On the use of discrete filters for large eddy simulation

By T. S. Lund[1]

1. Motivation and objectives

The equations for large eddy simulation (LES) are derived formally by applying a low pass-filter to the Navier-Stokes equations. This filtering must be repeated at each time step in the solution procedure since the non-linear terms continually generate frequencies higher than the assumed cutoff. In spite of this requirement, an explicit filtering operation has rarely been performed in practice. There are a few good reasons for this discrepancy, and perhaps the most compelling of these is the prior lack of filter operators that commute with differentiation. Without commuting operators, the act of filtering alters the Navier-Stokes equations through the addition of 'commutation error terms' (see Ghosal and Moin, 1995). Fortunately the commutation issue has recently been resolved by Vasilyev and Lund (this volume) who constructed filters that commute with differentiation to any specified order of accuracy for arbitrary boundary conditions.

The use of explicit filters opens the possibility to improve the fidelity and consistency of the LES procedure. By removing (or strongly damping) a band of the highest frequencies allowed by the mesh, it is possible to reduce truncation and aliasing errors. The filter is also well defined, which facilitates a comparison with (filtered) experimental data. In order to realize these benefits, however, the filtering process must be implemented correctly, and the filter itself should have satisfied a few constraints in addition to those required by commutation. The purpose of this paper is to outline the general procedure for explicit filtering and to specify the constraints on the filter shape. A second objective of this paper is to revisit some of the issues related to filtering in the dynamic model calculation and to propose a general method of estimating the test filter width.

2. Accomplishments

2.1 Explicit filtering procedure

Application of a commuting filter to the Navier-stokes equations leads to

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}. \tag{2}$$

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The correlation $\overline{u_i u_j}$ is unknown in LES and is typically treated by computing the product of the filtered velocities and modeling the remainder, i.e.

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j + \underbrace{(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)}_{\tau_{ij}}. \tag{3}$$

If this decomposition is substituted into the filtered momentum equation, a closed equation for $\bar{u}_i$ is obtained provided a model for τ_{ij} is supplied. This equation can be advanced in time from an initial $\bar{u}_i$ field, *and no explicit filtering operation is required during the solution process.* While this observation seems a bit unsettling, it is often argued that the wavenumber-dependent characteristic of finite-differencing errors act as an effective 'implicit filter'. This argument is based on the following equivalence between a finite difference and the exact derivative of a filtered variable (See Rogallo & Moin, 1984)

$$\left.\frac{\delta u}{\delta x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} = \frac{d}{dx}\int_{x_{i-1}}^{x_{i+1}} u\,dx = \left.\frac{d\bar{u}}{dx}\right|_i. \tag{4}$$

While this equivalence is undoubtedly genuine, there are two significant problems with extending the above observation to filtering as it applies to the solution of the LES equations. First, the equivalence requires a connection between the exact derivative of the filtered variable and the finite difference of the *unfiltered variable.* Thus a strict application of this law to the filtered Navier-Stokes equations would require that the original filterings be removed when the finite difference approximation is made. In order to avoid this problem, one can consider applying a second filter to the Navier-Stokes equations and allow this one to be removed when the finite differences are taken. As we shall see, this argument can not be made rigorous, either, due to the second complication that has to do with the multi-dimensionality associated with the Navier-Stokes equations. The filter used to derive the LES equations must be a three-dimensional operation that represents averaging the velocity field over a small volume in space. The filter implied by the finite difference operator, on the other hand, represents an average in a single coordinate direction. Thus each term in the LES equations is effectively acted on by a different one-dimensional filter when finite differences are used. In particular, the actual equation being solved is

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \widetilde{\bar{u}_i \bar{u}_1}^{x_1}}{\partial x_1} + \frac{\partial \widetilde{\bar{u}_i \bar{u}_2}^{x_2}}{\partial x_2} + \frac{\partial \widetilde{\bar{u}_i \bar{u}_3}^{x_3}}{\partial x_3} = -\frac{\partial \widetilde{\bar{p}}^{x_i}}{\partial x_i} - \frac{\partial \widetilde{\tau_{i1}}^{x_1}}{\partial x_1} - \frac{\partial \widetilde{\tau_{i2}}^{x_2}}{\partial x_2} - \frac{\partial \widetilde{\tau_{i3}}^{x_3}}{\partial x_3} + \frac{1}{Re}\left[\frac{\partial^2 \widehat{\bar{u}_i}^{x_1}}{\partial^2 x_1} + \frac{\partial^2 \widehat{\bar{u}_i}^{x_2}}{\partial^2 x_2} + \frac{\partial^2 \widehat{\bar{u}_i}^{x_3}}{\partial^2 x_3}\right], \tag{5}$$

where $\widetilde{()}^{x_i}$ and $\widehat{()}^{x_i}$ are the effective one-dimensional filters associated with the first and second difference approximations respectively. It should be clear that the above equation can not be derived from the Navier-Stokes equations since the various effective filters are not distributed uniformly. We conclude that although there is an inherent filtering operation associated with finite-difference approximations, their use does not lead to a well-defined effective three-dimensional filter.

With the issues associated with finite differences aside, there is another difficulty associated with the use of the decomposition given in Eq. (3). The problem with this formulation is that the non-linear product $\bar{u}_i \bar{u}_j$ generates frequencies beyond the characteristic frequency that defines $\bar{u}_i$. These high frequencies alias back as resolved ones and therefore act as fictitious stresses. In principle the subgrid-scale model, τ_{ij}, could exactly cancel this effect, but it is unlikely that such a model could be arranged. The obvious way to control the frequency content of the non-linear terms is to filter them. This strategy would result in the following alternative decomposition:

$$\overline{u_i u_j} = \overline{\bar{u}_i \bar{u}_j} + \underbrace{(\overline{u_i u_j} - \overline{\bar{u}_i \bar{u}_j})}_{\tau'_{ij}} . \qquad (6)$$

If this relation together with a subgrid-scale model for τ'_{ij} is substituted into Eq. (2), one again obtains a closed equation for $\bar{u}_i$, but this time with an additional *explicit filtering* operation applied to the non-linear term. We now see that the implicit filtering implied by the finite-difference operators shown in Eq. (5) is similar, although the one-dimensional filterings are not nearly as effective at controlling the frequency content of the solution.

While the decomposition of Eq. (6) has several advantageous properties from the point of view of explicit filtering, there is one significant side effect that should be mentioned. It can be shown that if Eq. (6) is substituted into Eq. (2), the resulting equation is in general not Galilean invariant. The residual takes the form $c_j d(\bar{\bar{u}}_i - \bar{u}_i)/dx_j$, where c_j is the uniform translation velocity. The error is seen to be proportional to the difference between the singly and doubly filtered velocity. This difference will be zero for a Fourier cutoff filter, but will not vanish in the general case. The spectral content of the error is proportional to $G(k)(1 - G(k))$ where $G(k)$ is the filter transfer function. This fact implies that error is only generated in the wavenumber band where $G(k)$ differs significantly from 0 or 1. It is also clear that the error is maximized at 25%. Thus it is possible to minimize the error by constructing the explicit filter to be as close as possible to a Fourier cutoff. It is also possible to eliminate the Galilean invariance error all together by switching to yet another alternative decomposition. This step amounts to adding a scale-similarity like term to the filtered Navier-Stokes equations. The difficulty in this approach is that the scale-similarity term generates higher frequencies and thus spoils the explicit filtering procedure. Clearly this issue will require further study. At the present time it appears best to continue with Eq. (6) but to use a filter that is as close as possible to a Fourier cutoff. We shall see that there are other compelling reasons to use this type of filter, and thus its use would be natural in practice.

In order to illustrate the explicit filtering procedure further, consider an Euler time stepping method applied to the LES equations:

$$\bar{u}_i^{n+1} = \bar{u}_i^n + \Delta t \left[-\frac{\partial \overline{\bar{u}_i \bar{u}_j}}{\partial x_j} - \frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \tau'_{ij}}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} \right]^n \tag{7}$$

Note that the frequency content of each term on the right-hand side is limited to the bar level (provided the subgrid-scale model is properly constructed). Thus in advancing from time level n to $n+1$, the frequency content of the solution is not altered. This fact implies that the additional filtering of the non-linear term (plus and analogous treatment of the subgrid-scale model) is sufficient to achieve an explicit filtering of the velocity field for all time. It is also important to note that the procedure outlined above is in general different from the 'filtering of the velocity field after each time step' procedure that has been alluded to in the literature, i.e.

$$\bar{u}_i^{*\,n+1} = \bar{u}_i^n + \Delta t \left[-\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} - \frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} \right]^n ,$$

$$\bar{u}_i^{\,n+1} = \overline{\bar{u}_i^{*\,n+1}}.$$

While this approach results in the correct treatment for the non-linear term, it is incorrect since the remaining terms are filtered twice. In particular, the additional filtering of the solution at the previous time level, $\overline{\bar{u}_i^n}$ is particularly harmful since the cumulative effect over several time steps implies multiple filterings of the velocity field, i.e.

$$\bar{u}_i^{n+1} = \overline{\overline{\bar{u}_i^{n-1}}} + \Delta t \overline{\overline{\bar{R}^{n-1}}} + \Delta t \overline{\bar{R}^n}.$$

In general, repeated application of the same filter implies a filter with increased width, and thus the procedure of filtering the velocity field after each time step results in a severe loss in spectral information [1].

With the correct explicit filtering procedure established (i.e. Eq. (7)), we are now in a position to address some of the more subtle issues involved, the first of which is commutivity. As discussed above, the issue of commutation between the filter and derivative operators arises mainly in deriving the LES equations from the Navier-Stokes system. Explicit filtering, on the other hand, involves the decomposition of Eq. (6) where the filtered product, $\overline{u_i u_j}$, is replaced with $\overline{\bar{u}_i \bar{u}_j} + \tau'_{ij}$. As we have seen this decomposition is not unique, and the decision to add the second bar to the non-linear term is not required in the basic derivation of the LES system, but rather is used simply as a convenient means to control the frequency content of the solution. Furthermore, Eq. (6) is a substitution for $\overline{u_i u_j}$, which appears inside the divergence operator. Thus, perhaps surprisingly, there does not appear to be any direct commutation requirement on the second filter. Of course, there is an indirect

[1] It is important to note that the above argument does not apply to the Fourier cutoff filter where repeated application has no cumulative effect. In this special case, filtering the velocity field at each time step is permissible and is equivalent to the general procedure listed in Eq. (7).

requirement if one requires the first and second bar filters to be identical (since the former was used in the derivation of the LES system). It is not clear whether consistency in this regard is really required in practice, however, and is appears possible to use the second alternative decomposition

$$\overline{u_i u_j} = \widetilde{\bar{u}_i \bar{u}_j} + \underbrace{(\overline{u_i u_j} - \widetilde{\bar{u}_i \bar{u}_j})}_{\tau''_{ij}}, \tag{8}$$

where $\widetilde{()} \simeq \bar{()}$ is a (perhaps non-commuting) approximation to the primary filter[2].

A second subtle issue concerning explicit filtering has to do with an associated false dissipation. The non-linear term in the classical LES decomposition (Eq. (3)) is energy conserving since $\bar{u}_i d(\bar{u}_i \bar{u}_j)/dx_j = d(\bar{u}_j 1/2 \bar{u}_i \bar{u}_i)/dx_j$, and thus an integral over the volume collapses to the surface fluxes via Gauss' theorem. Unfortunately this situation is changed when an explicit filter is applied to the non-linear term. The second filter on the non-linear product prohibits the redistribution of velocity components used to obtain a divergence form and one is left with $\bar{u}_i d(\overline{\bar{u}_i \bar{u}_j})/dx_j = d(\bar{u}_i \overline{\bar{u}_i \bar{u}_j})/dx_j - (d\bar{u}_i/dx_j)\overline{\bar{u}_i \bar{u}_j}$. The second term on the right-hand side does not vanish in general when integrated over the volume and in fact bears some resemblance to the turbulent production. More quantitative information regarding the false dissipation can be obtained by looking at the Fourier-space energy equation for isotropic turbulence which reads

$$\frac{dE(k)}{dt} = \underbrace{\langle -ik\hat{u}^*_m P_{mi} G(k) \sum_{\substack{p \\ k=p+q}} \sum_q \hat{u}_i(p)\hat{u}_j(q)\rangle}_{T(k)} - \frac{2}{Re} k^2 E(k),$$

where $E(k) = 1/2\langle \hat{u}^*_i \hat{u}_i \rangle$ is the spectral energy density, $\hat{u}_i$ is the Fourier transform of the velocity (bar omitted for simplicity), P_{li} is the divergence-free projection operator, $G(k)$ is the transfer function associated with the explicit filter, $()^*$ denotes complex conjugate, and $\langle\rangle$ is a shell average. It is clear that the explicit filter affects only the non-linear transfer term, $T(k)$. This term will be conservative if its integral vanishes, i.e. $\int_0^\infty T(k)dk = 0$. It can be shown that the integral will indeed vanish if the filter function $G(k)$ is a Fourier cutoff that passes frequencies up to some limit k_{max} and if the velocity field is truncated at this level before the transfer term is constructed (Kraichnan, 1976). For non-sharp filters the transfer will not integrate to zero since the weighting introduced by a smoothly-varying $G(k)$ destroys the symmetries required to achieve complete cancellation. Further analysis reveals that the residual transfer arises only out of interactions with wavenumber components where $G \neq 1$. The sign of this residual transfer is not fixed kinematically but is constrained to be negative for developed turbulence with a normal down-scale

[2] The formulation with an approximate second filter is probably always required in practice since even 'commuting' filters only do so to a specified order of accuracy (see Vasilyev and Lund, this volume)

energy cascade. Thus non-sharp filters lead to a false dissipation that is proportional to the degree to which the filter departs from a sharp-cutoff. For this reason, it is important to use filters that are close approximations to a Fourier cutoff.

It is worthwhile to note that the approximately-commuting filters developed by Vasilyev and Lund (this volume) become increasingly better approximations to a Fourier cutoff as the commutation error is reduced. Thus use of these filters will allow for a consistent explicit filtering scheme (first and second filters the same) and will introduce only a small amount of false dissipation.

2.2 Accurate estimation of discrete filter width

The previous discussion was concerned with explicit filtering of the non-linear terms as a means to improve the fidelity of the LES approach. In this subsection we consider a rather distinct filtering operation that is used in the dynamic modeling procedure. In order to estimate subgrid-scale model coefficients, the dynamic model uses a 'test filtering' operation to isolate the stresses produced by a band of the smallest resolved motions. Fitting model expressions to these stresses then provides a mechanism to determine any unknown model coefficients. The only parameter in the dynamic procedure is the ratio of the test to primary filter width, α, which is usually taken to be $\alpha = 2.0$ (there is very little sensitivity to this parameter). It goes without saying that the numerical value of the filter width ratio used in the dynamic procedure must match the properties of the test filter actually used in the calculation. While this seems like a trivial point, there can be some ambiguity in determining the test filter width. Any errors in this regard will have a negative effect on the solution and should be avoided. There has also been some discussion in the literature regarding the importance of the test filter shape. While the dynamic model derivation presupposes that the test filter is similar in form to the primary filter, there have been several attempts to improve on matters by 'optimizing' the test filter shape (Najjar and Tafti, 1996, Spyropoulos and Blaisdell, 1993). As we shall see most of these latter attempts involve fortuitous results that arise from use of an inconsistent filter width. In order to assess the effect of test filter shape, a numerical experiment was designed to investigate this issue. The isotropic decay experiment of Comte-Bellot and Corrsin (1971) was simulated using LES on a 32^3 mesh. The pseudo-spectral code of Rogallo (1981) was used with the volume-averaged form of the dynamic Smagorinsky model forming the closure. The test filter type was varied and the resulting kinetic energy decay histories compared with the (filtered) experimental data. The kinetic energy history provides a good measure of the accuracy of the subgrid-scale model since the model provides the bulk of the dissipation at this coarse resolution. The test results are shown in Fig. 1. It is clear that the filter type has almost no effect on the results. This fact is reassuring since it provides additional evidence on the robustness of the dynamic model. It also raises an interesting point that, although the derivation would suggest otherwise, there does not seem to be any practical requirement for the test and primary filters to be of the same form (Fourier cutoff in this case).

As mentioned above, most of the perceived sensitivity to test filter type noted in the literature has to do with the use of an incorrect value for the filter width

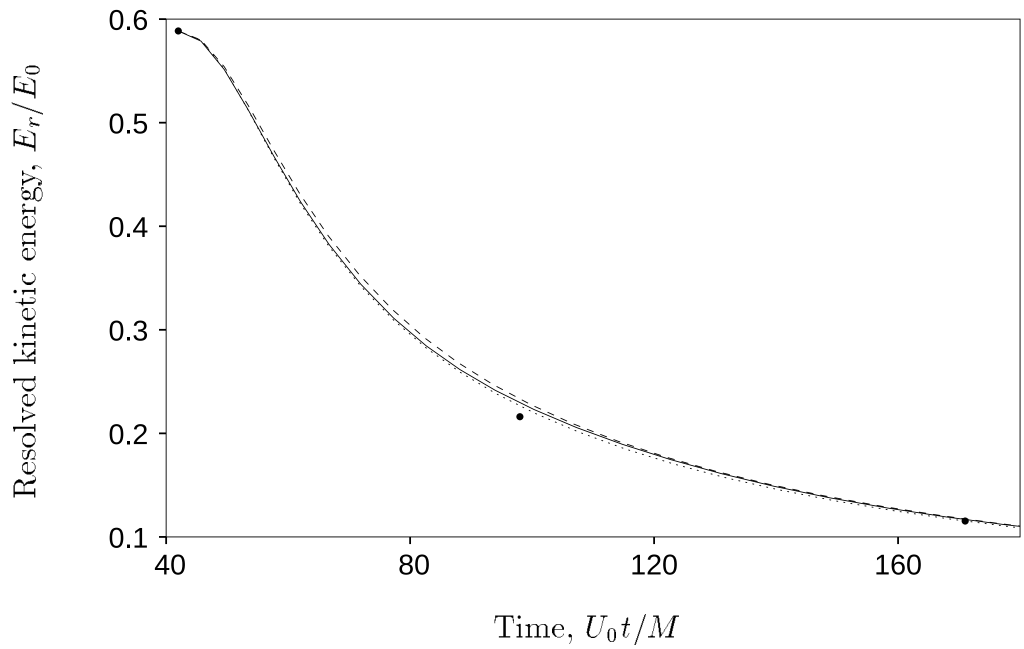

FIGURE 1. Resolved kinetic energy decay history computed with the dynamic model using various test filter types: ——— , Fourier cutoff; ---- , Gaussian; ········ , physical space top-hat; • , experimental of Comte-Bellot and Corrsin (1971).

ratio. This difficulty is usually associated with the inability to estimate the test filter width properly. These difficulties can be avoided by following the procedures listed below.

The width of a positive-definite filter is described best in terms of its standard deviation (Leonard, 1973):

$$\Delta_f = \sqrt{12 \int_{-\infty}^{\infty} x^2 G(x) dx}, \tag{9}$$

where the factor of 12 assures that the width of a physical-space top-hat filter is equal to the interval over which the filter kernel is non-zero. While this formula has been available for quite some time, it does not seem to have been transferred to the realm of discrete filters, which are much more common in practice. The general discrete filter

$$\bar{u}_i = \sum_{j=-(N-1)/2}^{(N-1)/2} W_j u_{i+j} \tag{10}$$

has an associated kernel that can be written as

$$G(x - x') = \sum_{j=-(N-1)/2}^{(N-1)/2} W_j \delta(x - x' + j\Delta). \tag{11}$$

When this expression is substituted into Eq. (9) and the integral performed, we arrive at the following discrete analog

$$\alpha \equiv \Delta_f/\Delta = \sqrt{12 \sum_{j=-(N-1)/2}^{(N-1)/2} j^2 W_j}, \tag{12}$$

where Δ is the computational grid spacing.

In order to illustrate the importance of an accurate estimation of the test filter width, we shall consider two different discrete approximations to a physical space top-hat filter of width 2Δ,

$$\bar{u}(x) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} u dx. \tag{13}$$

In a discrete system we consider $\bar{u}(x_j) \equiv \bar{u}_j$ where j is the mesh index. The integral is evaluated over the interval from x_{j-1} to x_{j+1} where only the three discrete values u_{j-1}, u_j, and u_{j+1} are available. If Simpson's rule is used to perform the quadrature, we obtain the sequence of weights $(W_{-1}, W_0, W_1) = (1/6, 2/3, 1/6)$. If these values are substituted in Eq. (12), we find $\Delta_f/\Delta = 2$ as expected. If the trapezoidal rule is used to evaluate the integral, however, we obtain $(W_{-1}, W_0, W_1) = (1/4, 1/2, 1/4)$ which, according to Eq. (12), have a width $\Delta_f/\Delta = \sqrt{6}$. Thus, perhaps surprisingly, the details of the discrete quadrature can affect the filter width. This is a subtle point that has been overlooked in several previous dynamic model simulations. If the weights associated with the Trapezoidal rule are used but the inconsistent value of the filter width ratio 2.0 is used, the dynamic modeling procedure will loose accuracy. Figure 2 illustrates this effect where kinetic energy decay histories are shown for three cases: (1) Simpson's rule, $\alpha = 2$; (2) Trapezoidal rule, $\alpha = 2$; and (3) Trapezoidal rule, $\alpha = \sqrt{6}$. The first and third cases use consistent values of the filter width ratio and are seen to lead to nearly identical results that are in good agreement with the experimental data. Case (2), on the other hand, uses an inconsistent value of the filter width ratio, and the results are clearly incorrect. If the subtle details of how to compute the filter width ratio correctly were not known, one might mistakenly attribute the poor performance of case (2) to the filter type itself. Unfortunately this type of confusion has appeared in the literature, and there are papers that recommend one filter over another (Najjar and Tafti, 1996, Spyropoulos and Blaisdell, 1993).

The foregoing discussion regarding discrete filters assumes that the discrete second moment used in Eq. (12) is non-zero. There are an important class of filters where this is not the case, however. In particular, Vasilyev and Lund (this volume) show that a filter with $n-1$ vanishing moments will commute to with differentiation to order n. Since the n^{th} filter moment is directly related to the n^{th} derivative of the filter transfer function at zero wavenumber, a filter with n vanishing moments also has n vanishing derivatives at the origin in wavenumber space. Thus by Taylor series, the transfer function remains very close to unity for sizable displacements in

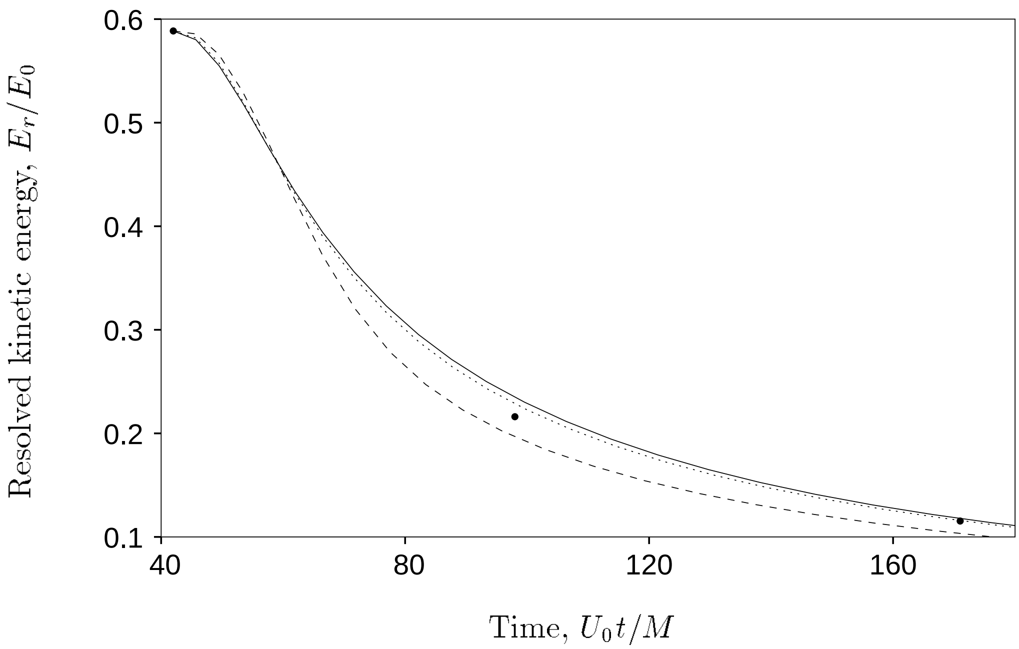

FIGURE 2. Effect of a mismatch between the true test filter width and the value used in the dynamic model calculation. ——— , Simpson's rule, $\alpha = 2.00$; ---- , Trapezoidal rule, $\alpha = 2.00$; ········ , Trapezoidal rule, $\alpha = \sqrt{6}$; • , experimental of Comte-Bellot and Corrsin (1971).

wavenumber, making these filters good approximations to a Fourier cutoff (at least for low to moderate wavenumbers).

The use of filters with vanishing moments presents a problem since the width can not be based on the second moment *a la* Eq. (12). While a similar expression based on a higher moment could be used, one would eventually encounter a filter where even this moment vanishes. Several more robust definitions of the filter width were investigated and these will be discussed below.

In order to facilitate the discussion, it will be convenient to consider the filter transfer function, which is obtained by taking the Fourier transform of Eq. (11), *viz.*

$$G(k) = \sum_{j=-(N-1)/2}^{(N-1)/2} W_j \cos(jk\Delta). \tag{14}$$

In deriving this result the weights are assumed to be symmetric with respect to j.

The first alternative method of determining the filter width takes advantage of the fact that some of the weights must be negative in order for the sum in Eq. (12) to vanish. More specifically, if a trigonometric interpolant is fit through the weights as a function of their index, an oscillatory distribution similar to the $\sin(\pi x/\Delta_f)/(\pi x)$ function characteristic of a Fourier cutoff is obtained. The position of the first zero crossing can then be used as an estimate of the filter width. The interpolating series is

$$W(x) = \frac{1}{N} \sum_{j=-(N-1)/2}^{(N-1)/2} G_j \exp\left(i2\pi \frac{j}{N} \frac{x}{\Delta} \right), \quad (15)$$

where the G_j are the discrete values of the filter transfer function:

$$G_j = G\left(\frac{2\pi j}{N\Delta} \right) = \sum_{l=-(N-1)/2}^{(N-1)/2} W_l \cos\left(l \frac{2\pi j}{N} \right). \quad (16)$$

Note that the interpolant is purely real since the G_j must be symmetric with respect to j.

The second alternative strategy works directly with the filter transfer function. In this case the filter width is taken to be proportional to the inverse wavenumber where the filter transfer function falls to 0.5. This rule gives $\alpha = \pi/(k_f \Delta)$.

The third alternative is to base the filter width on the second moment of the *filter transfer function* rather than on the second moment of the filter kernel. Defining the second moment as

$$M^2 = \int_0^{\pi/\Delta} k^2 G(k) dk \quad (17)$$

we may estimate the filter width from

$$\alpha = \left[\frac{\pi^3}{3M^2\Delta^3} \right]^{\frac{1}{3}}. \quad (18)$$

The constants in this formula were chosen so that it predicts the correct width in the case of an exact Fourier cutoff. The second moment for a discrete filter is found by substituting Eq. (14) into Eq. (17) and performing the integration. These operations lead to

$$M^2 = \frac{1}{\Delta^3} \left[\frac{\pi^3}{3} W_0 + 4\pi \sum_{j=1}^{(N-1)/2} \frac{(-1)^j}{j^2} W_j \right], \quad (19)$$

where weights are assumed to be symmetric with respect to j. Combining Eqs. (18) and (19) we obtain the final result

$$\alpha = \left[W_0 + \frac{12}{\pi^2} \sum_{j=1}^{(N-1)/2} \frac{(-1)^j}{j^2} W_j \right]^{-\frac{1}{3}}. \quad (20)$$

As an illustration of a filter that has a vanishing second moment in physical space, consider filter C discussed by Najjar and Tafti (1996). The stencil contains 7 points, and the weights are (1/256)(1, -18, 63, 164, 63, -18, 1). The interpolating function

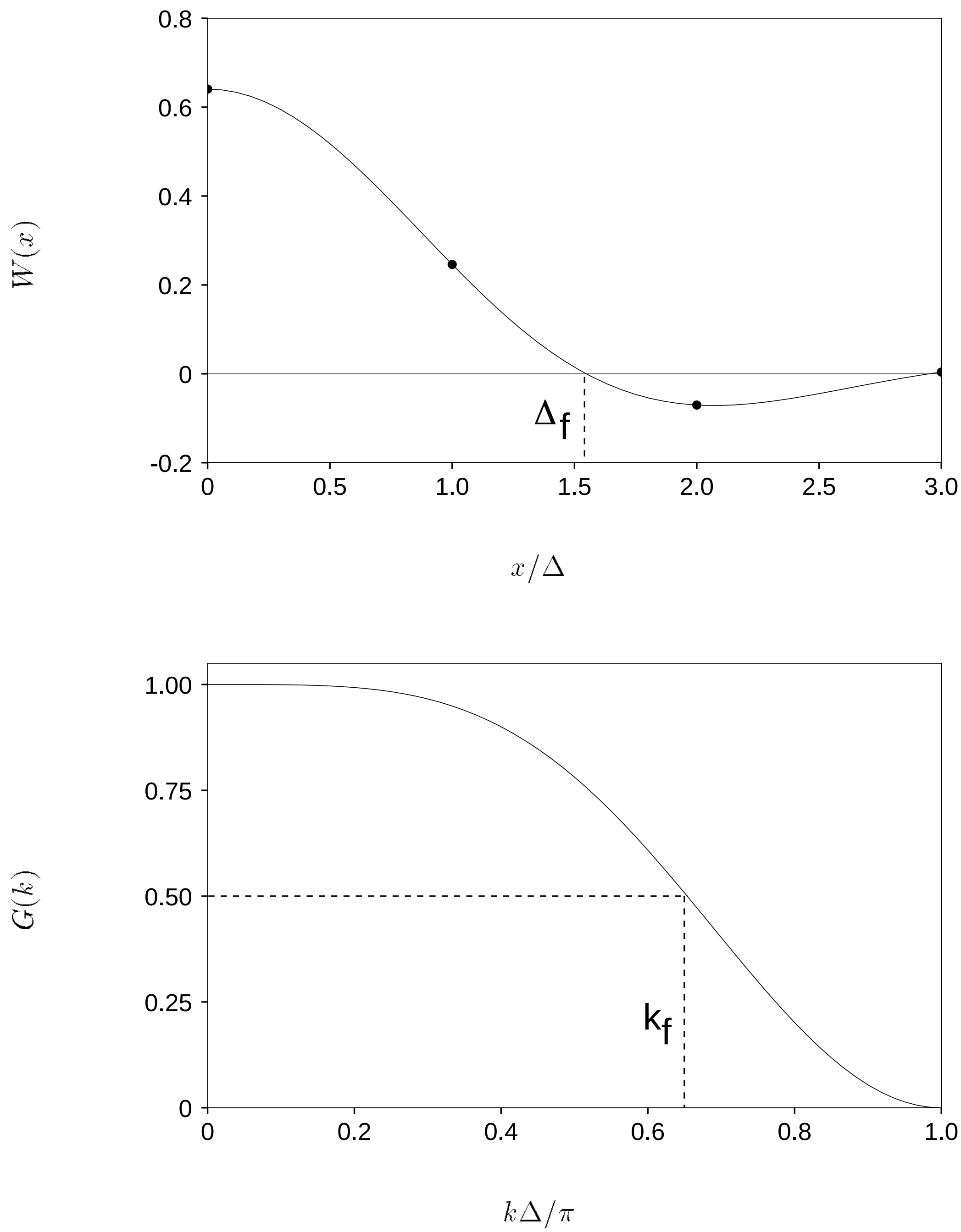

FIGURE 3. Filter weights and associated transfer function for filter C of Najjar and Tafti (1996).

for the weights as well as the filter transfer function are shown in Fig. 3. The position of the first zero-crossing in physical space gives the estimate $\alpha \simeq 1.55$, the location of the $G = 0.5$ point gives $\alpha \simeq 1.53$, and the second moment of the transfer function gives $\alpha \simeq 1.46$. While the three methods give nearly the same result, we shall see that there is a slight advantage to width predicted by the second moment. Other

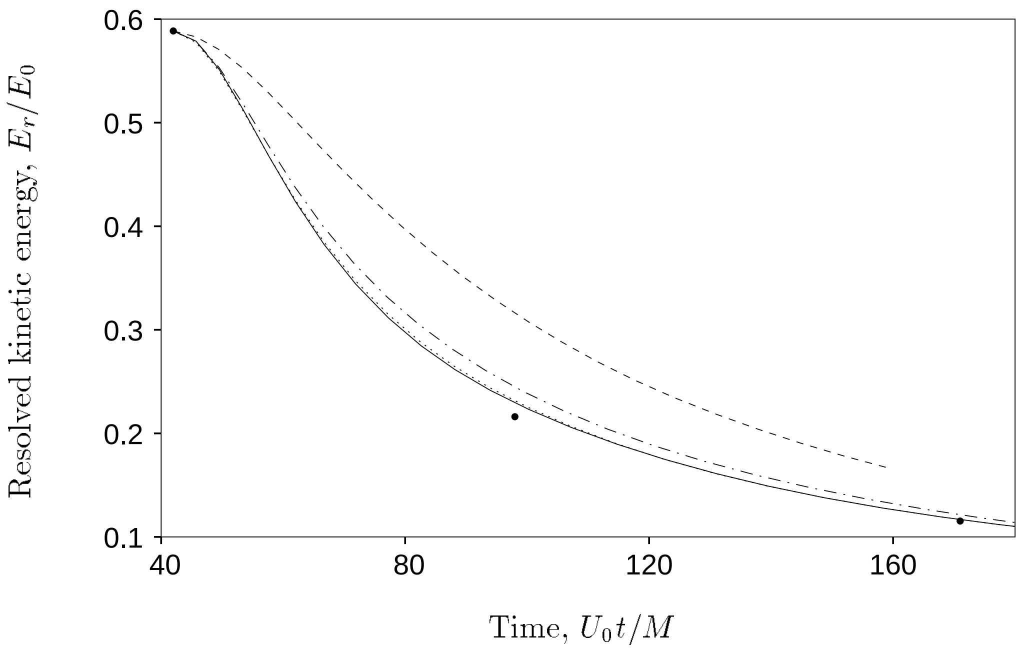

FIGURE 4. Results for a filter with a vanishing second moment in physical space. ——— , Fourier cutoff; ---- filter C, $\alpha = 2.00$; —·— , filter C, $\alpha = 1.55$; ········ , filter C, $\alpha = 1.46$. Filter C has weights (1/256)(1, -18, 63, 164, 63, -18, 1) and is described in Najjar and Tafti (1996).

filters were investigated where differences among the three estimates of the width were greater, and in each case the second moment rule predicted the width most accurately. The Najjar and Tafti filter was chosen for the purpose of illustration since these authors incorrectly assigned a value of $\alpha = 2.0$ to this filter. The effect of this mismatch is shown in Fig. 4 where the kinetic energy decay history is plotted for the Najjar and Tafti filter using several different values of α. The value $\alpha = 2.0$ is clearly incorrect while the estimate $\alpha \simeq 1.46$ given by the second moment of the transfer function is the most accurate. While Najjar and Tafti observed some improvement in their computational results when filter C was used with $\alpha = 2$, this was most likely a fortuitous effect brought on by a cancellation of errors. Chances are that, if $\alpha = 1.46$ were used instead, the results would have been nearly identical to their other cases where the value of α was consistent with the filter used.

3. Conclusions and future plans

Explicit filtering can be used in the LES solution procedure as a means of reducing truncation and aliasing errors. The required operation involves only filtering the non-linear terms and amounts to a slightly different definition of subgrid-scale stress. The procedure is in general different from filtering the entire velocity field at each time step, which could lead to a severe damping of even the largest scales. The explicit filter must commute with differentiation only if one insists that the primary and secondary filters be identical. If strict consistency in this regard is not required, more general filters can be considered, and the use of approximately commuting filters can be justified on a higher level.

The dynamic modeling procedure was shown to be extremely robust with respect to the test filter type but quite sensitive to mismatches between the true filter width ratio and the value used in the calculation of the model coefficient. Much of the apparent sensitivity to test filter type reported in the literature is related to inaccurate estimates for the test filter width and not to the shape of the filter itself. General rules were developed for accurate estimation of test filter width, and these are related to the second moments of either the filter kernel or its associated transfer function.

Future work will focus on the use of explicit filters in actual large eddy simulations. This work is in progress, and some preliminary results for three-dimensional explicit filtering in turbulent channel flow simulations have been obtained. The indication from these tests is that, while explicit filtering definitely improves the solution, some issues have arisen regarding the required reformulation of the dynamic model as well as the smearing effect of filtering in the inhomogeneous wall-normal direction. Current work is focusing on resolving these issues and in assessing the overall effectiveness of the explicit filtering strategy.

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