaller than the second part by a factor of M_t, its expression is interesting in the sense that it does not include the mean velocity. Each term in the square bracket can be divided into the two terms: $(1/\bar{\rho})\partial^2 P/\partial x_i^2$ and $-(1/\bar{\rho}^2)(\partial\bar{\rho}/\partial x_i)(\partial P/\partial x_i)$. A term similar to the latter can be seen in the K equation (7). The importance of this term in the K equation was discussed by Yoshizawa (1995). Similarly the transport equation for the Reynolds stress contains such a term. Therefore, the gradients of mean density and pressure can affect the Reynolds stress.

2.6 Pressure-dilatation correlation

The pressure-dilatation correlation has been investigated as a typical compressibility effect. In this work we obtained a model expression as

$$\begin{aligned}\left\langle p'\frac{\partial u_i'}{\partial x_i}\right\rangle &= C_{pd1}\frac{\rho_n^2}{M_t}\gamma P\frac{\varepsilon}{K} + C_{pd2}\frac{\rho_n^4}{M_t^3}\gamma P\frac{\varepsilon}{K} \\ &- C_{pd3}\frac{\rho_n^2}{M_t}\left(\gamma P\frac{\partial U_i}{\partial x_i} + 3\frac{\gamma P}{K}\frac{DK}{Dt} - \frac{5}{4}\frac{\gamma P}{\varepsilon}\frac{D\varepsilon}{Dt}\right) + 2\rho_n^2\frac{\gamma P}{\bar{\rho}}\frac{D\bar{\rho}}{Dt} \\ &- \rho_n^2\gamma\frac{DP}{Dt} + C_{pd8}\frac{\rho_n^2}{M_t}\frac{K^2}{\varepsilon}\gamma\bar{\rho}^2\frac{\partial}{\partial x_i}\left(\frac{P}{\bar{\rho}^3}\frac{\partial\bar{\rho}}{\partial x_i} - \frac{\gamma-1}{\gamma}\frac{1}{\bar{\rho}^2}\frac{\partial P}{\partial x_i}\right).\end{aligned} \tag{38}$$

By assuming some relations for basic model constants such as $C_{\omega\rho}$ and $C_{\omega c}$, we found that the constant C_{pd1} vanishes. If the assumption does not hold exactly, the constant can have a small nonzero value.

Using the simplified approach Yoshizawa (1995) proposed a model as

$$\left\langle p'\frac{\partial u_i'}{\partial x_i}\right\rangle = -C_{pdY1}\bar{\rho}\varepsilon\chi + C_{pdY2}\bar{\rho}K\chi\frac{\partial U_i}{\partial x_i} + C_{pdY3}\bar{\rho}K\chi\frac{1}{E}\frac{DE}{Dt}. \tag{39}$$

The third term on the right-hand side corresponds to the two terms that include $D\bar{\rho}/Dt$ and DP/Dt in the present model. Each term in (39) is proportional to χ whereas terms in (38) show a different dependence on ρ_n and M_t. Using the first and third terms in his model, Yoshizawa (1995) explained the property of the pressure-dilatation correlation whose value is positive for decaying isotropic turbulence and negative for homogeneous shear turbulence. The present model contains terms with

Dk/Dt and $D\varepsilon/Dt$. The terms can also explain the different sign of the correlation because of the difference in the development of energy in the two flows.

Sarkar (1992) modeled the pressure dilatation in the form of a power series in M_t as follows

$$\left\langle p' \frac{\partial u_i'}{\partial x_i} \right\rangle = C_{pdS1} M_t \bar{\rho} \left(\langle u_i' u_j' \rangle - \frac{2}{3} K \delta_{ij} \right) \frac{\partial U_i}{\partial x_j} + C_{pdS2} M_t^2 \bar{\rho} \varepsilon_s + C_{pdS3} M_t^2 \bar{\rho} K \frac{\partial U_i}{\partial x_i}. \tag{40}$$

This model is different from the above two models in that it does not contain the density variance. The first term on the right-hand side has a similar factor to the production term in the K equation. Yoshizawa (1995) illustrated that such a term can overestimate the pressure-dilatation correlation in a turbulent channel flow in which the shear is strong but the correlation is very small. On the other hand, the present and Yoshizawa's models contain the density variance; it is expected to explain the small value of the correlation.

2.7 Comparison to DNS data

Blaisdell *et al.* (1991) performed DNS of decaying isotropic and homogeneous shear turbulence. Using the DNS data we compare models for the dilatation dissipation. Although the TSDIA assumes inertial-range spectra, the simulations are at low Reynolds numbers and do not show an inertial range. The DNS results must include some low Reynolds number effects. The values of model constants in this paper may change for higher Reynolds number flows. Nonetheless, we believe that by comparing the models to the DNS we can better understand compressible turbulence.

We examined four simulations of isotropic turbulence and nine simulations of homogeneous shear flow. Here, we will show results of three simulations; their initial conditions are given in Table I. The parameter χ_c in Table I denotes the ratio of the compressible to total velocity variance $\langle u_{ci}' u_{ci}' \rangle / \langle u_j' u_j' \rangle$.

Case	Flow	M_t	ρ_n	χ_c
idc128	isotropic	0.3	0	0
ie128	isotropic	0.3	0.15	0.25
sha192	shear	0.4	0	0

Table 1. Initial conditions for DNS of isotropic and homogeneous shear turbulence by Blaisdell *et al.* (1991).

Figures 1 and 2 show the time history of the ratio $\varepsilon_d/\varepsilon$ for cases idc128 and ie128. The initial values of M_t are the same for the two cases whereas those of ρ_n and χ_c are different. The solid lines denote the DNS results, the dashed lines denote the values predicted by Sarkar's model (32), and the dotted lines denote those by the present model (29). The model constant in Sarkar's model is given by $C_{edS} = 1$.

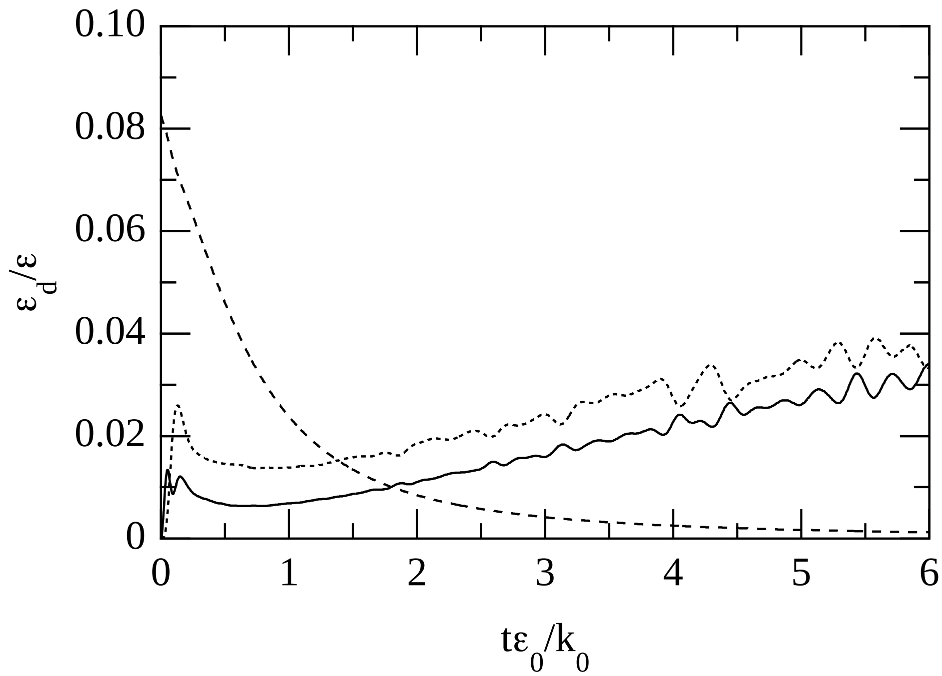

FIGURE 1. Time history of the ratio of dilatation dissipation to total dissipation for case idc128: ——— , DNS; - - - - , Sarkar's model; ········ , present model.

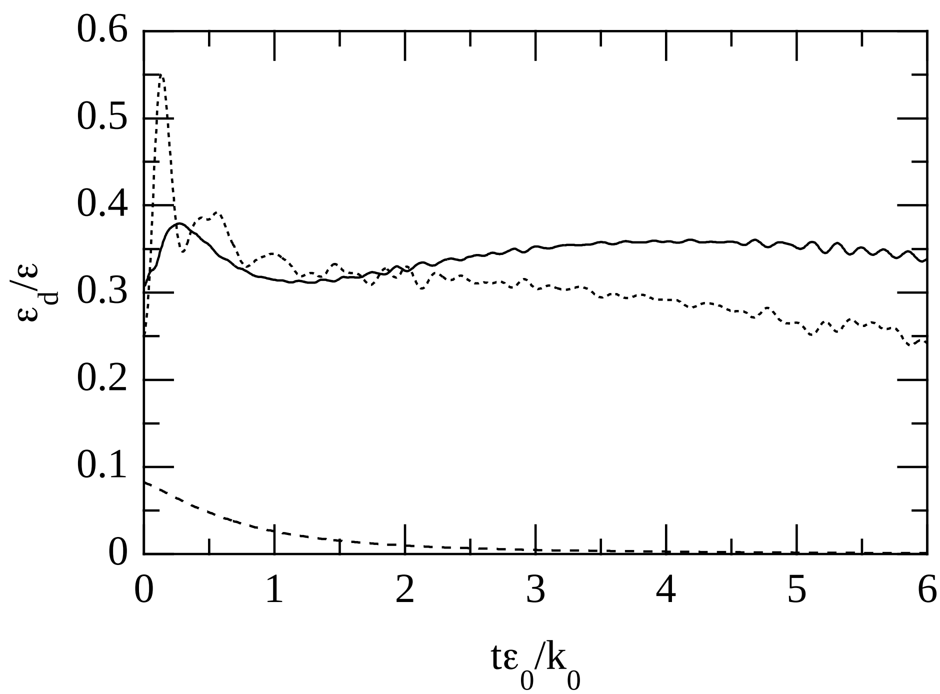

FIGURE 2. Time history of the ratio of dilatation dissipation to total dissipation for case ie128: ——— , DNS; - - - - , Sarkar's model; ········ , present model.

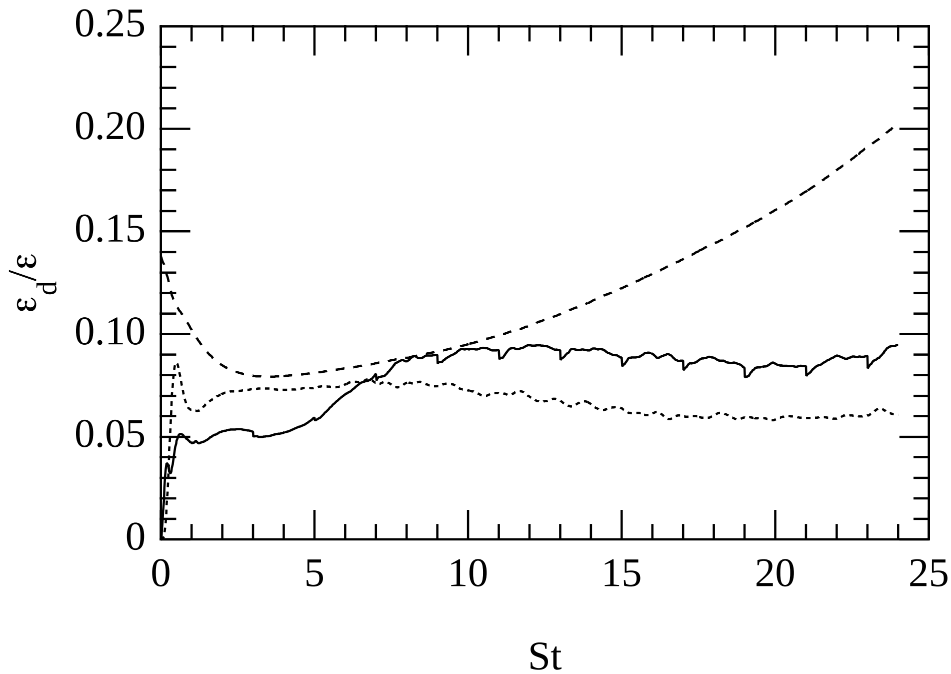

FIGURE 3. Time history of the ratio of dilatation dissipation to total dissipation for case sha192: ——— , DNS; ---- , Sarkar's model; ········ , present model.

On the other hand, values of constants in the present model have not been obtained yet because the values of the basic constants, such as $C_{\sigma c}$ and α, are not known. Here, to examine overall agreement with DNS data, the model constants are set at $C_{ed1} = 1$ and $C_{ed7} = 0$ in (29). In Figs. 1 and 2 the DNS results for the two cases are very different; the value of $\varepsilon_d/\varepsilon$ for ie128 in Fig. 2 is much greater than that for idc128 in Fig. 1. Since Sarkar's model contains only M_t, the predicted values for the two cases are almost the same; they decrease in time monotonically. On the other hand, the present model contains M_t and ρ_n^2; it predicts different values of $\varepsilon_d/\varepsilon$ for the two cases. The value for idc128 increases in time like the DNS result. The model explains the effect of the initial condition in terms of the density variance. Similar results were obtained for the other two simulations using a higher turbulent Mach number, $M_t = 0.7$ (not shown). Fujiwara (1996) also illustrated the initial condition effects solving the $K - \varepsilon - F$ model where F is the sum of the nondimensional density variance and compressible kinetic energy.

Contrary to isotropic turbulence the effect of initial conditions were shown to disappear for homogeneous shear turbulence. Time histories of $\varepsilon_d/\varepsilon$ for simulations with different initial conditions tend to overlap as time increases. Here, we show results of a case denoted sha192; in this case the largest number of grid points was used and its results are considered the most reliable. Figure 3 shows the time history of $\varepsilon_d/\varepsilon$ for case sha192. The difference between the present and Sarkar's models is smaller than that for isotropic turbulence. However, the DNS result shows almost a constant value after $St = 10$ whereas Sarkar's model predicts a continually increasing value after $St = 3$. The present model shows the same tendency as the DNS although the value is smaller. Other simulations with $M_t = 0.5$ extend to $St = 15$ and show qualitatively similar profiles as in Fig. 3. Therefore, the parameter ρ_n is concluded to be important for modeling the dilatation dissipation.

3. Future plans

Model expressions obtained in this work need to be examined further by comparing to DNS of homogeneous and inhomogeneous turbulence. Since the TSDIA is a method based on derivative expansions, expressions contain several terms including higher-order terms. Some terms should be selected so that model expressions are simple but contain essential compressibility effects. Model constants also should be estimated by DNS.

We assumed inertial-range spectra of the density and velocity variances. The spectral forms are not as established as those for incompressible flows. If details of inertial-range spectra are obtained in other theories or experiments, we can include them into this analysis. The relationship to incompressible models in the limit of zero Mach number also needs to be considered to improve the models.

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