Invariant modeling in large-eddy simulation of turbulence

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1. Motivation and objectives

Since the derivation of the Smagorinsky model (Smagorinsky 1963), much research has been dedicated to developing more reliable and physically plausible large-eddy models for turbulence. Speziale (1985) made the first attempt to derive realizable large-eddy models. He argued that any subgrid-scale (SGS) model in large-eddy simulation (LES) of turbulence should be Galilean invariant, a fundamental invariance property (also called symmetry) of the Navier-Stokes equations. In his investigation he found that many models violate this symmetry. The most widely used model, the Smagorinsky model, is Galilean invariant.

However, Galilean invariance is only one of several symmetries of the Navier-stokes equations. It will be seen later that several of the symmetries are violated by common SGS models, the bulk of which contain the local grid size of the computation as a length scale. From a theoretical point of view, having an external length scale in the turbulent model which is not related to any turbulent quantity violates certain symmetries of Navier-Stokes equations. This has serious implications for the overall performance of the model, which will be pointed out below. In particular, certain scaling laws cannot be realized by the modeled equations in wall-bounded flows (see Appendix A).

A differential equation admits a symmetry if a transformation can be obtained which leaves the equation unchanged in the new variables. It is said the equation is invariant under the transformation. Symmetries or invariant transformations are properties of the equations and not of the boundary conditions, which are usually not invariant. Symmetries and their consequences form some of the most fundamental properties of partial differential equations and illustrate many important features of the underlying physics. The Navier-Stokes equations admit several symmetries, each of them reflecting axiomatic properties of classical mechanics: time invariance, rotation invariance, reflection invariance, two scaling invariances, pressure invariance, material indifference, and generalized Galilean invariance, which encompasses frame invariance with respect to finite translation and classical Galilean invariance. Each of these symmetries is explained in Section 2.

For example, all known similarity solutions of the Euler and the Navier-Stokes equations for laminar flows can be derived from symmetries (see Pukhnachev 1972). Turbulent flows admit a wide variety of solutions derivable from symmetries. Some of them, like jets and wakes, have global character (see e.g. Townsend 1976; Cantwell 1981) others only apply locally, e.g. in wall-bounded flows. Recently several new scaling laws for turbulent wall-bounded flows were derived in Oberlack (1997a,b) using symmetry methods, and all of these are local self-similar regions. A well known example, which is also among Oberlack's results, is the logarithmic law of the wall which has a restricted validity near the wall region but can be found in many geometrically different flows. All the known local and global turbulent scaling laws can be derived from symmetries.

In order to reproduce all global and local self-similar turbulent solutions with an SGS model in LES of turbulence, it is a necessary condition that all the above mentioned invariance properties of the Navier-Stokes equations should be built into the SGS model. This implies certain restrictions for the functional form of the model.

In LES of turbulence not only the SGS model is constrained by symmetries, but so also is the filer function. Vreman, Geurts & Kuerten (1994) investigated whether certain filter functions preserve the classical realizability constraint by Schumann (1977). The key result in their analysis is that the filter kernel has to be positive in order to ensure positive turbulent subgrid kinetic energy. They concluded that the spectral cut-off filter is not suitable for LES since the kernel is negative for certain values of its argument. Symmetries of the Navier-Stokes equations imply further constraints for the filter function to be derived below. Moreover, it will be shown that the form of filter function is consistent with the finding of Vreman *et al.*.

The paper is organized as follows: In Section 2. all the known symmetries of the Navier-Stokes equations are discussed. In Section 3 the concept of spatial averaging is reexamined, and its implications for the SGS model and the filter function are derived. In Section 4 several examples of proposed SGS models will be investigated as to whether they obey or violate certain symmetries of the Navier-Stokes equations. Section 5 gives a summary and conclusions of the paper. In Appendix A the effect on near-wall scaling laws will be investigated for the case when the SGS model does not satisfy the proper scaling symmetries.

2. Symmetries of the Navier-Stokes equations

The Navier-Stokes and the continuity equations for an incompressible fluid written in primitive variables in a Cartesian coordinate system are

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k^2} \qquad \text{and} \qquad \frac{\partial u_k}{\partial x_k} = 0, \tag{1}$$

where $\boldsymbol{x}$, t, $\boldsymbol{u}$, p, and ν are, respectively, the spatial coordinate, time, the velocity vector, the pressure normalized by the density and the kinematic viscosity. Equations (1) admit several symmetries, each reflecting fundamental properties of classical mechanics. In the following a list of all known symmetry transformations will be given which preserve the functional form of (1) written in the new variables, subsequently denoted by "*".

I. Time invariance

An arbitrary time shift of the amount a

$$t^* = t + a, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \boldsymbol{u}^* = \boldsymbol{u}, \;\; p^* = p, \;\; \nu^* = \nu \tag{2}$$

has no effect on the functional form of (1).

II. Rotation invariance

Rotating the coordinate system and the velocity vector

$$t^* = t, \;\; x_i^* = A_{ij}x_j, \;\; u_i^* = A_{ij}u_j, \;\; p^* = p, \;\; \nu^* = \nu \tag{3}$$

by a finite but arbitrary angle in space, where $\boldsymbol{A}$ is the rotation matrix with $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$ and $|\boldsymbol{A}| = 1$, preserves the form of Eq. (1) in the new variables after multiplying the momentum equation with $\boldsymbol{A}$. The superscripts T, $\boldsymbol{I}$, and $|\cdot|$ denote, respectively, the transpose of a matrix, the unit tensor, and the determinant.

III. Reflection invariance

The reflection symmetry in any direction x_α is given by

$$\begin{aligned} &t^* = t, \;\; x_\alpha^* = -x_\alpha, \;\; u_\alpha^* = -u_\alpha, \\ &x_\beta^* = x_\beta \;\; , u_\beta^* = u_\beta \;\text{ with }\; \beta \neq \alpha, \;\; p^* = p, \;\; \nu^* = \nu, \end{aligned} \tag{4}$$

where the index α can be any of 1, 2, and 3, and β refers to the remaining two.

IV. Generalized Galilean invariance

Substituting

$$\boldsymbol{u}^* = \boldsymbol{u} + \frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}, \;\; p^* = p - \boldsymbol{x} \cdot \frac{\mathrm{d}^2\hat{\boldsymbol{x}}}{\mathrm{d}t^2} \quad \text{and} \quad \nu^* = \nu \tag{5}$$

into (1), where $\hat{\boldsymbol{x}}(t)$ is any twice differentiable time dependent vector-function, does not alter the functional form of (1). (5) covers two classical symmetries: (i) Invariance with respect to finite translation in space is obtained for $\hat{\boldsymbol{x}}(t) = \boldsymbol{b}$, where $\boldsymbol{b}$ is a constant and (ii) the classical Galilean invariance is recovered if $\hat{\boldsymbol{x}}(t)$ is a linear function in time.

All symmetries (2)-(5) are also admitted by the incompressible Euler equations.

V. Scaling invariance

Considering $\nu = 0$, the two-parameter transformation

$$t^* = \xi t, \;\; \boldsymbol{x}^* = \gamma\boldsymbol{x}, \;\; \boldsymbol{u}^* = \frac{\gamma}{\xi}\boldsymbol{u}, \;\; p^* = \left(\frac{\gamma}{\xi}\right)^2 p \tag{6}$$

is an invariant transformation of Eqs. (1), where ξ and γ are arbitrary positive real numbers. If $\nu \neq 0$ and ν is considered a parameter, then (6) is only a scaling invariance provided $\gamma^2 = \xi$.

Considering ν as an additional independent variable, the full two-parameter scaling invariance (6) for $\nu \neq 0$ is recovered if ν is scaled as

$$\nu^* = \frac{\gamma^2}{\xi}\nu. \tag{7}$$

The two scaling groups corresponding to γ and ξ refer to the fact that in classical mechanics time and space can be measured arbitrarily. Hence, scaling symmetries are equivalent to dimensional analysis.

VI. Pressure invariance

An arbitrary time variation of the background pressure, here denoted by $\varphi(t)$, does not affect an incompressible flow. The corresponding symmetry transformation is given by

$$t^* = t, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \boldsymbol{u}^* = \boldsymbol{u}, \;\; p^* = p + \varphi(t), \;\; \nu^* = \nu. \tag{8}$$

VII. Material indifference

Consider the Navier-Stokes equations in a constantly rotating frame with a rotation rate Ω_3 about the x_3 direction and where all velocities only depend on x_1, x_2, and t. The particular choice of the axis of rotation is not restrictive because of the transformations (3) and (4). The transformation which leaves (1) form-invariant is given by

$$\begin{aligned} &t^* = t, \;\; x_i^* = B_{ij}(t)x_j, \;\; u_i^* = B_{ij}(t)u_j + \dot{B}_{ik}(t)x_k, \\ &p^* = p + 2\,\Omega_3 \int_Q (u_1 dx_2 - u_2 dx_1) - \frac{3}{2}\Omega_3^2(x_1^2 + x_2^2), \;\; \nu^* = \nu \end{aligned} \tag{9}$$

where $\boldsymbol{B}(t)$ is the rotation matrix with $\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{B}^T\boldsymbol{B} = \boldsymbol{I}$, $|\boldsymbol{B}| = 1$, $\dot{B}_{ik}B_{jk} = \varepsilon_{3ij}\Omega_3$ and Ω_3 is a constant. The line integral along the arbitrary curve Q in the pressure transformation represents the usual two-dimensional stream-function. The property of material indifference can be reversed if turbulence undergoes rotation-like advection. This can be accomplished either by system rotation or stream-line curvature. In this case turbulence tends to become two-dimensional with the axis of independence aligned with the axis of rotation.

All the symmetry transformations have been obtained by group analysis, except for the reflection symmetry (4), which does not form a continuous group. Pukhnachev (1972) computed the first complete list of all continuous point symmetries (2), (3), (5), (6), and (8) by Lie group methods (see e.g. Ibragimov 1994,1995). Ünal (1994) added the scaling of viscosity (7). The transformation (9) is a well known property of two-dimensional flows (see e.g. Batchelor 1967). From group theoretical methods, it was first derived by Cantwell (1978). He computed it using Lie group analysis applied to the scalar stream-function equation of the two-dimensional Navier-Stokes equations. In this approach the symmetry (9) is a classical point symmetry while in primitive variables it is a non-local symmetry. A corresponding symmetry in three dimensions may not exist. In Oberlack (1997c) it was shown that the three-dimensional Navier-Stokes equations in vector-stream-function formulation admit only those symmetries which can be derived from the Navier-Stokes equations in primitive variables. Recently, additional non-classical symmetries have been obtained by Ludlow & Clarkson (1997). However, these symmetries are not invariant transformations in the classical sense but instead can only be used to obtain self-similar solutions of the Navier-Stokes equations.

3. Invariant modeling and filtering

In contrast to the classical Reynolds averaging, in large-eddy simulation of turbulence the averaging procedure is a spatial filtering defined as

$$\mathcal{L}[\cdot](\boldsymbol{x}) = \int_V G(\boldsymbol{x}, \boldsymbol{y})[\cdot](\boldsymbol{y}) d^3 y. \tag{10}$$

The kernel G is normalized as

$$\int_V G(\boldsymbol{x}, \boldsymbol{y}) d^3 y = 1 \tag{11}$$

and G is assumed to be sufficiently smooth and decays rapidly enough for large distances $\boldsymbol{y}$ so that the integrals converge.

In the present context f represents the instantaneous variables $\boldsymbol{u}$ and p. f is decomposed as

$$f = \bar{f} + f' \tag{12}$$

where

$$\bar{f} = \mathcal{L}[f](\boldsymbol{x}). \tag{13}$$

Introducing the decomposition (12) for both the velocity and pressure into Eqs. (1) and applying the filter (10) leads to the equation of motion for the large-eddies

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k^2} - \frac{\partial \tau_{ik}}{\partial x_k} \qquad \text{and} \qquad \frac{\partial \bar{u}_k}{\partial x_k} = 0. \tag{14}$$

The SGS stress τ_{ik} is given by

$$\tau_{ik} = L_{ik} + C_{ik} + R_{ik}, \tag{15}$$

where

$$L_{ik} = \overline{\bar{u}_i \bar{u}_k} - \bar{u}_i \bar{u}_k, \tag{16}$$

$$C_{ik} = \overline{u_i' \bar{u}_k} + \overline{\bar{u}_i u_k'}, \tag{17}$$

$$R_{ik} = \overline{u_i' u_k'}. \tag{18}$$

L_{ik}, C_{ik}, and R_{ik} are, respectively, referred to as the Leonard stress, the subgrid-scale cross-stress, and the subgrid-scale Reynolds stress. If explicit filtering is employed, the Leonard stress may be computed from the flow field, and closure models only need to be introduced for C_{ik} and R_{ik}. Though the decomposition of $\boldsymbol{\tau}$ is arbitrary, (16)-(18) is a very common notation in LES. A different decomposition has been proposed by Germano (1986) because it was found by Speziale (1985) that both L_{ik} and C_{ik} are not Galilean invariant as discussed below.

The principal assertion of this work is given by the following statement: *To derive a physically consistent large-eddy model for turbulence the filtered Navier-Stokes equations (14) with the SGS closure model must admit the same symmetries as the Navier-Stokes equations (as given in section 2).*

This has certain implications for the form of the model for (16)-(18) and puts restrictions on the filter kernel G in (10) to be derived in the next two sub-sections.

3.1 Implications for the subgrid-scale stresses

Suppose the filter (10) preserves the invariance properties of Navier-Stokes equations, then one can deduce from (14) and (15) that

$$\frac{\partial \bar{u}_i^*}{\partial t^*} + \bar{u}_k^* \frac{\partial \bar{u}_i^*}{\partial x_k^*} = -\frac{\partial \bar{p}^*}{\partial x_i^*} + \nu^* \frac{\partial^2 \bar{u}_i^*}{\partial {x_k^*}^2} - \frac{\partial \tau_{ik}^*}{\partial x_k^*} \qquad \text{and} \qquad \frac{\partial \bar{u}_k^*}{\partial x_k^*} = 0 \tag{19}$$

where

$$\tau_{ik}^* = L_{ik}^* + C_{ik}^* + R_{ik}^*, \tag{20}$$

and

$$L_{ik}^* = \overline{\bar{u}_i^* \bar{u}_k^*} - \bar{u}_i^* \bar{u}_k^*, \tag{21}$$

$$C_{ik}^* = \overline{{u'}_i^* \bar{u}_k^*} + \overline{\bar{u}_i^* {u'}_k^*}, \tag{22}$$

$$R_{ik}^* = \overline{{u'}_i^* {u'}_k^*}. \tag{23}$$

and "*" refers to any of the symmetry transformation variables in Section 2. The following is a list of all constraints for the SGS model to properly reproduce all symmetries of the equations of motion.

I. Time invariance

From (2) it can be deduced that the resolved and the unresolved quantities transform as

$$t^* = t + a, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \bar{\boldsymbol{u}}^* = \bar{\boldsymbol{u}}, \;\; {\boldsymbol{u}'}^* = \boldsymbol{u}', \;\; \bar{p}^* = \bar{p}, \;\; {p'}^* = p', \;\; \nu^* = \nu, \tag{24}$$

which leads to the transformation rule for the stresses

$$\boldsymbol{\tau}^* = \boldsymbol{\tau} \;\text{ or }\; \boldsymbol{L}^* = \boldsymbol{L}, \;\; \boldsymbol{C}^* = \boldsymbol{C} \;\text{ and }\; \boldsymbol{R}^* = \boldsymbol{R}. \tag{25}$$

Any model which is autonomous in time complies with this restriction. This is almost always guaranteed since common models are expressed as functionals of $\boldsymbol{x}$ and $\bar{\boldsymbol{u}}$ only.

II. Rotation invariance

From (3) one can conclude that the rotation invariance for the large scale and small scale quantities are given by

$$t^* = t, \;\; x_i^* = A_{ij} x_j, \;\; \bar{u}_i^* = A_{ij} \bar{u}_j, \;\; {u'_i}^* = A_{ij} u'_j, \;\; \bar{p}^* = \bar{p}, \;\; {p'}^* = p', \;\; \nu^* = \nu. \tag{26}$$

As a consequence, the stress tensor (15) and its components (16)-(18) need to transform as

$$\tau_{ik}^* = A_{im} A_{kn} \tau_{mn},$$
$$L_{ik}^* = A_{im} A_{kn} L_{mn}, \;\; C_{ik}^* = A_{im} A_{kn} C_{mn} \;\text{ and }\; R_{ik}^* = A_{im} A_{kn} R_{mn}. \tag{27}$$

This is always guaranteed if the model is formulated in a "tensorially correct" manner. The author is unaware of any existing model that violates this property.

III. Reflection invariance

Considering reflection in the x_α-direction, one can infer from (4) that the filtered and subgrid quantities transform as

$$\begin{aligned} &t^* = t, \;\; x_\alpha^* = -x_\alpha, \;\; \bar{u}_\alpha^* = -\bar{u}_\alpha, \;\; {u'_\alpha}^* = -u'_\alpha, \\ &x_\beta^* = x_\beta \;\; , \bar{u}_\beta^* = \bar{u}_\beta \;\; , {u'_\beta}^* = u'_\beta \;\text{with}\; \beta \neq \alpha, \;\; p^* = p, \;\; \nu^* = \nu, \end{aligned} \tag{28}$$

where α and β are defined according to the definitions below (4). Hence, the reflection symmetry is preserved if

$$\tau_{ik}^* = \omega \tau_{ik}, \;\; \text{where} \; \begin{cases} \omega & = & -1 & \text{for } i = \alpha \;\vee\; k = \alpha \;\wedge\; i \neq k \\ \omega & = & 1 & \text{else} \end{cases} . \tag{29}$$

Similarly, one has the additional restrictions

$$\begin{array}{lcl} L_{ik}^* & = & \omega L_{ik}, \\ C_{ik}^* & = & \omega C_{ik}, \\ R_{ik}^* & = & \omega R_{ik} \end{array} \;\; \text{where} \; \begin{cases} \omega & = & -1 & \text{for } i = \alpha \;\vee\; k = \alpha \;\wedge\; i \neq k \\ \omega & = & 1 & \text{else} \end{cases} . \tag{30}$$

It appears that all common SGS models comply with reflection symmetry.

IV. Generalized Galilean invariance

Generalizing Speziale (1985), (5) and (12) are used to obtain

$$t^* = t, \;\; \boldsymbol{x}^* = \boldsymbol{x} + \hat{\boldsymbol{x}}(t), \;\; \bar{\boldsymbol{u}}^* = \bar{\boldsymbol{u}} + \frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}, \;\; \boldsymbol{u}'^* = \boldsymbol{u}', \;\; \bar{p}^* = \bar{p} - \boldsymbol{x} \cdot \frac{\mathrm{d}^2 \hat{\boldsymbol{x}}}{\mathrm{d}t^2}, \;\; p'^* = p', \;\; \nu^* = \nu. \tag{31}$$

From the latter result and (20), one can verify that

$$\boldsymbol{\tau}^* = \boldsymbol{\tau}. \tag{32}$$

As pointed out by Speziale (1985), a corresponding simple transformation does not exist for (16)-(18). Using (31) in (21)-(23), we find

$$L_{ik}^* = L_{ik} - \frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} \overline{u'_k} - \overline{u'_i} \frac{\mathrm{d}\hat{x}_k}{\mathrm{d}t} \tag{33}$$

$$C_{ik}^* = C_{ik} + \frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} \overline{u'_k} + \overline{u'_i} \frac{\mathrm{d}\hat{x}_k}{\mathrm{d}t} \tag{34}$$

$$R_{ik}^* = R_{ik}. \tag{35}$$

Hence, L_{ik} and C_{ik} are not form-invariant, but their sum is. Germano (1986) tackled the latter problem by redefining the turbulent stresses. He introduced modified definitions for the quantities $\boldsymbol{L}$, $\boldsymbol{C}$, and $\boldsymbol{R}$ where each separate term is Galilean invariant. Since the decomposition is not unique, it appears to be preferable to test the entire SGS model for $\boldsymbol{\tau}$ for Galilean invariance.

The requirement of Galilean invariance has nicely been demonstrated by Härtel & Kleiser (1997), who have compared Galilean and non-Galilean invariant models and different filter functions. The most striking result of their computation was a negative dissipation if the model was not Galilean invariant.

V. Scaling invariance

From (6), (7) and (12) one finds

$$t^* = \xi t,\ \ \boldsymbol{x}^* = \gamma \boldsymbol{x},\ \ \bar{\boldsymbol{u}}^* = \frac{\gamma}{\xi}\bar{\boldsymbol{u}},\ \ \boldsymbol{u}'^* = \frac{\gamma}{\xi}\boldsymbol{u}',$$
$$\bar{p}^* = \left(\frac{\gamma}{\xi}\right)^2 \bar{p},\ \ p'^* = \left(\frac{\gamma}{\xi}\right)^2 p',\ \ \nu^* = \frac{\gamma^2}{\xi}\nu. \tag{36}$$

Applying these results to (20) yields

$$\boldsymbol{\tau}^* = \left(\frac{\gamma}{\xi}\right)^2 \boldsymbol{\tau}. \tag{37}$$

Similarly one can deduce from (21)-(23) that

$$\boldsymbol{L}^* = \left(\frac{\gamma}{\xi}\right)^2 \boldsymbol{L},\ \ \boldsymbol{C}^* = \left(\frac{\gamma}{\xi}\right)^2 \boldsymbol{C} \ \text{ and } \ \boldsymbol{R}^* = \left(\frac{\gamma}{\xi}\right)^2 \boldsymbol{R} \tag{38}$$

has to be valid for any SGS model. It will be shown later that (37) is violated by the classical Smagorinsky model. In Appendix A it will be demonstrated by investigating the two-point correlation equations that this symmetry breaking produces incorrect statistical results, particularly in the near-wall region.

VI. Pressure invariance

The pressure invariance (8) should also be observed by the filtered quantities which leads to

$$t^* = t,\ \ \boldsymbol{x}^* = \boldsymbol{x},\ \ \bar{\boldsymbol{u}}^* = \bar{\boldsymbol{u}},\ \ \boldsymbol{u}'^* = \boldsymbol{u}',\ \ \bar{p}^* = \bar{p} + \varphi(t),\ \ p'^* = p',\ \ \nu^* = \nu, \tag{39}$$

Since SGS models are usually modeled in terms of velocities, the pressure invariance does not give any restrictions on the stresses $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{\tau}$.

VII. Material indifference

From (9) and (12) one can conclude that

$$\begin{gathered} t^* = t,\ \ x_i^* = B_{ij}(t)x_j,\ \ \bar{u}_i^* = B_{ij}(t)\bar{u}_j + \dot{B}_{ij}(t)x_j,\ \ u_i'^* = B_{ij}(t)u_j', \\ \bar{p}^* = \bar{p} + 2\,\Omega_3 \int_Q (\bar{u}_1 dx_2 - \bar{u}_2 dx_1) - \tfrac{3}{2}\Omega_3^2(x_1^2 + x_2^2), \\ p'^* = p' + 2\,\Omega_3 \int_Q (u_1' dx_2 - u_2' dx_1),\ \ \nu^* = \nu. \end{gathered} \tag{40}$$

where $\boldsymbol{B}(t)$ obeys the definitions given below (9). Using the above relations in (20), one finds that an SGS model captures material indifference if

$$\tau_{ik}^{*} = B_{im}B_{kn}\tau_{mn}. \tag{41}$$

As in (33)-(35), the separated stresses (16)-(18) are not form invariant. Using (40) in (21)-(23) it can be concluded that the separated stresses L_{ik} and C_{ik} are not form invariant under constant rotation rate and hence

$$L_{ik}^{*} = B_{im}B_{kn}L_{mn} - B_{im}\overline{u'_m}\dot{B}_{kn}x_n - \dot{B}_{im}x_m B_{kn}\overline{u'_n}, \tag{42}$$

$$C_{ik}^{*} = B_{im}B_{kn}C_{mn} + B_{im}\overline{u'_m}\dot{B}_{kn}x_n + \dot{B}_{im}x_m B_{kn}\overline{u'_n}, \tag{43}$$

$$R_{ik}^{*} = B_{im}B_{kn}R_{mn}. \tag{44}$$

However, the sum of L_{ik} and C_{ik} is invariant. Employing the modified definition of the stresses as introduced by Germano (1986) leads to form invariant stresses under constant rotation rate in the sense that the last two terms on the right-hand side of (42) and (43) disappear.

3.2 Implications for the filter kernel G

In order to incorporate the symmetries of the Navier-Stokes equations in the large-eddy model, one needs to show that the transformation properties of $\boldsymbol{u}$ and p are preserved for the filtered quantities $\bar{\boldsymbol{u}}$ and $\bar{p}$. This restricts the form of the filter kernel as will be shown subsequently.

Time invariance (2) is always preserved no matter which filter kernel is chosen in (10) because t does not explicitly appear in G.

Generalized Galilean invariance (5) implies a restriction on the form of the filter kernel. Consider the Galilean invariance of the filtered velocities $\bar{\boldsymbol{u}}^{*} = \bar{\boldsymbol{u}} + \mathrm{d}\hat{\boldsymbol{x}}/\mathrm{d}t$ given in (31). Employing the definition of the filter (10), one obtains

$$\int_{V^*} G(\boldsymbol{x}^*, \boldsymbol{y}^*)\boldsymbol{u}^*(\boldsymbol{y}^*)d^3y^* = \int_V G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})d^3y + \frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}. \tag{45}$$

Since the instantaneous unfiltered velocities admit the generalized Galilean invariance, (5) can be substituted into the left-hand side. This yields

$$\int_V G(\boldsymbol{x}+\hat{\boldsymbol{x}}, \boldsymbol{y}+\hat{\boldsymbol{x}})\left[\boldsymbol{u}(\boldsymbol{y}) + \frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}\right]d^3y = \int_V G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})d^3y + \frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}. \tag{46}$$

Because of (11), $\mathrm{d}\hat{\boldsymbol{x}}/\mathrm{d}t$ cancels on both sides, and hence for arbitrary $\boldsymbol{u}$ the integrals are equal, provided

$$G(\boldsymbol{x}+\hat{\boldsymbol{x}}, \boldsymbol{y}+\hat{\boldsymbol{x}}) = G(\boldsymbol{x}, \boldsymbol{y}). \tag{47}$$

This functional equation can be transformed by differentiating with respect to $\hat{\boldsymbol{x}}$. The resulting first order partial differential equation has the unique solution

$$G = G(\boldsymbol{x} - \boldsymbol{y}). \tag{48}$$

An additional restriction on G is given due to frame invariance with respect to a fixed rotation. From the rotation invariance of the filtered velocities $\bar{u}_i^* = A_{ij}\bar{u}_j$ given in (26) and the definition of the filter, one can deduce that

$$\int_{V^*} G(\boldsymbol{x}^* - \boldsymbol{y}^*) u_i^*(\boldsymbol{y}^*) d^3y^* = A_{ij} \int_V G(\boldsymbol{x} - \boldsymbol{y}) u_j(\boldsymbol{y}) d^3y. \tag{49}$$

Employing (3), which results in $d^3y^* = d^3y$, the two integrals are equal except for the filter kernel. Hence, in order for (26) to hold for arbitrary u_i, the condition

$$G(\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{y})) = G(\boldsymbol{x} - \boldsymbol{y}) \tag{50}$$

must be satisfied. For arbitrary $\boldsymbol{A}$, the latter functional equation has the unique solution

$$G = G(|\boldsymbol{x} - \boldsymbol{y}|). \tag{51}$$

This corresponds to a known result from tensor invariant theory (see e.g. Spencer 1971): a scalar function depending on vectors or tensors can only depend on their scalar invariants. Tensor invariant theory is widely used in Reynolds averaged modeling; e.g. the scalar coefficient in the pressure-strain model depend only on scalar invariants. In (51) G depends only on the magnitude of the separation vector, which is the only invariant of a single vector. An additional consequence of (51) is that the averaging volume V in (10) is restricted to a sphere with center $\boldsymbol{x}$.

The last restriction on G follows from scaling invariance (36). For the present purpose the filter function G is not normalized, denoted by the superscript "u". Using (10) one can conclude from (36) that

$$\frac{\int_{V^*} G^{\mathrm{u}}(|\boldsymbol{x}^* - \boldsymbol{y}^*|) \boldsymbol{u}^*(\boldsymbol{y}^*) d^3y^*}{\int_{V^*} G^{\mathrm{u}}(|\boldsymbol{x}^* - \boldsymbol{y}^*|) d^3y^*} = \frac{\gamma}{\xi} \frac{\int_V G^{\mathrm{u}}(|\boldsymbol{x} - \boldsymbol{y}|) \boldsymbol{u}(\boldsymbol{y}) d^3y}{\int_V G^{\mathrm{u}}(|\boldsymbol{x} - \boldsymbol{y}|) d^3y} \tag{52}$$

and a corresponding relation for the pressure, not shown here, needs to hold. Using (6) the spatial scaling factor γ remains in the argument of G^{u} on the left-hand side of (52). As a result, γ can only cancel out for arbitrary $\boldsymbol{u}$ if G^{u} has the following form

$$G^{\mathrm{u}}(|\boldsymbol{x} - \boldsymbol{y}|) = A|\boldsymbol{x} - \boldsymbol{y}|^{\alpha} \tag{53}$$

where A and α are arbitrary constants. Using (11), the final form of the filter G is obtained

$$\mathcal{L}[\cdot](\boldsymbol{x}) = \frac{\alpha + 3}{4\pi l^{\alpha+3}} \int_{\mathcal{R}_l} |\boldsymbol{x} - \boldsymbol{y}|^{\alpha} [\cdot](\boldsymbol{y}) d^3y, \tag{54}$$

where $\mathcal{R}_l$ refers to a sphere with center $\boldsymbol{x}$ and radius l. (54) preserves all the symmetries in section 2. If the integration argument is sufficiently smooth, the integral converges for all $\alpha > -3$.

The time invariance, the reflection invariance, the pressure invariance, and the material indifference, even though not explicitly considered during the derivation, are consistent with (54).

The constraint for the filter function needs to hold for any filter operation used in LES. However, in practice it is only relevant for schemes that utilize an explicit filter, e.g. in the test filter used in Germano *et al.* (1991). In some LES models the actual form of the filter kernel does not appear explicitly in the computation, and the constraints for the filter derived above are irrelevant.

The restrictions on the filter kernel derived in this sub-section are rarely met by the filtering procedures used in practical applications. For computational convenience, explicit filtering at a given location is often performed by averaging values from adjacent grid points. As a result, in many applications, such as near-wall shear flows, the grid is highly anisotropic, and condition (26) is violated. To investigate this matter of grid dependence in LES, some empirical tests were performed by Scotti *et al.* (1997) to determine whether anisotropic meshes have an effect on isotropic turbulence. They show that on an anisotropic pencil-like grid, isotropic turbulence was severely influenced in an unphysical manner. However, by isotropization of the test-filter many of the features of isotropic turbulence could be restored. This result suggests that the isotropic filter kernel (51) may restore some of the physical properties of turbulence in large-eddy simulations.

4. Invariant properties of proposed large-eddy models

Almost all of the existing SGS models for large-eddy simulation of turbulence which have been proposed have the functional form:

$$\tau_{ik} = \mathcal{F}_{ik}[\bar{\boldsymbol{u}}; \boldsymbol{x}]. \tag{55}$$

In order to capture all of the invariance properties of the Navier-Stokes equations, (55) should reflect the same symmetries. Hence it is a necessary condition to have

$$\tau^*_{ik} = \mathcal{F}_{ik}[\bar{\boldsymbol{u}}^*; \boldsymbol{x}^*] \tag{56}$$

for all the transformations listed in Section 3.1. Nearly all SGS models proposed in the literature conform with time translation, rotation, and reflection invariance. However, as was first investigated by Speziale (1985), several SGS models (Biringen & Reynolds 1981, Moin & Kim 1982, Bardina, Ferziger & Reynolds 1983) are not Galilean invariant and, therefore, are also not invariant under the generalized Galilean transformation (5). In the present investigation it will be shown that several of the proposed SGS models are not scale invariant and not material indifferent. However, it will be demonstrated that a certain class of models, namely the dynamic models, obey all invariance properties derived in Section 3.

One of the most widely-used models in LES, the Smagorinsky model (Smagorinsky 1963), violates scale invariance but captures all other known symmetries. It is given by

$$\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = -C\Delta^2|\bar{\boldsymbol{S}}|\bar{S}_{ik} \qquad \text{where} \qquad \bar{S}_{ik} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_i}\right). \tag{57}$$

Δ is the filter width which is usually taken to be a function of the local grid spacing. In order to see the shortcoming of (57), Eqs. (36), (37), and (57) are used in (56) to yield

$$\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = -C\Delta^2|\bar{\boldsymbol{S}}|\bar{S}_{ik}\gamma^{-2}. \tag{58}$$

The latter expression is not form invariant since it is dependent on the arbitrary scaling parameter γ. The reason for this problem is the explicit external length scale that has been introduced into the model, which is not related to any turbulent length scale. This imposed length scale is particularly damaging in turbulent wall-bounded flows. To overcome this problem empirical wall-damping functions have been adopted to obtain reasonable results in the near-wall region. Wall-damping functions are widely used in conjunction with Reynolds averaged models. There, it has long been known that this approach is not frame invariant, and several new ideas have been put forward to overcome this problem.

Several new near-wall self-similar solutions or scaling laws have been derived in Oberlack (1997a,b) which rely heavily on the scaling symmetry. All near-wall scaling laws may be captured in a large-eddy simulation of turbulence when the symmetry properties of the Navier-Stokes equations are preserved by the model. In Appendix A it is shown by analyzing the two-point correlation equation that the Smagorinsky model is not able to capture important near-wall scaling laws. It can be concluded that any model which contains a fixed external length scale, and which does not account for the proper turbulent length scale, will violate the scaling symmetry. Since the Smagorinsky model is only written in terms of the strain rate $\bar{\boldsymbol{S}}$, material indifference is guaranteed.

A model which violates both scale invariance and material indifference is the structure-function model by Métais & Lesieur (1992). The latter problem has already been reported by Meneveau (1996). The proposed SGS model is of the form

$$\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = C^{SF}\Delta\langle(\bar{\boldsymbol{u}}(\boldsymbol{x}+\boldsymbol{r}) - \bar{\boldsymbol{u}}(\boldsymbol{x}))^2\rangle^{1/2}\bar{S}_{ik} \tag{59}$$

where C^{SF}, and $\langle\rangle$ are, respectively, a model constant and a spatial average. Using the condition (56) in conjunction with the transformation (40) and (41) yields

$$\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = C^{SF}\Delta\langle(\bar{\boldsymbol{u}}_{(m)}(\boldsymbol{x}+\boldsymbol{r}) - \bar{\boldsymbol{u}}_{(m)}(\boldsymbol{x}) - \varepsilon_{3l(m)}\Omega_3 r_l)^2\rangle^{\frac{1}{2}}\bar{S}_{ik}. \tag{60}$$

The latter expression is not of the form (59) since it contains an additional rotation term. Hence, the structure-function model is not materially indifferent. As for the Smagorinsky model, one can also show that (59) is not scale invariant.

A class of SGS models which have a similar deficiency are those explicitly containing the rotation rate

$$\bar{R}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} - \frac{\partial\bar{u}_j}{\partial x_i}\right). \tag{61}$$

Lund & Novikov (1992) derived the most general form of SGS model comprising all possible combinations of the strain and the rotation rate tensors. They proposed a model of the form

$$\begin{aligned}\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = \Delta^2 \Bigg[& C_1|\bar{\boldsymbol{S}}|\bar{S}_{ik} + C_2\left(\bar{S}_{im}\bar{S}_{mk} - \frac{\delta_{ik}}{3}\bar{S}_{mn}\bar{S}_{mn}\right) \\ & + C_3\left(\bar{R}_{im}\bar{R}_{mk} - \frac{\delta_{ik}}{3}\bar{R}_{mn}\bar{R}_{mn}\right) + C_4\left(\bar{S}_{im}\bar{R}_{mk} - \bar{R}_{im}\bar{S}_{mk}\right) \\ & + \frac{C_5}{|\bar{\boldsymbol{S}}|}\left(\bar{S}_{im}\bar{S}_{mn}\bar{R}_{nk} - \bar{R}_{im}\bar{S}_{mn}\bar{S}_{nk}\right)\Bigg]. \end{aligned} \tag{62}$$

For the same reason as the previous two models, (62) is also not scaling invariant. Violation of material indifference can be shown by computing the rotation rate (61) under the transformation (40) which yields

$$\bar{R}^*_{ij} = B_{ik}B_{jl}R_{kl} + \varepsilon_{kij}\Omega_k \tag{63}$$

Using this in (62), the required form of (41) under constant rotation cannot be recovered since the frame rotation term, i.e. the last term of (63), does not cancel out.

An SGS model which captures *all* the invariance requirements derived in Section 3 is the dynamic subgrid-scale model of Germano *et al.* (1991). They proposed a procedure which, used in conjunction with the classical Smagorinsky model, results in the following SGS model

$$\tau_{ik} - \frac{1}{3}\delta_{ik}\tau_{mm} = \frac{(\widetilde{\bar{u}_m\bar{u}_n} - \tilde{\bar{u}}_m\tilde{\bar{u}}_n)\bar{S}_{mn}}{\left(\frac{\tilde{\Delta}}{\Delta}\right)^2|\tilde{\bar{\boldsymbol{S}}}|\tilde{\bar{S}}_{mn}\bar{S}_{mn} - \widetilde{|\bar{\boldsymbol{S}}|\bar{S}_{pq}}\bar{S}_{pq}}|\bar{\boldsymbol{S}}|\bar{S}_{ik}. \tag{64}$$

Here, all the tilded quantities refer to the "test"-filter

$$\tilde{h}(\boldsymbol{x}) = \int_V \tilde{G}(\boldsymbol{x},\boldsymbol{y})h(\boldsymbol{y})d^3y, \tag{65}$$

which corresponds to the filter length $\tilde{\Delta}$ and $\tilde{\Delta} > \Delta$. The test-filter quantities are explicitly computed from the flow field. The resolved quantities are still denoted by an overbar. The dynamic model contains the ratio of two length scales, which is a dimensionless number, and therefore no external length scale is imposed to break symmetries. Consequently, the scaling invariance (37) is recovered, as can be shown by using (36), provided the proper filter function is utilized. It is straightforward to prove that frame invariance, generalized Galilean invariance, and material indifference are also captured by the dynamic model.

Since its publication by Germano *et al.* (1991), several modified versions of the dynamic model have been proposed. The model by Lilly (1992) keeps the Smagorinsky model as the base model, but the dynamic procedure is modified. Zang *et* *al.* (1993) used the mixed model, first introduced by Bardina *et al.* (1983), as a new base model. In addition, they employed Lilly's modification of the dynamic procedure. Yoshizawa *et al.* (1996) developed a new base model and also adopted Lilly's modification of the dynamic procedure. The dynamic mixed model by Zang *et al.* is further extended by Salvetti & Banerjee (1995). This new model contains two parameters which are both computed with a modified dynamic procedure. It can easily be shown that all the latter modified versions of the dynamic model capture the symmetry requirements developed in Section 3. It should be noted that the dynamic procedure only restores scaling invariance, which may be violated by certain base models. Other deficiencies such as the violation of Galilean invariance or material indifference cannot be repaired by the dynamic procedure.

So far, it was tacitly assumed that the symmetries are not broken by the filtering process. However, some of the common filter functions are not consistent with the symmetries of the Navier-Stokes equations. One of these is the Gaussian filter

$$G = \frac{1}{\pi^{3/2}\Delta^3} \exp\left[-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{\Delta^2}\right], \tag{66}$$

since it does not match the form (54). The scaling symmetry is violated by (66).

Another common filter function which is not consistent with the form of (54) is the spectral cut-off filter. In physical space it is given by

$$G = \prod_{i=1}^{3} \frac{\sin\left[\frac{\pi}{\Delta}(x_i - y_i)\right]}{\pi (x_i - y_i)}. \tag{67}$$

(67) violates both rotation and scaling invariance. It has already been pointed out by Vreman *et al.* (1994) that the latter filter should not be utilized as it may lead to unrealizable results. In Liu *et al.* (1994) it was shown by analyzing experimental results of a turbulent jet that (67) has a very prejudicial influence on the overall statistical behavior of SGS models.

The classical isotropic top-hat filter

$$G = \begin{cases} \dfrac{3}{4\pi\Delta^3} & \text{if } |\boldsymbol{x}-\boldsymbol{y}| < \Delta \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

is of the form (54) with $\alpha = 0$. Hence, it preserves all symmetry requirements of Navier-Stokes equations.

5. Summary and conclusions

The Navier-Stokes equations admit certain symmetries, that is, there are certain form-invariant transformations which preserve the equations. These symmetries are one of the most fundamental properties of the equations of motion. They reflect many features of classical mechanics. It was shown recently that certain statistical properties of turbulent shear flows follow from these symmetries (Oberlack 1997a/b).

To capture those statistical features of the Navier-Stokes equations that are associated with symmetries, the symmetries should be built into the SGS models and the filter functions in LES of turbulence. This leads to necessary conditions on the functional form of the SGS model and the filter kernel.

One particular symmetry, scale invariance, is violated by the most common SGS model, the Smagorinsky model, because it contains the grid size as an explicit length scale. This seriously impairs the ability of the model to describe turbulence. In particular, in near-wall turbulent flows it is known that the Smagorinsky model performs poorly and wall damping functions have to be used. In Appendix A it is shown that the violation of the scaling symmetry excludes important turbulent near-wall scaling laws such as the log law and the algebraic law. Other models such as the structure function model by Métais & Lesieur (1992) violate material indifference.

It appears that the dynamic Smagorinsky model by Germano *et al.* (1991) and its successors (e.g. Lilly 1992, Zang *et al.* 1993, Yoshizawa *et al.* 1996, Salvetti & Banerjee 1995) conserve the symmetries of Navier-Stokes equations. In fact, numerical simulations have shown (see Germano *et al.* 1991) that the dynamic model captures the proper near-wall behavior without introducing any artificial wall treatment such as damping functions.

The symmetry restrictions for the filter function are severe in the sense that only a very confined class of filters is allowed. For example, only a spherical filter function admits finite rotation invariance. The consequences of anisotropic filter functions may be illustrated by a simple example. Consider a simulation of homogeneous turbulent shear flow where explicit filtering is employed. The integration domain of the filter function may have the form of a box whose edges are aligned with the grid, which is chosen to be parallel to the mean flow. In homogeneous shear the dominant turbulent structures have a certain inclination to the mean flow. If the grid and the filter were instead chosen to be parallel to this inclination, averaging would take place over different flow structures. As a consequence, large scale quantities such as the Reynolds stress tensor would exhibit different growth rates. Since a model should be frame independent, the latter result is in contradiction to the basic physics of the problem.

However, the practical implications may be less severe than they appear. Since explicit filtering takes place on very few mesh points, the numerical truncation error may be of the same order of magnitude as the error caused by a non-spherical filter. Numerical tests for different applications need to be performed to determine how closely the filter form given by (54) has to be matched. A first test towards this requirement has been carried out by Scotti *et al.* (1997). An isotropized test-filtering in conjunction with the dynamic model on a highly anisotropic pencil-like mesh considerably improved the LES of isotropic turbulence.

An approach to overcome the very restricted form of the filter function may be to introduce the strain rate into the filter function. Since the strain rate introduces three additional directions corresponding to its principle axes, a more complex geometry for the filter volume may be in order.

Another consequence of the required spherical form of the filter appears to be its use in combination with wall-bounded flows. Close to solid walls the requirement that the filter be spherical filter is always violated, and hence certain symmetries are broken. However, the symmetries listed in Section 2 are only properties of the Navier-Stokes equations. Symmetries are always broken by arbitrary boundary conditions. One can conclude that a non-spherical filter near a solid wall is not a restriction of LES, but a consequence of boundary conditions for turbulence models in general.

It appears that future improvements for LES models should be along the lines of the dynamic model since it mimics fundamental properties of the Navier-Stokes equations. Despite its known superior performance, it has problems with stability since the model coefficient in the SGS model may become negative. If the flow under investigation possesses a homogeneous direction, averaging of the model coefficient in that direction seems to stabilize the simulation. In more complex geometries a clipping procedure is introduced which sets a negative model coefficient to zero. However, the first approach may violate rotation invariance since a preferred direction has been introduced. The clipping approach seems to obey all the symmetry properties of the Navier-Stokes equations but appears to be unrelated to Navier-Stokes equations.

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Appendix A. Two-point correlation equation of LES models containing an explicit external length scale

To investigate why SGS models containing an explicit length scale are inconsistent with certain near-wall scaling laws, two-point correlation equations derived from LES models are analyzed. The standard Reynolds decomposition is given by

$$\bar{\boldsymbol{u}} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}', \quad \bar{p} = \langle p \rangle + p', \tag{A1}$$

where the instantaneous velocity $\bar{\boldsymbol{u}}$ and the pressure $\bar{p}$ is assumed to be computed by a LES in conjunction with a certain SGS model and $\langle \cdot \rangle$ denotes an ensemble average. Using this, several two-point quantities may be defined

$$R_{ij}(\boldsymbol{x},\boldsymbol{r}) = \langle u_i'(\boldsymbol{x})\, u_j'(\boldsymbol{x}^{(1)}) \rangle, \tag{A2}$$

$$R_{(ik)j}(\boldsymbol{x},\boldsymbol{r}) = \langle u_i'(\boldsymbol{x}) u_k'(\boldsymbol{x}) u_j'(\boldsymbol{x}^{(1)}) \rangle, R_{i(jk)}(\boldsymbol{x},\boldsymbol{r}) = \langle u_i'(\boldsymbol{x}) u_j'(\boldsymbol{x}^{(1)}) u_k'(\boldsymbol{x}^{(1)}) \rangle \tag{A3}$$

$$P_j(\boldsymbol{x},\boldsymbol{r}) = \langle p'(\boldsymbol{x})\, u_j'(\boldsymbol{x}^{(1)}) \rangle, \quad Q_j(\boldsymbol{x},\boldsymbol{r}) = \langle u_j'(\boldsymbol{x})\, p'(\boldsymbol{x}^{(1)}) \rangle \tag{A4}$$

$$S_{(ik)j}(\boldsymbol{x},\boldsymbol{r}) = \langle \tau_{ik}(\boldsymbol{x}) u_j'(\boldsymbol{x}^{(1)}) \rangle, \quad T_{i(jk)}(\boldsymbol{x},\boldsymbol{r}) = \langle u_i'(\boldsymbol{x}) \tau_{jk}(\boldsymbol{x}^{(1)}) \rangle. \tag{A5}$$

Using the latter definitions the two-point correlation equations are derived from (14)

$$\frac{\mathrm{D}R_{ij}}{\mathrm{D}t} = -R_{kj}\,\frac{\partial\langle u\rangle_i}{\partial x_k} - R_{ik}\,\frac{\partial\langle u\rangle_j}{\partial x_k}\bigg|_{\boldsymbol{x}+\boldsymbol{r}} - \left[\langle u\rangle_k(\boldsymbol{x}+\boldsymbol{r},t) - \langle u\rangle_k(\boldsymbol{x},t)\right]\frac{\partial R_{ij}}{\partial r_k}$$
$$- \left[\frac{\partial P_j}{\partial x_i} - \frac{\partial P_j}{\partial r_i} + \frac{\partial Q_i}{\partial r_j}\right] - \frac{\partial R_{(ik)j}}{\partial x_k} + \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right]$$
$$- \frac{\partial S_{(ik)j}}{\partial x_k} + \frac{\partial}{\partial r_k}\left[S_{(ik)j} - T_{i(jk)}\right], \tag{A6}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \langle u\rangle_k \partial/\partial x_k$. The tensors in (A2)-(A5) are functions of the physical and correlation space coordinates, $\boldsymbol{x}$ and $\boldsymbol{r} = \boldsymbol{x}^{(1)} - \boldsymbol{x}$ respectively. The vertical line denotes the derivative to be taken with respect to $\boldsymbol{x}$ and evaluated at $\boldsymbol{x}+\boldsymbol{r}$.

In Oberlack (1997a) the 22-component of the two-point correlation equations emerging from the Navier-Stokes equations (A6 with $S_{(ik)j} = T_{i(jk)} = 0$) for parallel mean flows of the form $\langle \boldsymbol{u}\rangle = (\langle u\rangle_1(x_2), 0, 0)^T$ was investigated. The entire system contains one physical and three correlation coordinates and consists of equation (A6) and two Poisson equations for P_j and Q_i (not shown here). It was shown that for four distinct mean velocity profiles similarity variables can be introduced so that the number of independent variables is reduced by one.

The most general self-similar solution, with all group parameters different from zero, is given by

$$\bar{u}_1 = C_1\left(x_2 + \frac{q_4}{q_1}\right)^{1-\frac{q_5}{q_1}} - \frac{q_7}{q_1 - q_5}, \tag{A7}$$

$$\tilde{r}_1 = \frac{r_1 + \frac{q_2}{q_1}}{x_2 + \frac{q_4}{q_1}},\ \ \tilde{r}_2 = \frac{r_2}{x_2 + \frac{q_4}{q_1}},\ \ \tilde{r}_3 = \frac{r_3 + \frac{q_3}{q_1}}{x_2 + \frac{q_4}{q_1}}, \tag{A8}$$

$$R_{22} = \left(x_2 + \frac{q_4}{q_1}\right)^{2\left(1-\frac{q_5}{q_1}\right)}\tilde{R}_{22}, \tag{A9}$$

$$P_2 = \left(x_2 + \frac{q_4}{q_1}\right)^{3\left(1-\frac{q_5}{q_1}\right)}\tilde{P}_2,\ \ Q_2 = \left(x_2 + \frac{q_4}{q_1}\right)^{3\left(1-\frac{q_5}{q_1}\right)}\tilde{Q}_2, \tag{A10}$$

$$R_{(2k)2} = \left(x_2 + \frac{q_4}{q_1}\right)^{3\left(1-\frac{q_5}{q_1}\right)}\tilde{R}_{(2k)2},\ \ R_{2(2k)} = \left(x_2 + \frac{q_4}{q_1}\right)^{3\left(1-\frac{q_5}{q_1}\right)}\tilde{R}_{2(2k)}. \tag{A11}$$

where the "~" correlation quantities only depend on (A8). From (A7)-(A11) one can conclude that scaling of the fluctuation velocity is according to

$$\boldsymbol{u}' = \left(x_2 + \frac{q_4}{q_1}\right)^{1-\frac{q_5}{q_1}}\hat{\boldsymbol{u}}'. \tag{A12}$$

The second similarity solution is given by $q_1 = q_5$, which corresponds to the log-law and (A7) changes to

$$\bar{u}_1 = \frac{q_7}{q_1} \ln\left(x_2 + \frac{q_4}{q_1}\right) + C_2, \tag{A13}$$

while the similarity coordinates (A8) are unaltered and the correlation functions R_{22}, P_2, Q_2, $R_{(2k)2}$, and $R_{2(2k)}$ are un-scaled.

If $q_1 = 0$ and $q_5 \neq 0$, the exponential law holds and the new similarity variables are given by

$$\bar{u}_1 = C_3 \exp\left(-\frac{q_5}{q_4} x_2\right) + \frac{q_7}{q_5}, \tag{A14}$$

$$\tilde{r}_1 = r_1 + \frac{q_2}{q_4} x_2, \quad \tilde{r}_2 = r_2, \quad \tilde{r}_3 = r_3 + \frac{q_3}{q_4} x_2, \tag{A15}$$

$$R_{22} = \mathrm{e}^{-2\frac{q_5}{q_4} x_2} \tilde{R}_{22}, \tag{A16}$$

$$P_2 = \mathrm{e}^{-3\frac{q_5}{q_4} x_2} \tilde{P}_2, \quad Q_2 = \mathrm{e}^{-3\frac{q_5}{q_4} x_2} \tilde{Q}_2, \tag{A17}$$

$$R_{(2k)2} = \mathrm{e}^{-3\frac{q_5}{q_4} x_2} \tilde{R}_{(2k)2}, \quad R_{2(2k)} = \mathrm{e}^{-3\frac{q_5}{q_4} x_2} \tilde{R}_{2(2k)}. \tag{A18}$$

where similar to (A7)-(A11) the "~" correlation quantities only depend on (A15). It can be concluded that the fluctuation velocities scale as

$$\boldsymbol{u}' = \mathrm{e}^{-\frac{q_5}{q_4} x_2} \hat{\boldsymbol{u}}'. \tag{A19}$$

Finally, if $q_1 = q_5 = 0$, the mean velocity is given by

$$\bar{u}_1 = q_7 x_2 + C_4 \tag{A20}$$

while the similarity variables (A15) are the same as for the exponential case, but the correlations R_{22}, P_2, Q_2, $R_{(2k)2}$, and $R_{2(2k)}$ stay un-scaled.

In order to see that some common SGS models are not consistent with the latter scaling laws if they contain an explicit external length scale, the Smagorinsky model will be investigated. Suppose (57) is substituted for $\boldsymbol{\tau}$ in (A2)-(A5), then $S_{(ik)j}$ and $T_{i(jk)}$ will read as follows

$$S_{(ik)j}(\boldsymbol{x}, \boldsymbol{r}) = -C\Delta^2 \langle |\bar{\boldsymbol{S}}|(\boldsymbol{x}) \bar{S}_{ik}(\boldsymbol{x}) u_j'(\boldsymbol{x}^{(1)}) \rangle, \tag{A21}$$

$$T_{i(jk)}(\boldsymbol{x}, \boldsymbol{r}) = -C\Delta^2 \langle u_i'(\boldsymbol{x}) |\bar{\boldsymbol{S}}|(\boldsymbol{x}^{(1)}) \bar{S}_{jk}(\boldsymbol{x}^{(1)}) \rangle. \tag{A22}$$

Here $\bar{\boldsymbol{S}}$ is computed from (57) while for $\bar{\boldsymbol{u}}$ the Reynolds decomposition (A1) is used.

Using (21)-(22) in Eqs. (A6) leads to a reduced set of possible self-similar solutions. From the above-mentioned four scaling laws, only two allow for self-similarity so that the number of independent variables reduces by one. These two scaling laws are the exponential law (A14)-(A18) and the linear law (A20). Both have been derived under the assumption that there is an external symmetry breaking length scale in the flow and no scaling with respect to the coordinates exists. It is straightforward to show that the algebraic law (A7) and the logarithmic law (A13) are no longer self-similar solutions of the system (A6) if (21)-(22) is employed.

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